Discrete Mathematics 310 (2010) 2637–2654

Contents lists available at ScienceDirect

Discrete Mathematics

journal homepage: www.elsevier.com/locate/disc

Dense graphs have $K_{3,t}$ minors

A.V. Kostochka [a,b,*], N. Prince [c]

[a] Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA

[b] Sobolev Institute of Mathematics, Novosibirsk, Russia

[c] Illinois Mathematics and Science Academy, Aurora, IL, 60506, USA

ARTICLE INFO

Article history:
Received 1 December 2008
Accepted 25 March 2010
Available online 21 April 2010

Keywords:
Bipartite minors
Dense graphs

ABSTRACT

Let $K^*_{3,t}$ denote the graph obtained from $K_{3,t}$ by adding all edges between the three vertices of degree t in it. We prove that for each $t \geq 6300$ and $n \geq t+3$, each n-vertex graph G with $e(G) > \frac{1}{2}(t+3)(n-2)+1$ has a $K^*_{3,t}$-minor. The bound is sharp in the sense that for every t, there are infinitely many graphs G with $e(G) = \frac{1}{2}(t+3)(|V(G)|-2)+1$ that have no $K_{3,t}$-minor. The result confirms a partial case of the conjecture by Woodall and Seymour that every $(s+t)$-chromatic graph has a $K_{s,t}$-minor.

1. Introduction

Graphs in this paper are undirected simple graphs. For a graph G, $V(G)$ is the set of its vertices, $E(G)$ is the set of its edges, $e(G) = |E(G)|$, and $\overline{G}$ is the complement of G. By $G[X]$ we denote the subgraph of G induced by the vertex set X. By $e_G(X, Y)$ we denote the number of edges connecting disjoint sets X and Y. We let $N_G(v)$ denote the set of neighbors of v in G and $N_G[v] = N_G(v) \cup \{v\}$. Similarly, for $X \subseteq V(G)$, we define $N(X) := \bigcup_{x \in X} N(x)$ and $N[X] := \bigcup_{x \in X} N[x]$. *Contraction of edge xy* in G is the operation of replacing the vertices x and y with a new vertex, denoted as $x * y$, that is adjacent to all neighbors of x, all neighbors of y, and to no other vertices. A *minor* of a graph G is a graph H that can be obtained from G by a sequence of vertex and edge deletions and edge contractions. A subgraph F of G is an H-*minor* in G if H can be obtained from F by a sequence of edge contractions and deletions.

A famous open problem concerning graph minors is the Hadwiger Conjecture.

Conjecture 1 (*Hadwiger*)**.** *Every k-chromatic graph has a K_k-minor.*

The Conjecture is known to be true for $k \leq 6$ but remains open for all larger values of k. In order to stimulate attacks on the conjecture, Woodall [18] and independently Seymour [15] suggested proving the following weaker statement.

Conjecture 2. *Every $(s+t)$-chromatic graph has a $K_{s,t}$-minor.*

Another way to approach Hadwiger's Conjecture is to search for sufficient conditions other than k-chromaticity that force a graph to contain a K_k-minor. Mader [7] proved that for each positive integer k, there exists a function $D(k)$ such that every graph with average degree at least $D(k)$ has a K_k-minor by demonstrating that $D(k) \leq 2^{\binom{k}{2}+1}$. Later, Kostochka [3,4] and Thomason [16] determined that $D(k) = \Theta(k\sqrt{\log k})$. More recently, Thomason [17] determined that

$$D(k) = (\alpha + o(1))k\sqrt{\log k},$$

where $\alpha = 0.6381726\ldots$ is given explicitly.

* Corresponding author at: Department of Mathematics, University of Illinois, Urbana, IL, 61801, USA.
E-mail addresses: kostochk@math.uiuc.edu (A.V. Kostochka), nprince@imsa.edu (N. Prince).

0012-365X/$ – see front matter
doi:10.1016/j.disc.2010.03.026

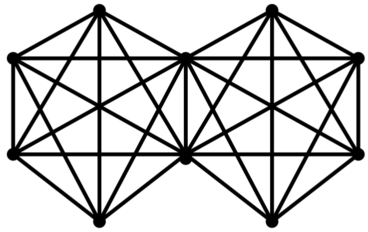

Fig. 1. Graph $M(2, 3, 4)$ has no $K_{3,4}$-minor.

Myers and Thomason extended the function D above to general graphs H, that is, they defined

$$D(H) = \inf\{d \mid 2e(G)/n(G) \geq d \text{ implies that } G \text{ has an } H\text{-minor}\}.$$

They determined [12,9] $D(H)$ for almost every H, showing, in particular, that for almost all H, the extremal graphs not containing H are quasi-random (built deterministically from randomly generated subcomponents). However, their methods work well only for *dense* graphs, i.e. for graphs H with average degree comparable with $|V(H)|$.

An example of a *sparse* H is $K_{s,t}$, where s is fixed and t is large with respect to s. For this reason, Myers [10,11] studied $D(K_{s,t})$ when s is fixed and t is large. Let $M(r, s, t)$ be the graph obtained by taking r copies of K_{s+t-1} arranged so that each two copies share the same fixed $s-1$ vertices (Fig. 1 shows $M(2, 3, 4)$). Myers [11] observed that $M(r, s, t)$ has no $K_{s,t}$-minor and that

$$e(M(r,s,t)) = \frac{1}{2}(t + 2s - 3)(n - s + 1) + \binom{s-1}{2}, \tag{1}$$

where $n = |V(M(r, s, t))| = rt + s - 1$. He proved the following.

Theorem 1 (*[11]*)**.** *Let $t > 10^{29}$ be a positive integer. Let G be a graph with $n \geq 3$ vertices such that*

$$e(G) > \frac{1}{2}(t + 1)(n - 1). \tag{2}$$

Then G has a $K_{2,t}$-minor.

The graphs $M(r, 2, t)$ witness that this bound is sharp when $|V(G)| \equiv 1 (\mathrm{mod}\ t)$.

In connection with Conjecture 2, recently, Chudnovsky et al. [1] proved that Theorem 1 in fact holds for all t. They used this result to prove that Conjecture 2 holds for $s = 2$ and each t.

Myers conjectured that a similar, more general statement is true for $K_{s,t}$-minors.

Conjecture 3. *Let s be a positive integer. Then there exists a constant $C(s)$ such that, for all positive integers t, if G has average degree at least $C(s) \cdot t$, then G has a $K_{s,t}$-minor.*

Let $K^*_{s,t} = K_{s+t} - E(K_t)$. In other words, $K^*_{s,t}$ is the graph obtained from $K_{s,t}$ by adding all $\binom{s}{2}$ possible edges into the s-vertex partite set. Myers noted that the average degree that forces G to contain a $K_{s,t}$-minor also likely forces a $K^*_{s,t}$-minor, that is, $D(K_{s,t}) = D(K^*_{s,t})$ when s is fixed and t is large.

Myers' Conjecture was proved independently in [5,6] using different methods. Kühn and Osthus [6] showed the following.

Theorem 2 (*[6]*)**.** *For every $\epsilon > 0$ and every positive integer s, there exists a number $t_0 = t_0(s, \epsilon)$ such that for all $t \geq t_0$, every graph of average degree at least $(1 + \epsilon)t$ contains $K^*_{s,t}$ as a minor.*

In [5], the following fact was proved.

Theorem 3. *Let s and t be positive integers with*

$$t > (240s \log_2 s)^{8s \log_2 s+1}.$$

Let G be a graph such that $e(G) \geq \frac{t+3s}{2}(n(G) - s + 1)$. Then G has a $K^*_{s,t}$-minor. Furthermore, for n large, there exists a graph G of order n and size at least $\frac{t+3s-5\sqrt{s}}{2}(n - s + 1)$ that has no $K_{s,t}$-minor.

From Theorem 3 we have that for huge t,

$$t + 3s - 5\sqrt{s} \leq D(K_{s,t}) \leq D(K^*_{s,t}) \leq t + 3s.$$

Hence, Myers' insight that $D(K_{s,t})$ is the same as $D(K^*_{s,t})$ is true asymptotically in s.

The second half of Theorem 3 shows that for $s > 100$, Myers' construction of $M(r, s, t)$ is not optimal. In Section 2, we provide another construction with fewer edges that shows that $M(r, s, t)$ is not optimal for $s \geq 6$. Note that while

Theorem 2 does not provide the dependence of $D(K_{s,t})$ on s, it applies for a much wider range of s than Theorem 3, namely for $s \leq C \cdot t/\log t$.

The goal of the present paper is to determine $D(K_{3,t})$ for $t > 6300$ exactly. We prove the slightly stronger version with $K^*_{3,t}$ in place of $K_{3,t}$.

Theorem 4. *Let $t \geq 6300$. Let G be a graph of order $n \geq 3$ with*

$$e(G) > \frac{1}{2}(t+3)(n-2) + 1. \tag{3}$$

Then G has a $K^*_{3,t}$-minor.

The graphs $M(r, 3, t)$ demonstrate the sharpness of Theorem 4 for the existence of minors for both $K^*_{3,t}$ and $K_{3,t}$.

Remark. If $t \geq 6300$ and for some $n \geq 3$, an n-vertex graph G satisfies (2), then adding a new vertex x adjacent to all vertices of G creates a graph G' with $n' = n + 1$ vertices that satisfies the conditions of Theorem 4. By this theorem, G' has a $K^*_{3,t}$-minor, and hence $G = G' - x$ has a $K^*_{2,t}$-minor. This implies Theorem 1 and the corresponding result of Chudnovsky et al. [1], restricted to $t \geq 6300$, in a slightly stronger form, namely, with the $K^*_{2,t}$-minor in place of the $K_{2,t}$-minor.

Seymour showed that Theorem 4 implies the validity of Conjecture 2 for $s = 3$ and $t \geq 6300$. With his kind permission, we present this proof here.

Corollary 5 (*Seymour*)**.** *Let $t \geq 6300$. Then every $(3+t)$-chromatic graph has a $K^*_{3,t}$-minor.*

Proof. Let G be a counter-example to this corollary with the smallest total number of edges and vertices. Then G is $(3+t)$-critical, and hence $\delta(G) \geq t+2$. If $\delta(G) \geq t+3$, then $e(G) \geq \frac{t+3}{2}|V(G)|$, and so by Theorem 4, G has a $K^*_{3,t}$-minor. Thus, G has a vertex v with $d_G(v) = t+2$. If $G[N(v)] = K_{t+2}$, then G contains K_{t+3} which contains $K^*_{3,t}$. So, $N(v)$ contains some non-adjacent vertices x and y.

Let G' be obtained from G by contracting the edges vx and vy. Since G' is a minor of G, it does not have a $K^*_{3,t}$-minor. Therefore, by the minimality of G, G' is $(t+2)$-colorable. Let f' be a proper $(t+2)$-coloring of G'. It naturally yields a proper $(t+2)$-coloring f of $G - v$ in which $f(x) = f(y)$. But then one of the $t+2$ colors is not used on $N(v)$, and we can use this color to color v, a contradiction to the definition of G. $\square$

We also show that Theorem 4 cannot be extended to $s \geq 6$. Namely, we prove the following two results.

Theorem 6. *Let s and t be integers satisfying $s \geq 6$ and $t > (2s)^{2s-1}$ such that $s+t$ is odd. Then for infinitely many $n > s+t$, there exists a graph $G(n, s, t)$ of order n with*

$$e(G(n, s, t)) > \frac{1}{2}(t + 2s - 3 + 3^{-s-1})(n - s + 1) + \binom{s-1}{2}$$

that has no $K_{s,t}$-minor.

The bound of this theorem is getting weaker when s grows. For larger s, the bound of the second part of Theorem 3 is better.

Theorem 7. *Let s and t be integers satisfying $t \geq s \geq 4$ such that $s+t$ is odd. Then for infinitely many $n > s+t$, there exists a graph $G(n, s, t)$ of order n with*

$$e(G(n, s, t)) > \frac{1}{2}\left(t + 2s - 2 - \frac{2s}{t}\right)(n - s + 1) + \binom{s-1}{2}$$

that has no $K^*_{s,t}$-minor.

The proof of our main theorem elaborates and refines the ideas of [5] and uses discharging to handle the most difficult case: the case of $n = t+5$.

The structure of the paper is the following. In the next section we prove Theorems 6 and 7. The subsequent five sections are devoted to the proof of Theorem 4. In Section 3 we cite and prove several auxiliary statements. In Sections 4–7, we consider several cases depending on how large the number n_0 of vertices of a minimum counter-example to our statement is. In Section 4, we set up the proof and handle the case $n_0 \leq 1.1t$, $n_0 \neq t+5$. In Sections 5 and 6 we consider the case $n_0 > 1.1t$. The singular case $n_0 = t+5$ is postponed to the last section. We conclude the paper with a couple of comments.

2. A lower bound for $s \geq 6$

For this section, it will be convenient to use the following definition of a $K_{s,t}$-minor of G. We say that G *has a* $K_{s,t}$*-minor* if there are a set $V_0 \in V(G)$ and a function $f: (V(G) - V_0) \to V(K_{s,t})$ such that $f^{-1}(v)$ induces a connected subgraph of G for all $v \in V(K_{s,t})$ and such that, for all $v_1v_2 \in E(K_{s,t})$, there is an $x_i \in f^{-1}(v_i)$ $(i = 1, 2)$ such that $x_1x_2 \in E(G)$.

We will need the following old result of Sauer [13]:

Theorem 8 ([13]). *Let $g \geq 5$ and $m \geq 4$. Then, for every even $n \geq 2(m-1)^{g-2}$, there exists an n-vertex m-regular graph of girth at least g.*

Lemma 9. *Let s and t be integers satisfying $s \geq 6$ and $t > (2s)^{2s-1}$ such that $s + t$ is odd. Then there exists a graph $G = G(s, t)$ of order $n = t + s + 3$ with*

$$e(G) \geq \frac{1}{2}(t + s + 3)\left(t + s - 2 + \frac{1}{3^s}\right)$$

that contains no $K_{s,t}$-minor.

Proof. Under the conditions of the lemma, the numbers $n = s + t + 3$, $g = s + 7$, and $m = 4$ satisfy the conditions of Theorem 8. Hence, there exists an n-vertex 4-regular graph $H = H(s, t)$ with girth at least $s+7$. Since the number of vertices at distance at most j from a given edge in H is less than $4 \cdot 3^j$, we can greedily find a set A of at least $\frac{n}{2 \cdot 3^s}$ edges in H at distance at least s from each other. Let $H' = H'(s, t) = H - A$. Let us prove that

$$|N_{H'}(U) - U| \geq 7 \quad \text{for every } U \subset V(H') \text{ with } s - 3 \leq |U| \leq s. \tag{4}$$

Indeed, let u satisfy $s - 3 \leq u \leq s$ and U be a set of u vertices in H'. Let $W = N_{H'}(U) - U$. Since the girth of H' is greater than s, $H'[U]$ has no cycles. So, if $H'[U]$ has x edges, k components, and ℓ vertices of degree 3, then $x + k = u$ and $e_{H'}(U, W) = 4u - \ell - 2x$. Furthermore, since vertices of degree 3 are at distance at least $s \geq u$ from each other, each component of $H'[U]$ has at most one such vertex, and hence $\ell \leq k = u - x$. Suppose $|W| \leq 6$. Then $|U \cup W| \leq s + 6$ and hence $H'[U \cup W]$ has no cycles. Therefore, $e(H'[U \cup W]) \leq |U \cup W| - 1 \leq u + 5$. On the other hand, by the above,

$$e(H'[U \cup W]) \geq x + (4u - \ell - 2x) = 3u + (u - \ell - x) \geq 3u.$$

So, $3u \leq u + 5$ and $u \leq 2$. It follows that $s \leq u + 3 \leq 5$. This contradiction implies (4).

Let $G = \overline{H'}$. Suppose that G has a $K_{s,t}$-minor. Let $S \subseteq V(G)$ be the set of vertices in the pre-image of the smaller partite set of this minor, and let $S' \subseteq S$ be the vertices that are not deleted or contracted with a neighbor to get the minor. By (4) with $U = S'$, there must be at least seven vertices with a non-neighbor in S', and at least one of these vertices x is the entire pre-image of a vertex of the larger partite set in the minor. This contradicts the fact that every vertex of S' is adjacent to x. Therefore G has no $K_{s,t}$-minor.

Since $\Delta(H') = 4$ and at least $\frac{n}{3^s}$ vertices of H' have degree 3,

$$2e(G) \geq n(n-5) + \frac{n}{3^s} = (t + s + 3)\left(t + s - 2 + \frac{1}{3^s}\right). \quad \square$$

Now we are ready to prove Theorem 6

Proof. Consider $G(s, t)$ and $H'(s, t)$ from the proof of Lemma 9. Since $\Delta(H'(s, t)) = 4$ and $s + t + 3 > 5s$, $H'(s, t)$ has an independent set I of size $s - 1$. Then I induces an $(s-1)$-clique in $G(s, t)$. Let $G'(r, s, t)$ be obtained from r copies of $G(s, t)$ by arranging them so that each two copies share the set I and nothing else. This is an analog of $M(r, s, t)$; only the bricks are different. By construction,

$$|V(G'(r, s, t))| = (s - 1) + r(s + t + 3 - (s - 1)) = (s - 1) + r(t + 4) \tag{5}$$

and

$$e(G'(r, s, t)) = r \cdot e(G(s, t)) - (r - 1)\binom{s-1}{2} = r\left(e(G(s, t)) - \binom{s-1}{2}\right) + \binom{s-1}{2}. \tag{6}$$

Since for $s \geq 6$ and $t > (2s)^{2s-1}$,

$$e(G(s, t)) \geq \frac{(t + s + 3)\left(t + s - 2 + \frac{1}{3^s}\right)}{2} > (t + 4)\frac{t + 2s - 3 + 3^{-s-1}}{2} + \binom{s-1}{2},$$

we get by (6) and (5) for $n = |V(G'(r, s, t))|$ that

$$e(G'(r, s, t)) > r(t + 4)\frac{t + 2s - 3 + 3^{-s-1}}{2} + \binom{s-1}{2} = (n - s + 1)\frac{t + 2s - 3 + 3^{-s-1}}{2} + \binom{s-1}{2}.$$

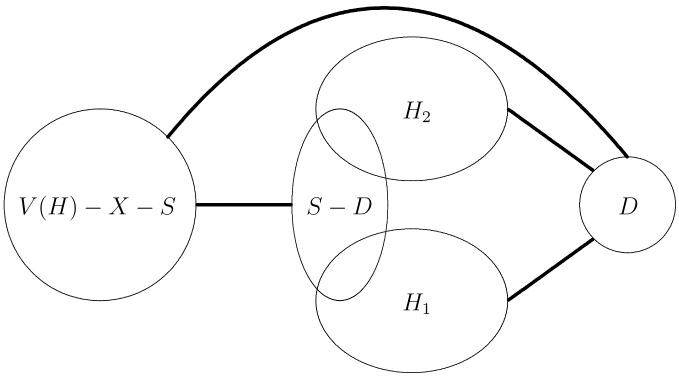

Fig. 2. Lemma 10.

Suppose that $G'(r, s, t)$ has a $K_{s,t}$-minor with partite sets X_s and X_t and $f\colon (V(G) - V_0) \to V(K_{s,t})$ is the corresponding function. Since $|I| < s$, the pre-image of some vertex in X_s avoids I and is contained in some copy C_1 of $G(s, t)$. Then the pre-image of each vertex in X_t has a vertex in C_1 and hence at least $t - s + 1$ pre-images of vertices in X_t are contained in $C_1 - I$. It follows that the pre-image of each vertex of our $K_{s,t}$ has a vertex in C_1. Since these pre-images induce connected subgraphs of $G'(r, s, t)$, each of the pre-images that is not completely in $C_1 - I$ contains a vertex in I. Since I induces a complete subgraph of $G'(r, s, t)$, $f^{-1}(v) \cap V(C_1)$ induces a connected subgraph for every $v \in V(K_{s,t})$. For the same reason, if the pre-images of some two vertices of $K_{s,t}$ are connected by an edge in $G'(r, s, t)$, then their intersections with $V(C_1)$ are also connected by an edge. It follows that C_1 also has a $K_{s,t}$-minor, a contradiction to Lemma 9.

Thus, for every n of the form $n = (s - 1) + r(t + 4)$, the graph $G(n, s, t) = G'(r, s, t)$ satisfies the statement of the theorem. Note that the difference between $e(G(n, s, t))$ and the right-hand side of (1) is linear in n. $\square$

The proof of Theorem 7 below essentially repeats that of Theorem 6; only the starting brick is different.

Proof of Theorem 7. Let $G(s, t)$ be the graph on $s + t + 1$ vertices whose complement is a perfect matching. Clearly, contracting an edge in $G(s, t)$ creates at most three all-adjacent vertices. Thus $G(s, t)$ has no $K^*_{s,t}$-minor and

$$e(G(s, t)) = \frac{(s + t + 1)(s + t - 1)}{2} > \frac{t + 2}{2}\left(t + 2s - 2 - \frac{2s}{t}\right) + \binom{s - 1}{2}.$$

Fix the vertex set I of an $(s - 1)$-clique in $G(s, t)$ and let $G'(r, s, t)$ be obtained from r copies of $G(s, t)$ by arranging them so that each two copies share the set I and no other vertices. Repeating the proof of Theorem 6 (with slightly different calculations) we obtain that $G'(r, s, t)$ has no $K^*_{s,t}$-minor and that for $n = |V(G'(r, s, t))|$,

$$e(G'(r, s, t)) > r(t + 2)\frac{t + 2s - 2 - \frac{2s}{t}}{2} + \binom{s - 1}{2} = (n - s + 1)\frac{t + 2s - 2 - \frac{2s}{t}}{2} + \binom{s - 1}{2}. \quad \square$$

3. Lemmas on connectivity and domination

If H is a graph and $X \subset V(H)$, we say that X is *k-separable* if $N[X] \neq V(H)$ and $|N(X) - X| \leq k$.

Lemma 10. *Let k be a positive integer and H be a graph such that each edge of H belongs to at least $3k/2$ triangles. If X is an inclusion-minimal k-separable set in H and $S = N(X) - X$, then $H[X \cup S]$ is $(1 + \lceil k/2 \rceil)$-connected.*

Proof. Assume that there is a separating set D of $H[X \cup S]$ with $|D| \leq \lceil k/2 \rceil$. Let H_1 be a component of $H[X \cup S] - D$ that has minimum size of intersection with S, and let $H_2 = H[X \cup S] - D - H_1$. Then the set $S_1 = D \cup (S \cap V(H_1))$ has at most $|D| + |S|/2 \leq k$ vertices. If $H_1 - S_1 \neq \emptyset$, then S_1 separates $H_1 - S_1$ from the rest of the graph (see Fig. 2), and $H_1 - S_1$ is properly contained in X, contradicting the minimality of X. Therefore $V(H_1) \subseteq S$. Let $y \in V(H_1)$. By the definition of S, y has some neighbor $x \in X$. Since xy belongs to at least $3k/2$ triangles, y is adjacent to at least $3k/2 + 1$ vertices in $X \cup S$. Since $|S| \leq k$ and $y \in S$, y has at least $k/2 + 2$ neighbors in X. It follows that y has a neighbor in $X - D$, a contradiction to $V(H_1) \subseteq S$. $\square$

Let U_1, U_2, and U_3 be disjoint sets of vertices in a graph G. Then a path P is a *(U_1, U_2)-path* if one end of P is in U_1 and the other is in U_2. Similarly P is a strict *$((U_1, U_2) - U_3)$-path* if one end of P is in U_1, the other is in U_2 and no internal vertex of P is in $U_1 \cup U_2 \cup U_3$. Furthermore, a pair (P_1, P_2) of paths is *(U_1, U_2, U_3)-connecting* if for some $i \in \{1, 2, 3\}$, one of the paths is a strict $((U_i, U_{i+1}) - U_{i+2})$-path and the other is a strict $((U_{i-1}, U_i) - U_{i+1})$-path (indices sum modulo 3). Note that the paths in a (U_1, U_2, U_3)-connecting pair may share an end (in the set U_i) and also internal vertices (outside of $U_1 \cup U_2 \cup U_3$).

Lemma 11. *Let G be a graph and let U_1, U_2, and U_3 be disjoint sets of vertices in G. If G contains a U_1, U_2-path P_1 and a U_1, U_3-path P_2 which is vertex-disjoint from P_1, then $P_1 \cup P_2$ contains a (U_1, U_2, U_3)-connecting pair of paths.*

Proof. For $i = 1, 2$, let P'_i be a shortest subpath of P_i that starts at U_1 and finishes at U_{1+i}. If neither of the P'_i intersects U_{4-i}, then the pair (P'_1, P'_2) is (U_1, U_2, U_3)-connecting. Suppose that P'_1 meets U_3. Let Q_1 be the subpath of P'_1 from U_1 to the first vertex in $U_3 \cap V(P_1)$ and Q_2 be the subpath of P'_1 from the last vertex in $U_3 \cap V(P_1)$ to U_2. Then the pair (Q_1, Q_2) is (U_1, U_2, U_3)-connecting. $\square$

For a graph G, a set $T \subseteq V(G)$ is *totally dominating* if every vertex of G has a neighbor in T. We say that a set $T \subseteq V(G)$ is *connected* if $G[T]$ is connected.

Lemma 12. *Let G be an n-vertex connected graph with minimum degree $k \geq 1$. Then:*

(a) *G contains a totally dominating set T with $|T| \leq \lfloor \log_{n/(n-k)} n \rfloor + 1$; and*
(b) *G contains a connected totally dominating set T' with $|T'| \leq 2 \log_{n/(n-k)} n$.*

Proof. Let $A \subseteq V(G)$. The total number of neighbors of vertices in A counted with multiplicities is at least $k|A|$. Hence

$$\text{there exists } v_A \in V(G) \text{ that is adjacent to at least } k|A|/n \text{ vertices in } A. \tag{7}$$

Consider the sequence $A_0, A_1, \ldots$, where $A_0 = V(G)$ and for $i \geq 1$, $A_i = A_{i-1} - N(v_{A_{i-1}})$. By (7), for every $i \geq 1$, $|A_i| \leq \frac{n-k}{n} |A_{i-1}|$. It follows that for $i_0 = \lfloor \log_{n/(n-k)} n \rfloor + 1$,

$$|A_{i_0}| \leq n \left(\frac{n-k}{n} \right)^{i_0} < n \left(\frac{n-k}{n} \right)^{\log_{n/(n-k)} n} = 1,$$

and so $A_{i_0} = \emptyset$. Hence $T = \{v_{A_0}, v_{A_1}, \ldots, v_{A_{i_0-1}}\}$ is totally dominating. This proves (a).

Let $C_1, \ldots, C_m$ be the vertex sets of the components of $G[T]$. Since T is totally dominating, each C_j has at least two vertices. It follows that $m \leq i_0/2$. Let $T' = T$ and $C_0 = C_1$. We do the following iteration for C_0: If C_0 dominates $V(G)$, then stop. Otherwise, choose any vertex w at distance exactly 2 from C_0. Let w' be the intermediate vertex on a shortest path from C_0 to w. By the choice of T, w has a neighbor $z \in T - C_0$. By definition, z belongs to some C_j. Add to T' vertices w and w' and let the new C_0 be the component of the new T' that contains $C_0 \cup C_j \cup \{w, w'\}$. This increases $|T'|$ by 2 and decreases the number of components in $G[T']$ by at least 1.

After at most $m - 1$ iterations, we obtain a connected totally dominating set T'. By construction, $|T'| \leq |T| + 2(m-1) \leq i_0 + 2(i_0/2 - 1) = 2i_0 - 2 \leq 2 \log_{n/(n-k)} n$. $\square$

Applying Lemma 12 s times, we have the following corollary.

Lemma 13. *Let s, k, and n be positive integers. Suppose $n > k \geq 1$. Let H be a graph of order n with $\delta(H) \geq k + 2(s-1) \log_{n/(n-k)} n$ and connectivity greater than $2(s-1) \log_{n/(n-k)} n$. Then $V(H)$ contains s disjoint subsets $A_1, \ldots, A_s$ such that, for every $i = 1, \ldots, s$,*

(i) *$H[A_i]$ is connected,*
(ii) *$|A_i| \leq 2 \log_{n/(n-k)} n$,*
(iii) *A_i dominates $H - \bigcup_{j=1}^{i-1} A_j$.*

Schönheim [14], Mills [8] and others, for $m \geq k \geq l$, studied the minimum number $C(m, k, \ell)$ of k-element subsets of an m-element set S that cover all ℓ-tuples of elements of S. We will use the following bounds on $C(m, k, \ell)$ due to Schönheim and Mills.

Lemma 14 (*[14,8]*). (a) *For all $m \geq k \geq \ell \geq 1$,*

$$C(m, k, \ell) \geq \left\lceil \frac{m}{k} C(m-1, k-1, \ell-1) \right\rceil; \tag{8}$$

(b) *if $m/k > 9/5$, then $C(m, k, 2) \geq 6$;*

(c) *if $m/k > 7/3$, then $C(m, k, 2) \geq 8$.*

Erdős et al. [2] proved a result whose partial case is the following.

Lemma 15 (*[2]*). *If $0 \leq k < (5n-3)/9$ and H is an n-vertex graph with maximum degree k and diameter 2, then $e(H) \geq 4n - 2k - 11$. In particular, if $k \leq 0.5n + \alpha$ and $\alpha \geq 0$, then $e(H) \geq 3n - 11 - 2\alpha$.*

4. Preliminaries and graphs of small order

We will prove Theorem 4 by contradiction. Suppose that the theorem is false. Then there exists a counter-example G_0 which is minimum with respect to $|V(G)| + |E(G)|$. Suppose that $n_0 = |V(G_0)|$. Our starting point is the following lemma concerning properties of such minimum counter-examples.

Lemma 16. *Let $t \geq 3$. Let G_0 be a graph minimum with respect to $|V(G)| + |E(G)|$ satisfying (3) such that $n_0 = |V(G_0)| \geq 3$ and G_0 has no $K^*_{3,t}$-minor. Then:*

(p0) $n_0 \geq t + 4$;
(p1) $\frac{1}{2}(t+3)(n_0-2) + 1 < e(G_0) \leq \frac{1}{2}(t+3)(n_0-2) + 2$;
(p2) *each edge of G_0 belongs to at least $(t+2)/2$ triangles;*
(p3) $\delta(G_0) \geq (4+t)/2$;
(p4) *G_0 is 3-connected.*

Proof. Since no n-vertex graph can have more than $\binom{n}{2}$ edges, (3) yields

$$\frac{1}{2}(t+3)(n_0-2) + 1 < \frac{n_0(n_0-1)}{2}.$$

For $n_0 \geq 3$ this is equivalent to $t+3 < n_0+1$, i.e. $n_0 \geq t+3$. Suppose that $n_0 = t+3$. Then (3) gives $2e(G_0) > n_0(n_0-2)+2$. It follows that at least three vertices have degree $n_0 - 1$, i.e., are all-adjacent ones in G_0. So, G_0 contains $K^*_{3,t}$. This proves (p0).

Property (p1) holds by (3) and the minimality of G_0. If an edge e of G_0 belongs to at most $(t+1)/2$ triangles, then after contracting e we obtain from G_0 a graph G'_0 with one vertex fewer and no more than $1 + (t+1)/2 = (t+3)/2$ fewer edges. So if G_0 satisfies (3), then G'_0 also satisfies (3) and by (p0) has at least $t + 4 - 1$ vertices. This contradicts the minimality of G_0. So, (p2) holds, and (p3) follows from (p2).

Let us prove (p4). Suppose otherwise. Then there is $S \subset V(G_0)$ such that $|S| \leq 2$ and $G_0 - S$ is disconnected. Then there are $V_1, V_2 \subset V(G_0)$ such that $V_1 - S, V_2 - S \neq \emptyset$, $V_1 \cup V_2 = V(G_0)$, $V_1 \cap V_2 = S$, and $V_1 - S$ has no neighbors in $V_2 - S$. Let $n_i = |V_i|$ for $i = 1, 2$ and $n_1 \geq n_2$. By (p3), $\delta(G_0) \geq \lceil (4+t)/2 \rceil \geq 4$. Hence $n_2 + |S| \geq 1 + \delta(G_0) \geq 5$, and so $n_2 \geq 3$.

Case 1: $|S| \leq 1$. By the minimality of G_0 and the fact that $n_1, n_2 \geq 3$, we have $e(G_0[V_i]) \leq \frac{1}{2}(t+3)(n_i-2) + 1$. So,

$$e(G_0) = e(G_0[V_1]) + e(G_0[V_2]) \leq \frac{(t+3)(n_1+n_2-4)}{2} + 2 \leq \frac{(t+3)(n_0-3)}{2} + 2,$$

a contradiction to (3).

Case 2: $|S| = 2$. Let $S = \{x, y\}$. For $i = 1, 2$, let G_i be obtained from $G_0[V_i]$ by adding edge xy, if it does not belong to G_0. Since Case 1 does not hold, each of $G_0[V_1]$ and $G_0[V_2]$ contains an x, y-path and so G_i is a minor of G_0 for $i = 1, 2$. Again by the minimality of G_0 and the fact that $n_1, n_2 \geq 3$, we have $e(G_i) \leq \frac{1}{2}(t+3)(n_i-2) + 1$. Furthermore, $e(G_0) \leq e(G_1) + e(G_2) - 1$, since either we count edge xy twice or have added extra edges. Therefore,

$$e(G_0) \leq e(G_1) + e(G_2) - 1 \leq \frac{(t+3)(n_1+n_2-4)}{2} + 1 \leq \frac{(t+3)(n_0-2)}{2} + 1,$$

a contradiction to (3), again. $\square$

In the course of our proof, we will increase the lower bound on n_0. For small n_0, the complement of G_0 has far fewer edges than G_0 and it is easier to understand its structure. Let $H_0 = \overline{G_0}$. Then phrasing (p1) and (p3) in terms of H_0, we get the following.

Lemma 17. *Let $t \geq 3$. Let G_0 be a minimum with respect to $|V(G)| + |E(G)|$ graph satisfying (3) such that $n_0 = |V(G_0)| \geq 3$ and G_0 has no $K^*_{3,t}$-minor. Let $d = n_0 - t$ and $H_0 = \overline{G_0}$. Then:*

(q1) $\frac{1}{2}(d-2)(n_0-2) - 1 \leq e(H_0) < \frac{1}{2}(d-2)(n_0-2)$;
(q2) $\Delta(H_0) \leq \frac{n_0+d}{2} - 3$.

Lemma 18. $n_0 \neq t + 4$.

Proof. Suppose $n_0 = t + 4$. By (q1) in Lemma 17, $e(H_0) < n_0 - 2$. Hence H_0 has at least three tree components, say C_1, C_2, and C_3. If all three are singletons, then G_0 contains $K^*_{3,t+1}$. If C_2 and C_3 are singletons and C_1 is not, then deleting the neighbor, say x, of a leaf in C_1 we will have three isolated vertices in $H_0 - x$, which correspond to three all-adjacent vertices in $G_0 - x$. Finally suppose that C_1 and C_2 are not singletons. For $i = 1, 2$, let y_i be a leaf in C_i and x_i be the neighbor of y_i in C_i. Then contracting in G_0 the edge x_1x_2 creates a $(t+3)$-vertex graph with all-adjacent vertices y_1, y_2, and $x_1 * x_2$. $\square$

The next statement has quite a long proof which we postpone to the last section.

Lemma 19. $n_0 \neq t+5$.

For the time being, we continue our proof assuming that Lemma 19 holds. To handle the cases $t+6 \le n_0 \le t+23$, we need a couple of auxiliary facts.

Lemma 20. *Let $t \ge 162$. Let G be a graph of order $n \ge t+6$ such that $e(\overline{G}) < 17n/6 - 2$ and $\Delta(\overline{G}) \le (n+3)/2$. Then G has a $K^*_{3,t}$-minor.*

Proof. For $n \ge t+6 \ge 168$, $(n+3)/2 < (5n-3)/9$ and $17n/6 - 2 \le 3n - 11 - 2(3/2)$, so by Lemma 15, $\overline{G}$ has two vertices x_1 and x_2 at distance at least 3. Note that $X = \{x_1, x_2\}$ is a connected dominating set in G. Let $G_1 = G - X$ and $n_1 = |V(G_1)| = n-2$. Again, for $n \ge 168$, $(n+3)/2 = (n_1+5)/2 < (5n_1-3)/9$ and $17n/6 - 2 \le 3n_1 - 11 - 2(5/2)$, so by Lemma 15, $\overline{G}_1$ has two vertices y_1 and y_2 at distance at least 3. Again, $Y = \{y_1, y_2\}$ is a connected dominating set in G_1. Let $G_2 = G_1 - Y$ and $n_2 = |V(G_2)| = n-4$. Since $n_0 \ge 168$, $(n+3)/2 = (n_2+7)/2 < (5n_2-3)/9$ and $17n/6 - 2 \le 3n_2 - 11 - 2(7/2)$, so by Lemma 15, $\overline{G}_2$ has two vertices z_1 and z_2 at distance at least 3. Contracting in G edges x_1x_2, y_1y_2 and z_1z_2, we get a graph containing $K^*_{3,n-6} \supseteq K^*_{3,t}$. □

Here is another fact in a similar spirit.

Lemma 21. *Let $k \ge 3$ and $n \ge 4(k^2+3k+6)$. Let H be an n-vertex graph with maximum degree at most $n/2+k$ and $e(H) \le (k+1.5)n/2 - k$. If at most two vertices of H have degree less than k, then H contains three disjoint pairs (x_i, y_i) of vertices such that $d_H(x_i, y_i) \ge 3$ for $i = 1, 2, 3$.*

Proof. The total number of pairs of distinct vertices at distance at most 2 in H is at most $e(H)$ plus the number of paths of length 2 in H. Denoting this value by $F(H)$, we have

$$F(H) \le e(H) + \sum_{v \in V(H)} \binom{d_H(v)}{2} = \frac{1}{2}\sum_{v \in V(H)} d_H(v) + \frac{1}{2}\sum_{v \in V(H)} d_H(v)(d_H(v)-1) = \frac{1}{2}\sum_{v \in V(H)} d_H^2(v).$$

Under the conditions of the lemma, the maximum of the last sum is attained when two vertices have degree 0 and all other vertices apart from at most one have degree either $k+n/2$ or k. Recall that $\sum_{v \in V(H)}(d_H(v)-k) \le 1.5n - 2k$. In this situation, the sum of the squares of the degrees of the vertices with degree greater than k is at most $3(k+n/2)^2$. Thus,

$$F(H) \le \frac{1}{2}\left[3(k+n/2)^2 + (n-5)k^2\right] = \frac{1}{2}\left[\frac{3n^2}{4} + n(3k+k^2) - 2k^2\right] < \frac{n^2}{2} - 3n.$$

It follows that H has at least $\binom{n}{2} - F(H) \ge 3n - n/2 = 2.5n$ pairs of vertices at distance at least 3. Hence some three of these pairs are disjoint. □

The next fact is based on Lemma 14.

Lemma 22. *Let $t \ge 231$. Let G be a graph of order $n \ge t+9$ such that $e(\overline{G}) \le 3.75n - 6$ and $\Delta(\overline{G}) \le n/2 + 6$. Then G has a $K^*_{3,t}$-minor.*

Proof. Let G satisfy the conditions of the lemma. Order the vertices $x_1, x_2, \ldots, x_n$ of $\overline{G}$ so that $d_{\overline{G}}(x_1) \le d_{\overline{G}}(x_2) \le \cdots \le d_{\overline{G}}(x_n)$. For $i = 1, \ldots, n$, let $N_i = N_{\overline{G}}(x_i)$.

Case 1: $d_{\overline{G}}(x_3) \le 5$. Each of the neighbors of x_1 in $\overline{G}$ has at most $\Delta(\overline{G}) - 1 \le n/2 + 5 < 5(n-8)/9$ neighbors in $V(G) - N_1 - \{x_1, x_2, x_3\}$. So, by Lemma 14(b) for $m = n-8$ and $k = \Delta(\overline{G}) - 1$, $V(G) - N_1 - \{x_1, x_2, x_3\}$ contains a pair (y_1, z_1) of vertices such that no vertex in $\overline{G}$ is adjacent to all of x_1, y_1 and z_1. This means that $\{x_1, y_1, z_1\}$ is a connected dominating set in G. Similarly, each of the neighbors of x_2 has at most $\Delta(\overline{G}) - 1 \le n/2 + 5 < 5(n-10)/9$ neighbors in $V(G) - N_2 - \{x_1, x_2, x_3, y_1, z_1\}$. Again, by Lemma 14(b), $V(G) - N_2 - \{x_1, x_2, x_3, y_1, z_1\}$ contains a pair (y_2, z_2) of vertices such that $\{x_2, y_2, z_2\}$ is a connected dominating set in G. Finally, since $\Delta(\overline{G}) - 1 \le n/2 + 5 < 5(n-12)/9$, by Lemma 14(b), $V(G) - N_3 - \{x_1, x_2, x_3, y_1, z_1, y_2, z_2\}$ contains a pair (y_3, z_3) of vertices such that $\{x_3, y_3, z_3\}$ is a connected dominating set in G. Contracting these three sets in G, we find a $K^*_{3,t}$-minor of G.

Case 2: $d_{\overline{G}}(x_3) \ge 6$. Then n and $H = \overline{G}$ satisfy conditions of Lemma 21 with $k = 6$. Thus by this lemma, $\overline{G}$ has three disjoint pairs of vertices at distance at least 3. Contracting the corresponding edges in G, we find a $K^*_{3,n-6}$-minor of G. □

Now we are ready to say more about our minimum counter-example G_0 to Theorem 4 and about its complement H_0.

Lemma 23. *Let $t \ge 432$. Then $n_0 \notin \{t+6, t+7, \ldots, t+22\}$.*

Proof. Recall that H_0 satisfies (q1) and (q2). Let $d = n_0 - t$.

Case 1: $6 \le d \le 7$. By (q1) and (q2), $e(H_0) < 5(n_0 - 2)/2$ and $\Delta(H_0) \le (n_0 + 1)/2$. So, we are done by Lemma 20.

Case 2: $d = 8$. By (q1) and (q2), $e(H_0) < 3(n_0 - 2)$ and $\Delta(H_0) \le n_0/2 + 1$. If $\Delta(H_0) \ge n_0/6 - 1$, consider $G'_0 = G_0 - v$, where v has maximum degree in H_0. Then $e(\overline{G}'_0) < 17n'_0/6 - 2$ and $\Delta(\overline{G}'_0) \le (n'_0 + 3)/2$, where $n'_0 = |V(G'_0)| = n_0 - 1$, and so G'_0 satisfies the conditions of Lemma 20. This yields our statement.

Suppose now that $\Delta(H_0) < n_0/6 - 1$. By (q1), H_0 has three vertices, x, y, z, of degree at most $d - 3 = 5$. Then each of x, y, z has fewer than $5(n_0/6 - 1)$ vertices at distance 1 or 2 from it. Hence, we can choose for each of them a vertex at distance at least 3 in H_0 so that all chosen vertices are distinct. Contracting the corresponding edges in G_0, we get K^*_{3,n_0-6}.

Case 3: $d = 9$. By (q1) and (q2), $e(H_0) < 3.5(n_0 - 2)$ and $\Delta(H_0) \le n_0/2 + 1.5$. Since $t \ge 231$, Lemma 22 yields the result.

Case 4: $d = 10$. By (q1) and (q2), $e(H_0) \le 4(n_0 - 2) - 1$ and $\Delta(H_0) \le n_0/2 + 2$. If H_0 has a vertex v of degree at least $(n_0 + 3)/4$, then applying Lemma 22 to $H_0 - v$ we are done. Suppose that $\Delta(H_0) \le (n_0 + 2)/4$. Since $e(H_0) < 4(n_0 - 2)$, H_0 has three vertices, x_1, x_2 and x_3 of degree at most 7. Let $N_i = N_{H_0}(x_i)$, $i = 1, 2, 3$. Since $\Delta(H_0) \le (n_0 + 2)/4$, for $i \in \{1, 2, 3\}$, the number of pairs of vertices in $V(H_0) - N_i - \{x_1, x_2, x_3\}$ that have a common neighbor in N_i is at most $7\binom{(n_0+2)/4-1}{2}$. Since this is much less than $\binom{n_0-10}{2}$, we can choose for $i \in \{1, 2, 3\}$, a pair $\{y_i, z_i\} \subset V(H_0) - N_i - \{x_1, x_2, x_3\}$ so that all there chosen pairs are disjoint and y_i and z_i have no common neighbor in N_i. Contracting in G_0 for $i \in \{1, 2, 3\}$, x_i with y_i and z_i, we obtain a K^*_{3,n_0-9}-minor of G_0.

Case 5: $d = 11$. By (q1) and (q2), $e(H_0) < 4.5(n_0 - 2)$ and $\Delta(H_0) \le n_0/2 + 2.5$. If at most two vertices in H_0 have degree less than 8, then we are done by Lemma 21 for $k = 8$. Suppose now that H_0 has three vertices, x_1, x_2 and x_3 of degree at most 7. Let $N_i = N_{H_0}(x_i)$, $i = 1, 2, 3$.

If H_0 has two vertices, v_1 and v_2, of degree at least $3(n_0 - 15)/7$, then $H_0 - v_1 - v_2$ has less than $4.5n_0 - 10 - 6(n_0 - 15)/7 + 1 < 3.75(n_0 - 2) - 6$ edges and satisfies the conditions of Lemma 22 with $n_0 - 2$ in place of n_0. So, in this case by Lemma 22, $G_0 - v_1 - v_2$ has a $K^*_{3,t}$-minor. Thus, we may assume that for some $v \in V(H_0)$, $\Delta(H_0 - v) < 3(n_0 - 15)/7$. By Lemma 14(c), some pair $\{y_1, z_1\}$ of vertices in $V(H_0) - \{x_1, x_2, x_3, v\} - N_1$ has no common neighbor in N_1. Similarly, some pair $\{y_2, z_2\}$ of vertices in $V(H_0) - \{x_1, x_2, x_3, v, y_1, z_1\} - N_2$ has no common neighbor in N_2, and some pair $\{y_3, z_3\}$ of vertices in $V(H_0) - \{x_1, x_2, x_3, v, y_1, z_1, y_2, z_2\} - N_3$ has no common neighbor in N_3. Thus contracting in $G_0 - v$ vertices x_i, y_i and z_i for $i = 1, 2, 3$, we find a K^*_{3,n_0-10}-minor of G_0.

Case 6: $12 \le d \le 14$. By (q1) and (q2), $e(H_0) < 6(n_0 - 2)$ and $\Delta(H_0) \le n_0/2 + 4$. If at most two vertices in H_0 have degree less than 11, then we are done by Lemma 21 for $k = 11$. Suppose now that H_0 has three vertices, x_1, x_2 and x_3, of degree at most 10. Let $N_i = N_{\overline{G_0}}(x_i)$, $i = 1, 2, 3$.

By Lemma 14(a) and (b), if $m \ge 9k/5$, then $m - 1 > 9(k - 1)/5$ and

$$C(m, k, 3) \ge \left\lceil \frac{m}{k} C(m - 1, k - 1, 2) \right\rceil \ge \left\lceil \frac{9}{5} 6 \right\rceil = 11. \tag{9}$$

Thus, since $|N_1| \le 10$ and

$$|V(H_0) - \{x_1, x_2, x_3\} - N_1| \ge n_0 - 13 \ge \frac{9}{5}\left(\frac{n_0}{2} + 3\right) \ge \frac{9}{5}(\Delta(H_0) - 1),$$

the set $V(H_0) - \{x_1, x_2, x_3\} - N_1$ contains a triple $\{y_1, z_1, u_1\}$ that has no common neighbors of all three of these vertices in N_1. Similarly, the set $V(H_0) - \{x_1, x_2, x_3, y_1, z_1, u_1\} - N_2$ contains a triple $\{y_2, z_2, u_2\}$ that has no common neighbors in N_2 and the set $V(H_0) - \{x_1, x_2, x_3, y_1, z_1, u_1, y_2, z_2, u_2\} - N_3$ contains a triple $\{y_3, z_3, u_3\}$ that has no common neighbors in N_3. For this we need $n_0 - 19 \ge \frac{9}{5}(\frac{n_0}{2} + 3)$ which holds for $n_0 \ge 244$. Now contracting in G_0 the three quadruples $\{x_i, y_i, z_i, u_i\}$, we find a K^*_{3,n_0-12}-minor of G_0.

Case 7: $15 \le d \le 22$. By (q1) and (q2), $e(H_0) < 10(n_0 - 2)$ and $\Delta(H_0) \le n_0/2 + 8$. Since $e(H_0) < 10(n_0 - 2)$, H_0 has three vertices, x_1, x_2 and x_3 of degree at most 19. Let $N_i = N_{\overline{G_0}}(x_i)$, $i = 1, 2, 3$. By Lemma 14(a) and by (9), if $m \ge 9k/5$, then $m - 1 > 9(k - 1)/5$ and

$$C(m, k, 4) \ge \left\lceil \frac{m}{k} C(m - 1, k - 1, 3) \right\rceil \ge \left\lceil \frac{9}{5} 11 \right\rceil = 20. \tag{10}$$

Now we repeat the second part of the proof of Case 6 with quadruples in place of triples and 5-tuples in place of quadruples. We will find a K^*_{3,n_0-15}-minor of G_0 if $n_0 - (19 + 3 + 4 + 4) \ge \frac{9}{5}(\frac{n_0}{2} + 8)$ which holds for $n_0 \ge 444$. □

Lemma 24. *Let $t \ge 432$. Then $n_0 \ge 1.1t$.*

Proof. Suppose that $d := n_0 - t \le 0.1t$. If $d \le 22$ then we are done by the previous lemma. So suppose that $d \ge 23$. By (q1), H_0 has three vertices, v_1, v_2, v_3, with degree at most $d - 3$. So, the degrees of v_1, v_2, v_3 in G_0 are at least $t + 2$. We will construct disjoint connected dominating sets D_1, D_2, D_3 as follows.

STEP 0: For $j = 1, 2, 3$, let $D_{1,j} = \{v_j\}$ and $Q_{1,j} = V(G_0) - N_{G_0}(v_j) - v_j$.

STEP i, $i \geq 1$: Consecutively, for $j = 1, 2, 3$, if $Q_{i,j} = \emptyset$, then let $D_j := D_{i+1,j} := D_{i,j}$. When all D_j are defined, then stop. If $Q_{i,j} \neq \emptyset$, then let $F_{i,j} := Q_{i,j} \cup \bigcup_{\ell=1}^{j-1} D_{i+1,\ell} \cup \bigcup_{\ell=j}^{3} D_{i,\ell}$, then choose in $V(G_0) - F_{i,j}$ a vertex $v_{i,j}$ that has the most neighbors in $Q_{i,j}$ and let $D_{i+1,j} := D_{i,j} \cup \{v_{i,j}\}$ and $Q_{i+1,j} := Q_{i,j} - N_{G_0}(v_{i,j}) - v_{i,j}$.

By definition, for all i and j, $Q_{i,j}$ is exactly the set of vertices of G_0 not dominated by $D_{i,j}$. Since we always choose $v_{i,j} \notin Q_{i,j}$, $G_0[D_{i,j}]$ is a connected subgraph of G_0 for all i and j. Also, $D_{i,j}$ and $D_{i,j'}$ are disjoint if $j' \neq j$. We now show by induction on i that for each $j \in \{1, 2, 3\}$, if $Q_{i,j} \neq \emptyset$, then for each $x \in Q_{i,j}$,

$$\frac{|V(G_0) - F_{i,j}|}{|N_{G_0}(x) - F_{i,j}|} < \frac{5}{2}. \tag{11}$$

It will be easier to prove a slightly stronger inequality

$$\frac{n_0 - |N_{G_0}(x)|}{|N_{G_0}(x) - F_{i,j}|} < \frac{3}{2}. \tag{12}$$

By (p3), since $n_0 < 1.1t$, we have $n_0 - |N_{G_0}(x)| < 1.1t - t/2 - 2 = 0.6t - 2$. By the definition of $Q_{i,j}$, for each $x \in Q_{i,j}$, $N_{G_0}(x) - F_{i,j} = N_{G_0}(x) - (F_{i,j} - D_{i,j})$. Since

$$|F_{i,j} - D_{i,j}| \leq 2i + |Q_{i,j}|, \tag{13}$$

we will estimate $|Q_{i,j}|$.

By the choice of v_1, v_2, and v_3, $|Q_{1,j}| \leq d - 3$ and hence $|F_{1,j} - D_{1,j}| \leq d - 1 < 0.1t - 1$. So,

$$|N_{G_0}(x) - F_{1,j}| \geq 0.5t + 2 - (0.1t - 1) \geq 0.4t + 1,$$

and (since $n_0 - |N_{G_0}(x)| < 0.6t - 2$) (12) (and hence (11), as well) holds for $i = 1$. Observe that

$$\text{if (11) holds for a pair } (i, j), \text{ then } v_{i+1,j} \text{ has more than } 2|Q_{i,j}|/5 \text{ neighbors in } Q_{i,j}. \tag{14}$$

Thus if (11) holds for a pair (i, j) and $|Q_{i,j}| \geq 3$, then by (14),

$$|F_{i+1,j} - D_{i+1,j}| \leq |F_{i,j} - D_{i,j}| + 2 - (|Q_{i,j}| - |Q_{i+1,j}|) \leq |F_{i,j} - D_{i,j}| + 2 - \lceil 2|Q_{i,j}|/5 \rceil \leq |F_{i,j} - D_{i,j}|.$$

It follows that the inequality

$$|F_{i,j} - D_{i,j}| \leq 0.1t - 1 \tag{15}$$

holds if $|Q_{i-1,j}| \geq 3$ and $|F_{i-1,j} - D_{i-1,j}| \leq 0.1t - 1$. Let i_0 be the smallest $i \geq 2$ such that $|Q_{i-1,j}| \leq 2$. If $Q_{i_0,j} = \emptyset$, then (12) is proved for all i. Suppose that $|Q_{i_0,j}| \geq 1$. Then inequalities (15) and (12) hold for $i = i_0 - 1$. This implies that $|Q_{i_0,j}| = 1$ and that

$$|F_{i_0,j} - D_{i_0,j}| \leq |F_{i_0-1,j} - D_{i_0-1,j}| + 2 - (|Q_{i_0,j}| - |Q_{i_0-1,j}|) \leq (0.1t - 1) + 2 - 1 = 0.1t.$$

So, if $Q_{i_0,j} = \{x_j\}$, $|N(x_j) - F_{i_0,j}| > 0.4t$ and hence $Q_{i_0+1,j} = \emptyset$. Thus in all cases (12) holds.

By (11) and (14), if $Q_{i,j} \neq \emptyset$, then

$$|Q_{i+1,j}| < \frac{3}{5}|Q_{i,j}|. \tag{16}$$

Let $k = \lceil \log_{5/3}(d - 3) \rceil$. By (16) applied k times,

$$|Q_{k+1,j}| < |Q_{1,j}| \left(\frac{3}{5}\right)^k \leq (d - 3) \left(\frac{3}{5}\right)^{\log_{5/3}(d-3)} = 1.$$

Hence $Q_{k+1,j} = \emptyset$ for each $j \in \{1, 2, 3\}$ and so our algorithm constructs by the end of Step k disjoint connected dominating sets D_1, D_2, and D_3. In particular, this means that G_0 has a $K^*_{3,n_0-3(k+1)}$-minor.

It is left to show that $3(k + 1) \leq d$, i.e., that $3\lceil \log_{5/3}(d - 3) \rceil \leq d - 3$. Since $d - 3 \geq 20$, it is enough to show that for integer $x \geq 20$, $\log_{5/3} x \leq \lfloor x/3 \rfloor$, which is true. For example, $\log_{5/3} 20 < 5.87$. □

5. Graphs with a dense subgraph of moderate order

Lemma 25. *Let $2 \leq s \leq 3$, $t \geq 500$, and let G be a $2s$-connected graph that contains a vertex subset U with*

$$t + 19(s - 1) \ln t \leq |U| \leq 2t + 20(s - 1) \ln t$$

such that $\delta(G[U]) \geq 2t/5 + 36(s-1) \ln t$. Then G has a $K^*_{s,t}$-minor such that the pre-image of each vertex of the minor intersects U.

Proof. Let $u = |U|$. Perform the following procedure on $G[U]$. Let $i = 1$ and $G_1 = G[U]$. Step i: If every component of G_i has connectivity greater than $10(s-1)\ln t$ and the number of components in G_i is exactly i, then stop. Otherwise, choose a set S_i with $|S_i| = \lfloor 10(s-1)\ln t\rfloor$ so that $G_i - S_i$ has more than i components and let $G_{i+1} = G_i - S_i$.

Let G' be the resulting graph. Let $H_1, H_2, \ldots, H_\ell$ be the components of the graph G' and let $U_i = V(H_i)$ and $u_i = |U_i|$ for $i = 1, \ldots, \ell$. We may assume that $u_1 \geq \cdots \geq u_\ell$. First, we show that

$$\ell \leq 4. \tag{17}$$

Suppose that (17) does not hold. Consider G_4. By construction, G_4 has at least four components. Since $\delta(G_4) \geq \delta(G) - 30(s-1)\ln t \geq 0.4t + 6(s-1)\ln t$, each component of G_4 has more than $0.4t + 6(s-1)\ln t$ vertices. So, if G_4 has at least five components, then $|V(G_4)| > 5(0.4t + 6(s-1)\ln t) = 2t + 30(s-1)\ln t$, a contradiction. Moreover, each component of G_4 that is not $10(s-1)\ln t$-connected has more than $2\delta(G_4) - 10(s-1)\ln t \geq 0.8t + 2(s-1)\ln t$ vertices. So, if there is such a component, then $|V(G_4)| > 0.8t + 2(s-1)\ln t + 3(0.4t + 6(s-1)\ln t) = 2t + 20(s-1)\ln t$, a contradiction. This proves (17).

Case 1: $\ell = 1$. This means that $H_1 = G[U]$, $u_1 = u$, and the connectivity of H_1 is greater than $10(s-1)\ln t$. Let us check that H_1 satisfies the conditions of Lemma 13 with $k = \lceil 0.4t + 4(s-1)\ln t\rceil$ and $n = u$. Indeed, in this case $u \leq 5k$; hence $u/(u-k) \geq 5/4$, and so for $t \geq 500$,

$$2(s-1)\log_{u/(u-k)} u \leq 2(s-1)\log_{5/4} u < 9(s-1)\ln(u).$$

Hence by this lemma, U contains s disjoint subsets $A_1, \ldots, A_s$ such that, for every $i = 1, \ldots, s$, (i) $G[A_i]$ is connected, (ii) $|A_i| \leq 2\log_{u/(u-k)} u$, and (iii) A_i dominates $H_1 - \bigcup_{j=1}^{i-1} A_j$.

Contracting each of $A_1, \ldots, A_s$ into a vertex, we find a $K^*_{s,u-2s\log_{u/(u-k)} u}$-minor of G. We want to prove that $u - 2s\log_{u/(u-k)} u \geq t$, i.e. that for $0.4t \leq k < t$,

$$f(u,k) = u\ln\frac{u}{u-k} - 2s\ln u - t\ln\frac{u}{u-k} \geq 0. \tag{18}$$

For this we show first that $f'_u(u,k) \geq 0$ when $0.4t \leq k < t$ and $u \geq t$. Indeed,

$$f'_u(u,k) = \ln\frac{u}{u-k} + \frac{u}{u} - \frac{u}{u-k} - \frac{2s}{u} - t\left(\frac{1}{u} - \frac{1}{u-k}\right) = \ln\frac{u}{u-k} - \frac{k(u-t)}{u(u-k)} - \frac{2s}{u}.$$

Hence,

$$(uf'_u(u,k))'_u = \ln\frac{u}{u-k} - \frac{k}{u-k} - \frac{k(t-k)}{(u-k)^2}.$$

Since $\ln(1+x) < x$ for $x = \frac{k}{u-k}$, function $uf'(u)$ decreases for $u > t$. So, to check the inequality $f'_u(u,k) \geq 0$ for $u \in (t, 2t]$, it is enough to check it for $u = 2t$. For $u = 2t$,

$$(f'_u(u,k))'_k = \frac{1}{u-k} - \frac{u-t}{u}\frac{u}{(u-k)^2} = \frac{t-k}{(u-k)^2} > 0.$$

So,

$$f'_u(2t,k) \geq f'_u(2t, 0.4t) = \ln\frac{2t}{1.6t} - \frac{0.4t\cdot t}{1.6t\cdot 2t} - \frac{2s}{2t} = \ln\frac{5}{4} - \frac{1}{8} - \frac{s}{t}.$$

Since $\ln 1.25 > 0.22$, $s \leq 3$ and $t > 200$, $f'_u(2t,k) > 0$ and $f(u,k)$ grows with u on $(t, 2t]$. Let $u_0 = t + \lceil 2s\log_{3/2} 1.2t\rceil$. Since $s \leq 3$ and $t \geq 500$, $u_0 \leq 1.2t$. Hence $\frac{u_0}{u_0-k} \geq 3/2$ and

$$u_0 - 2s\log_{u_0/(u_0-k)} u_0 \geq t + 2s\log_{3/2} 1.2t - 2s\log_{3/2} u_0 \geq t.$$

Since $\log_{3/2} 1.2t \leq 2.6\ln t$ for $t \geq 500$, this proves Case 1 for $u \leq 2t$. If $u \in [2t, 2t + 20(s-1)\ln t]$, then it is enough to show that $f_1(u) = u - 2s\log_{5/4} u \geq t$. Since for $u \geq 2t \geq 1000$, $f'_1(u) = 1 - \frac{2s}{u\ln 5/4} > 1 - \frac{9s}{u} > 0$, we have for such u

$$f_1(u) \geq f_1(2t) > 2t - 9s\ln 2t \geq 2t - 27\ln 2t > t.$$

This finishes Case 1.

Case 2: $\ell = 2$. Then $\delta(G') \geq \delta(G[U]) - 10(s-1)\ln t \geq 0.4t + 26(s-1)\ln t$. For $j = 1, 2$, let $u_j = |V(H_j)|$. If $u_1 > t + 6(s-1)\ln t$, then we simply repeat the proof of Case 1 for H_1. The only difference would be the lower bound on $\delta(G')$, but the new bound is sufficient for the argument. Suppose that $u_2 \leq u_1 \leq t + 6(s-1)\ln t$. Since G is $2s$-connected, there are s pairwise disjoint paths $P_1, \ldots, P_s$ connecting $V(H_1)$ with $V(H_2)$. We may assume that for $i = 1, \ldots, s$ and $j = 1, 2$, $V(P_i) \cap V(H_j) = \{x_{i,j}\}$. Let $H'_j = H_j - \{x_{1,j}, \ldots, x_{s,j}\}$. Then for $j = 1, 2$, $\delta(H'_j) \geq \delta(G') - s > 0.4t + 25(s-1)\ln t$. So each of H'_j satisfies the conditions of Lemma 13 with $k = \lceil 0.4t + 3(s-1)\ln t\rceil$ and $n_j = u'_j = u_j - s$. Hence

$V(H_1') \cup V(H_2')$ contains disjoint subsets $A_{1,1}, A_{2,1}, \ldots, A_{s,2}$ such that for every $i = 1, \ldots, s$ and $j = 1, 2$, (i) $G[A_{i,j}]$ is connected, (ii) $|A_{i,j}| \le 2\log_{u_j/(u_j-k)} u_j$, and (iii) $A_{i,j}$ dominates $H_j' - \bigcup_{q=1}^{i-1} A_{q,j}$.

Since $u_j \le 2.5k$, by (ii),

$$\sum_{j=1}^{2}\sum_{i=1}^{s} |A_{i,j}| \le 4s\log_{5/3} t + 6(s-1)\ln t < 4s\cdot 2\ln t + 6(s-1)\ln t < 8.5s\ln t. \tag{19}$$

For $j = 1, 2$, choose in $V(H_j') - \bigcup_{i=1}^{s} A_{i,j}$ vertices $y_{1,j}, \ldots, y_{s,j}$ so that $x_{i,j}y_{i,j} \in E(G')$ for $i = 1, \ldots, s$. We can do this because each $x_{i,j}$ has at least $0.4t + 26(s-1)\ln t$ neighbors in H_j. For $i = 1, \ldots, s$, let $B_i = A_{i,1} \cup A_{i,2} \cup V(P_i) \cup \{y_{i,1}, y_{i,2}\}$. By the dominating properties of $A_{i,j}$, $y_{i,j}$ has a neighbor in $A_{i,j}$. Hence each of $B_1, \ldots, B_s$ induces a connected subgraph in G and dominates the set $X = V(H_1') \cup V(H_2') - \bigcup_{j=1}^{2}\bigcup_{i=1}^{s}(A_{i,j} \cup \{y_{i,j}\})$. Under the assumptions of the case, by (19),

$$|X| \ge |U| - |S_1| - 2s - \sum_{j=1}^{2}\sum_{i=1}^{s}|A_{i,j}| - 2s \ge |U| - 10(s-1)\ln t - 8.5s\ln t - 4s \ge |U| - 18s\ln t.$$

So, if $|U| \ge t + 18s\ln t$, then the case is proved. Suppose $|U| < t + 18s\ln t$. Then $u_1 < |U| - \delta(G') \le t + 18s\ln t - (2t/5 + 36(s-1)\ln t) \le 0.6t$ and $u_2 \le u_1$. So, repeating the above argument, instead of (19), we get

$$\sum_{j=1}^{2}\sum_{i=1}^{s}|A_{i,j}| \le 4s\log_3 0.6t < 4s(\ln t + \ln 0.6) < 4s\ln t - 2s. \tag{20}$$

Hence

$$|X| \ge |U| - 10(s-1)\ln t - (4s\ln t - 2s) - 4s \ge |U| - 18(s-1)\ln t - 2s \ge t.$$

Case 3: $\ell = 3$. Since $\delta(G') \ge 0.4t + 16(s-1)\ln t$, $u_1 \ge u_2 \ge u_3 \ge 1 + 0.4t + 16(s-1)\ln t$. For $j = 1, 2, 3$, choose $F_j \subset U_j$ with $|F_1| = 2s$ and $|F_2| = |F_3| = s$. Since G is $2s$-connected, G contains $2s$ vertex-disjoint paths $P_1, \ldots, P_{2s}$ from F_1 to $F_2 \cup F_3$. By Lemma 11, $P_1 \cup \ldots \cup P_{2s}$ contains s vertex-disjoint (U_1, U_2, U_3)-connecting pairs of paths $(Q_{i,1}, Q_{i,2})$, $i = 1, \ldots, s$. Let $Q = \bigcup_{i=1}^{s}\bigcup_{j=1}^{2} V(Q_{i,j})$. For $j = 1, 2, 3$, let $H_j' = H_j - Q$ and $u_j' = |V(H_j')|$. Then for $j = 1, 2, 3$, $\delta(H_j') \ge \delta(G') - 2s > 0.4t + 15(s-1)\ln t$ and hence $u_j' > 1 + 0.4t + 15(s-1)\ln t$.

As in Case 2, $u_1 \le t + 6s\ln t$. Moreover, $u_2 \le (|U| - 20(s-1)\ln t - u_3)/2 \le 0.8t$ and $u_3 \le (|U| - 20(s-1)\ln t)/3 \le 2t/3$. So, each of H_j' $(j = 1, 2, 3)$ satisfies the conditions of Lemma 13 with $k_1 = \lceil 0.4t + 2.5s\ln t\rceil$, $u_1'/(u_1' - k_1) \ge 5/3$, $k_2 = k_3 = \lceil 0.4t\rceil$, $u_2'/(u_2' - k_2) \ge 2$ and $u_3'/(u_3' - k_3) \ge 5/2$. Hence $V(H_1') \cup V(H_2') \cup V(H_3')$ contains disjoint subsets $A_{1,1}, A_{2,1}, \ldots, A_{s,3}$ such that, for every $i = 1, \ldots, s$ and $j = 1, 2, 3$, (i) $G[A_{i,j}]$ is connected, (ii) $|A_{i,j}| \le 2\log_{u_j'/(u_j'-k_j)} u_j'$, and (iii) $A_{i,j}$ dominates $V(H_j') - \bigcup_{q=1}^{i-1} A_{q,j}$.

By construction, $\sum_{j=1}^{3}\sum_{i=1}^{s}|A_{i,j}| < 2s(\log_{5/3} u_1 + \log_2 u_2 + \log_{5/2} u_3)$. Since $u_1 \le 1.2t$ (cf. Case 1) and $u_3 \le u_2 < t$,

$$\log_{5/3} u_1' + \log_2 u_2' + \log_{5/2} u_3' \le \ln t\left(\frac{1 + \ln 1.2/\ln t}{\ln 5/3} + \frac{1}{\ln 2} + \frac{1}{\ln 5/2}\right),$$

so for $t \ge 500$ we have

$$\sum_{j=1}^{3}\sum_{i=1}^{s}|A_{i,j}| < 2s\cdot 4.6\ln t = 9.2s\ln t. \tag{21}$$

By the definition of (U_1, U_2, U_3)-connecting pairs, for $j = 1, 2, 3$ and $i = 1, \ldots, s$, $(Q_{i,1} \cup Q_{i,2}) \cap U_j \ne \emptyset$. Let $x_{i,j} \in (Q_{i,1} \cup Q_{i,2}) \cap U_j$. Since each $x_{i,j}$ has at least $0.4t + 16(s-1)\ln t$ neighbors in H_j, for $j = 1, 2, 3$ we can choose in $V(H_j') - \bigcup_{i=1}^{s} A_{i,j}$ vertices $y_{1,j}, \ldots, y_{s,j}$ so that $x_{i,j}y_{i,j} \in E(G')$ for $i = 1, \ldots, s$. For $i = 1, \ldots, s$, let $B_i = V(Q_{i,1} \cup Q_{i,2}) \cup \bigcup_{j=1}^{3}(A_{i,1} \cup \{y_{i,j}\})$.

By the dominating properties of $A_{i,j}$ and the choice of $y_{i,j}$, each of $B_1, \ldots, B_s$ induces a connected subgraph in G and dominates the set

$$X = V(H_1') \cup V(H_2') \cup V(H_3') - \bigcup_{j=1}^{3}\bigcup_{i=1}^{s}(A_{i,j} \cup \{y_{i,j}\}).$$

Furthermore, by (21),

$$\begin{aligned}|X| &\ge 3(1 + 0.4t + 15(s-1)\ln t) - |Q \cap (U_1 \cup U_2 \cup U_3)| - \sum_{j=1}^{3}\sum_{i=1}^{s}|A_{i,j} \cup \{y_{i,j}\}|\\ &\ge 3 + 1.2t + 45(s-1)\ln t - 12 - 9.2s\ln t - 3s = 1.2t + 35.8(s-1)\ln t - 9 - 3s > t.\end{aligned}$$

Case 4: $\ell = 4$. Since $\delta(G') \geq 0.4t + 6(s-1)\ln t$, $u_1 \geq \cdots \geq u_4 \geq 1 + 0.4t + 6(s-1)\ln t$. Hence $u_2 \leq (|U| - 30(s-1)\ln t - u_3 - u_4)/2 < 0.6t$ and $u_4 \leq u_3 \leq u_2$. We can now repeat the proof for Case 3 with H_2, H_3 and H_4 in place of H_1, H_2 and H_3 and with $k_2 = k_3 = k_4 = 0.4t$. The disadvantage is a slightly smaller minimum degree, but the advantage is that $\frac{u_j}{u_j - 0.4t} \geq \frac{0.6t}{0.6t - 0.4t} = 3$, and so instead of (21) we will have

$$\sum_{j=2}^{4}\sum_{i=1}^{s}|A_{i,j}| < 2s \cdot 3\ln u_j' < 6s\ln t.$$

This finishes the proof of the lemma. □

6. The final argument

We are now ready to prove Theorem 4. Recall that G_0 is our smallest counter-example to the theorem.

Case 1: G_0 is 6-connected.

Case 1.1: G_0 has a vertex v with $t + 19\ln t \leq d(v) \leq 2t + 20\ln t$. Then $G_0 - v$ is 5-connected and by (p2), $\delta(G_0[N(v)]) > t/2$. Since $t \geq 3000$, $2t/5 + 36\ln t \leq t/2$, so $G_0 - v$ with $U = N(v)$ satisfies the conditions of Lemma 25 for $s = 2$. Hence by this lemma, $G_0 - v$ has a $K^*_{2,t}$-minor such that the pre-image of each vertex of the minor intersects $N(v)$. Adding v, we will get a $K^*_{3,t}$-minor.

Case 1.2: G_0 has no vertices v with $t + 19\ln t \leq d(v) \leq 2t + 20\ln t$. Let $V_{\rm sm}$ be the set of vertices of degree less than $t + 19\ln t$. We first show that

$$|V_{\rm sm}| \leq t + 38\ln t. \tag{22}$$

Assume that $|V_{\rm sm}| = \ell > t + 38\ln t$. Order $v_1, \ldots, v_\ell$, the vertices in $V_{\rm sm}$, so that for all $1 \leq i < j \leq \ell$,

$$\left|\bigcup_{q=1}^{i} N[v_q]\right| \geq \left|\bigcup_{q=1}^{i-1} N[v_q]\right| \cup N[v_j]. \tag{23}$$

In other words, having already defined $v_1, \ldots, v_{i-1}$, we choose as v_i a vertex v with maximum $|N[v] - \bigcup_{q=1}^{i-1} N[v_q]|$. If (22) does not hold, then $|\bigcup_{q=1}^{\ell} N[v_q]| > t + 38\ln t$. Let i_0 be the largest i such that $|\bigcup_{q=1}^{i} N[v_q]| \leq t + 38\ln t$. Let us check that

$$\left|\bigcup_{q=1}^{i_0+1} N[v_q]\right| \leq 2t + 20\ln t. \tag{24}$$

By the definition of $V_{\rm sm}$, $i_0 \geq 1$, and if $i_0 = 1$, then (24) holds. So, let $i_0 \geq 2$. If (24) does not hold, then by the definition of i_0, $|N[v_{i_0+1}] - \bigcup_{q=1}^{i_0} N[v_q]| > t - 18\ln t$. But then by the ordering of $v_1, \ldots, v_{i_1}$, $|N[v_{i_0}] - \bigcup_{q=1}^{i_0-1} N[v_q]| > t - 18\ln t$ and $|N[v_{i_0-1}] - \bigcup_{q=1}^{i_0-2} N[v_q]| > t - 18\ln t$, so $|\bigcup_{q=1}^{i_0} N[v_q]| > 2t - 36\ln t$. But for $t \geq 6300$, $2t - 36\ln t > t + 38\ln t$, a contradiction to the definition of i_0. Thus (24) holds. Then G_0 and $U = \bigcup_{q=1}^{i_0+1} N[v_q]$ satisfy the conditions of Lemma 25 for $s = 3$, since $t \geq 6300$. This proves (22).

Since every vertex not in $V_{\rm sm}$ has degree at least $2t + 20\ln t$, by (p1),

$$\frac{t|V_{\rm sm}|}{2} + 2t(n_0 - |V_{\rm sm}|) \leq \sum_{v \in V(G_0)} d_{G_0}(v) \leq (t+3)(n_0 - 2) + 4 < (t+3)n_0.$$

It follows that $n_0 - \frac{3n_0}{t} < \frac{3|V_{\rm sm}|}{2}$ and hence by (22),

$$n_0 < \frac{3t|V_{\rm sm}|}{2(t-3)} \leq 2t\frac{3(t + 38\ln t)}{4(t-3)} < 2t.$$

Thus if $n_0 \geq t + 38\ln t$, then we apply Lemma 25 for $s = 3$ to G_0 and $U = V(G_0)$. If $n_0 < t + 38\ln t$, then, since $t \geq 6300$, $n_0 < t + 0.1t$, and by Lemma 24, the theorem holds for G_0.

Case 2: G_0 is not 6-connected. Let X be a separating set in G_0 with $|X| \leq 5$. Let V_1 and V_2 be vertex sets of some two connected components of $G_0 - X$. By definition, each of V_1 and V_2 is 5-separable, and hence 9-separable. For $j = 1, 2$, let W_j be an inclusion-minimal 9-separable subset of V_j, let $S_j = N(W_j) - W_j$, and let $G_j = G_0[W_j \cup S_j]$. By (p2), for $j = 1, 2$, $\delta(G_j) > t/2$ and by Lemma 10 for $k = 9$, graph G_j is 6-connected.

Case 2.1: $|V(G_1)| \geq t + 38\ln t$. If $|V(G_1)| \leq 2t$, then G_1 with $U = V(G_1)$ satisfies the conditions of Lemma 25 for $s = 3$. So, we may assume that

$$|V(G_1)| > 2t. \tag{25}$$

If W_1 contains a vertex v with $t + 19 \ln t \le d(v) \le 2t + 40 \ln t$, then we simply repeat the argument of Cases 1.1 and 1.2 with G_1 in place of G_0. If not, we let V_{sm} be the set of vertices in W_1 of degree less than $t+19 \ln t$. Note that we do not include vertices of S_1 into V_{sm}. Repeating the proof of (22) word by word, we get that it holds for our new definition of V_{sm}. By the minimality of G_0, $e(G_1) \le \frac{t+3}{2}(|W_1| + |S_1| - 2) + 1$. So, since every vertex in $W_1 - V_{sm}$ has degree at least $2t + 40 \ln t$,

$$\frac{(t+3)(|V_{sm}| + |S_1|)}{2} + 2t(|W_1 - V_{sm}|) \le \sum_{w \in S_1 \cup W_1} d_{G_1}(w) < (t+3)(|W_1| + |S_1|).$$

Since $|S_1| \le 9$, this and (22) yield for $t > 2000$

$$|W_1| \le \frac{3t(|V_{sm}| + 3 + 9/t)}{2(t-3)} < \frac{3t(t + 38 \ln t + 3 + 9/t)}{2(t-3)} < \frac{3t(1.2t - 5)}{2(t-3)} < 1.8t < 2t - 9.$$

Hence $|W_1| + |S_1| < 2t$, a contradiction to (25). This proves Case 2.1.

Case 2.2: For $j = 1, 2$, $|V(G_j)| \le t + 38 \ln t$. For each $w \in W_j$, $d(w) \le |W_j| + |S_j| - 1$. On the other hand, by the minimality of G_0, $e(G_0) - e(G_0 - W_j) \ge (t+3)|W_j|/2$. Thus, since $|S_j| \le 9$,

$$(t+3)|W_j| \le \sum_{w \in W_j} d(w) + |S_j||W_j| \le |W_j|(|W_j| + 2|S_j| - 1) \le |W_j|(|W_j| + 17),$$

and hence $|W_j| \ge t - 14$. Thus, if at most two vertices of G_j have degree greater than $t - 12$, then

$$\begin{aligned}(t+3)|W_j| &\le \sum_{w \in W_j \cup S_j} d_{G_j}(w) \le 2(|W_j| + |S_j| - 1) + (t-12)(|W_j| + |S_j| - 2)\\ &\le (t-10)|W_j| + 16 + 7(t-12).\end{aligned}$$

It follows that $13|W_j| \le 7(t-12) + 16$, a contradiction to $|W_j| \ge t - 14$. So, G_j contains some three vertices $v_{1,j}$, $v_{2,j}$ and $v_{3,j}$ of degree at least $t - 11$ in G_j.

By (p4), there are three vertex-disjoint S_1, S_2-paths P_1, P_2, and P_3. We may assume that for $i = 1, 2, 3$ and $j = 1, 2$, the only common vertex of P_i and S_j is $p_{i,j}$. We also may assume that if $p_{i,j} \in \{v_{1,j}, v_{2,j}, v_{3,j}\}$, then $p_{i,j} = v_{i,j}$. Let $F_j = \{v_{1,j}, v_{2,j}, v_{3,j}, p_{1,j}, p_{2,j}, p_{3,j}\}$ for $j = 1, 2$. If $p_{i,j} \ne v_{i,j}$ and $p_{i,j}v_{i,j} \notin E(G_j)$, then $p_{i,j}$ and $v_{i,j}$ have at least

$$d_{G_j}(p_{i,j}) + d_{G_j}(v_{i,j}) - |V(G_j)| \ge \frac{t+3}{2} + (t-11) - (t + 38 \ln t) > 10$$

common neighbors. Thus, we can choose distinct vertices $q_{1,j}, q_{2,j}, q_{3,j} \in V(G_j) - F_j$ so that $q_{i,j}$ is a common neighbor of $p_{i,j}$ and $v_{i,j}$ if $p_{i,j} \ne v_{i,j}$ and $p_{i,j}v_{i,j} \notin E(G_j)$. For $j = 1, 2$, let $F'_j = F_j \cup \{q_{1,j}, q_{2,j}, q_{3,j}\}$ and let M_j be the set of common neighbors of $v_{1,j}$, $v_{2,j}$ and $v_{3,j}$ in $V(G_j) - F'_j$. By definition, for $t \ge 6000$,

$$|M_j| \ge \sum_{i=1}^{3} d_{G_1}(v_{i,j}) - 2|V(G_j)| - |F'_j| \ge 3(t-11) - 2(t + 38 \ln t) - 9 = t - 76 \ln t - 42 > 7t/8.$$

For $i = 1, 2, 3$, let $B_i = V(P_i) \cup \{v_{i,1}, v_{i,2}, q_{i,1}, q_{i,2}\}$. Then $G_0[B_i]$ is connected and contracting each B_i into a vertex, we get a $K_{3,7t/4}$-minor of G_0, where the pre-images of the remaining vertices are the vertices in $M_1 \cup M_2$. Since $K_{3,t+2}$ has a $K^*_{3,t}$-minor, this finishes the proof of the theorem. $\square$

7. Case $n_0 = t + 5$

In this section we deliver the postponed proof of Lemma 19 that $n_0 \ne t + 5$. We assume that $n_0 = t + 5$ and will eventually get a contradiction. As was noted, in this case it is easier to consider the complement, H_0, of our counter-example G_0 than G_0 itself. By (q1),

$$e(H_0) < 1.5n_0 - 3. \tag{26}$$

Lemma 26. *H_0 is connected.*

Proof. Suppose first that each component of H_0 has at least three vertices. Let x be a vertex of degree at most 2 in H_0. Contracting in G_0 the neighbors of x in H_0 with vertices in other components of H_0, we find a $K_{3,n-5}$-minor. Similarly, if H_0 has a K_2-component C with $V(C) = \{y_1, y_2\}$, then it has another vertex x of degree at most 2 in $H_0 - C$. If $N_{H_0}(x) \subseteq \{z_1, z_2\}$, then we contract in G_0 the edges y_1z_1 and y_2z_2. Suppose finally that H_0 has an isolated vertex x. Let $H_1 = H_0 - x$ and $n_1 = |V(H_1)| = n_0 - 1$. Since $e(H_1) = e(H_0) < 1.5(n_1 - 1)$, by Lemma 15, H_1 contains two disjoint pairs (y_1, y_2) and (z_1, z_2) of vertices at distance at least 3. Thus, contracting in G_0 edges y_1y_2 and z_1z_2, we get a graph containing $K^*_{3,t}$. $\square$

Lemma 27. *If a vertex of H_0 is adjacent to two degree-1 vertices, then there are no other degree-1 vertices in H_0.*

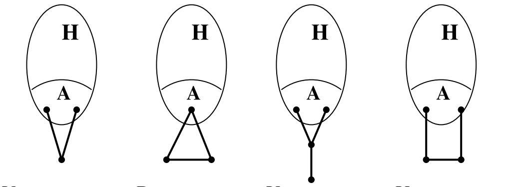

Fig. 3. Some kinds of tiny components.

Proof. Suppose that a vertex $x \in V(H_0)$ is adjacent to degree-1 vertices v_1 and v_2, and that there is another degree-1 vertex v_3 adjacent to a vertex y (possibly, $y = x$). Then v_1, v_2 and v_3 are isolated in $H_0 - x - y$, i.e. G_0 contains $K^*_{3,t}$. $\square$

For $A \subset V(H_0)$, a component C of $H_0 - A$ is *tiny* if $|V(C)| \le 2$ and $e(C, A) \le 2$.

We need a couple of statements on tiny components. For this, let us first give names to some of these components. We will say that a tiny component C of $H_0 - A$ is:

(c1) a V-component if $|V(C)| = 1$ and $e(C, A) = 2$,
(c2) a D-component if $|V(C)| = 2$ and both vertices in C are adjacent to the same vertex in A,
(c3) a Y-component if $|V(C)| = 2$ and two vertices in A are adjacent to the same vertex in C,
(c4) a U-component if $|V(C)| = 2$ and the vertices in C are adjacent to distinct vertices in A.

(See Fig. 3.)

Lemma 28. *For every $x \in V(H_0)$, the number of tiny components in $H_0 - x$ is at most 2.*

Proof. Suppose that $H_0 - x$ has tiny components C_1, C_2, and C_3. If $|V(C_1)| = |V(C_2)| = |V(C_3)| = 1$, then we have $K^*_{3,t+1}$ in $G_0 - x$, so suppose $V(C_1) = \{v_1, v_2\}$. If $|V(C_2)| = |V(C_3)| = 1$, then we have $K^*_{3,t}$ in $G_0 - x - v_1$, so suppose $V(C_2) = \{w_1, w_2\}$. In this case the graph G'_0 obtained from $G_0 - x$ by contracting edge v_1w_1 has three all-adjacent vertices: v_2, w_2, and $v_1 * w_1$. $\square$

Lemma 29. *For every $x_1, x_2 \in V(H_0)$, the number of tiny components in $H_0 - x_1 - x_2$ that are not Y-components is at most 2.*

Proof. Suppose that $H_0 - x_1 - x_2$ has tiny components C_1, C_2, and C_3 that are not Y-components. For convenience, suppose that $V(C_i) = \{v_{i,1}, \dots, v_{i,|V(C_i)|}\}$ for $i = 1, 2, 3$. If all of them are singletons, then G_0 contains $K^*_{3,t}$. So, we may assume that $|V(C_1)| = 2$.

Case 1: Vertex x_i has no neighbors in C_1 for some $i \in \{1, 2\}$. If, say, C_3 is a singleton, then contracting in $G_0 - x_{3-i}$ the edge $v_{1,1}x_i$ we get a $(3 + t)$-vertex graph with three all-adjacent vertices, namely $v_{1,2}$, $v_{1,1} * x_i$, and the vertex in C_3. So we may assume that $|V(C_2)| = |V(C_3)| = 2$. If for some $\ell \in \{2, 3\}$ and $j \in \{1, 2\}$, $v_{\ell,j}x_i \notin E(H_0)$, then contracting in $G_0 - x_{3-i}$ the edge $v_{1,1}v_{\ell,3-j}$ we get a $(3 + t)$-vertex graph with three all-adjacent vertices: $v_{1,2}$, $v_{\ell,j}$, and $v_{1,1} * v_{\ell,3-j}$. Thus both C_2 and C_3 are D-components in $H_0 - x_i$. Switching the roles of x_i and x_{3-i}, we again get the same case and repeating the proof get a contradiction.

Case 2: Both x_1 and x_2 have neighbors in C_1. In other words, C_1 is a U-component (recall that we forbid Y-components). We may assume that $x_1v_{1,1}, x_2v_{1,2} \in E(H_0)$.

Case 2.1: Some vertex u is at distance at least 3 from some x_i in H_0. If C_2 and C_3 are singletons, then contracting in $G_0 - x_{3-i}$ the edge x_iu we get a $(3 + t)$-vertex graph with three all-adjacent vertices: $v_{2,1}$, $v_{3,1}$, and $x_i * u$. So, we may assume that $|V(C_2)| = 2$. Then contracting in G_0 the edges x_iu and $v_{1,3-i}v_{2,3-i}$ we get a $(3 + t)$-vertex graph with three all-adjacent vertices: $v_{1,i}$, $v_{2,i}$, and $x_i * u$.

Case 2.2: Each vertex in H_0 is at distance at most 2 from x_1 and from x_2. Let $N_{i,j}$ denote the set of vertices in H_0 that are at distance i from x_1 and at distance j from x_2. By definition, $v_{1,1} \in N_{1,2}$ and $v_{1,2} \in N_{2,1}$. We observe some properties of vertices in $N_{i,j}$.

$$\text{Each } u \in N_{2,2} \text{ of degree at most 2 has a neighbor in } N_{1,1}. \tag{27}$$

Indeed, suppose that $u \in N_{2,2}$ has at most two neighbors, say w_1 and w_2, and that $w_1, w_2 \notin N_{1,1}$. Since Case 2.1 does not hold, we may assume that $w_1 \in N_{1,2}$ and $w_2 \in N_{2,1}$. Then contracting in G_0 edges $w_1v_{1,2}$ and $w_2v_{1,1}$ we get a $(3 + t)$-vertex graph with three all-adjacent vertices: u, $w_1 * v_{1,2}$, and $w_2 * v_{1,1}$.

$$\text{No two vertices } u_1, u_2 \in N_{2,2} \text{ of degree 1 in } H_0 \text{ have a common neighbor.} \tag{28}$$

Indeed, assume that w is the only neighbor of $u_1, u_2 \in N_{2,2}$. Then contracting in G_0 the vertices w, $v_{1,1}$, and $v_{1,2}$ into the new vertex z we get a $(3 + t)$-vertex graph with three all-adjacent vertices: z, u_1, and u_2.

$$\text{Neither of } x_1 \text{ and } x_2 \text{ has a neighbor of degree 1.} \tag{29}$$

Indeed, suppose that x_1 has a neighbor w of degree 1. If C_2 and C_3 are singletons, then $G_0-x_1-x_2$ has all-adjacent vertices w, $v_{2,1}$, and $v_{3,1}$. So, we may assume that $|V(C_2)| = 2$. Then contracting in $G_0 - x_1$ edge $v_{1,2}v_{2,2}$ we get a $(3+t)$-vertex graph with three all-adjacent vertices: $v_{1,1}$, $v_{2,1}$, and w.

Now we use discharging to find a contradiction. At the beginning, each edge has charge 1 and so the total charge is $e(H_0)$. The edges give their charges to vertices according to the following rules.

(R1) If both ends of an edge e are in $N_{2,2}$ or both are in $N_{1,1} \cup N_{1,2} \cup N_{2,1}$, then e gives $1/2$ to either of its ends.
(R2) If exactly one of the ends of e is in $\{x_1, x_2\}$, then e gives 1 to the other end.
(R3) If $e = xy$, $x \in N_{2,2}$, and $y \in N_{1,2} \cup N_{2,1}$, then e gives $1/2$ to either of its ends.
(R4) If $e = xy$, $x \in N_{2,2}$, and $y \in N_{1,1}$, then e gives 1 to x. Moreover, if $d_{H_0}(x) = 1$, then y forwards 0.5 from its charge of 2 received from the edges x_1y and x_2y by Rule (R2) to x.

We claim that the resulting charge of each vertex apart from x_1 and x_2 is at least $3/2$, so the total charge is at least $3(n-2)/2$, a contradiction to (26). To prove the claim, consider all possible cases. If $w \in N_{2,2}$ has degree at least 3, then by (R1), (R3), and (R4), it receives at least $1/2$ from each incident edge. If $w \in N_{2,2}$ has degree exactly 2, then by (27) and (R4), at least one of the incident with w edges gives 1 to w, so w gets at least $3/2$ in total. If $w \in N_{2,2}$ has degree 1, then by (R4), w gets 1 from the incident edge and $1/2$ from the neighbor. If $w \in N_{1,1}$, then it gets 2 from the edges x_1w and x_2w by Rule (R2), and by (28) and Rule (R4), gives $1/2$ to the at most one neighbor of degree 1 in $N_{2,2}$.

Suppose that $w \in N_{1,2} \cup N_{2,1}$. Then w gets 1 from the edge connecting w with $\{x_1, x_2\}$. Moreover, by (29), w has another incident edge which gives $1/2$ to w either by (R1) or by (R3). This proves the claim and thus the lemma. □

Lemma 30. *If $n_0 \geq 200$, then H_0 has no dominating set with at most $\sqrt{n_0/2} - 2$ vertices.*

Proof. Suppose that H_0 has a dominating set S with $|S| = s \leq \sqrt{n_0/2} - 2$. Let $S' = V(H_0) - S$ and $H'_0 = H_0[S']$. Let m be the number of tree components in H'_0. Since S dominates S', it intersects at least $|S'|$ edges. So, $e(H'_0) < 3n_0/2 - 3 - |S'| < 0.5n_0 - 3 + s$. It follows that $m \geq 3 + 0.5n_0 - 2s$. Let c_i denote the number of tree components of H'_0 with i vertices and $c_{i,j}$ denote the number of tree components of H'_0 with i vertices that are connected with S by exactly j edges.

Claim 1: $\sum_{i=1}^{n_0}((s-1)c_{i,1} + c_{i,2}) \leq 2\binom{s}{2}$.

Proof. Since S is dominating, $c_{i,1} + c_{i,2} = 0$ for every $i \geq 3$ and so $\sum_{i=1}^{n_0}((s-1)c_{i,1} + c_{i,2})$ counts only tiny components of $H_0 - S$. For the same reason, $H_0 - S$ has no Y-components. By Lemma 29, the sum over all pairs $\{x_1, x_2\} \subseteq S$ of the number of tiny components of $H_0 - \{x_1, x_2\}$ is at most $2\binom{s}{2}$. Furthermore, each component of $H_0 - S$ that has only one neighbor in S is counted $(s-1)$ times in this sum. This proves the claim.

The number of edges in all components of H'_0 is at least $|V(H'_0)| - m = n_0 - s - m$. Thus by Claim 1, the total number of edges in H_0 is at least

$$e(H'_0) + \sum_{i=1}^{n}\sum_{j=1}^{n^2} jc_{i,j} \geq (n_0 - s - m) + 3m - \sum_{i=1}^{n_0}(2c_{i,1} + c_{i,2}) \geq n_0 - s + 2m - 2\binom{s}{2}.$$

Since $m \geq 3 + 0.5n_0 - s$ and $s \leq \sqrt{n_0/2} - 2$, for $n_0 \geq 200$ this is at least

$$2n_0 + 6 - 4s - s^2 \geq 2n_0 + 10 - (s+2)^2 \geq 2n_0 + 10 - \frac{n_0}{2} > \frac{3n_0}{2},$$

a contradiction. □

Lemma 31. *Each 2-vertex in H_0 has a neighbor of degree greater than $\sqrt{n_0/2} - 4$.*

Proof. Suppose that neighbors of $v \in V(H_0)$ are x and y of degree at most $\sqrt{n_0/2} - 4$. By Lemma 30, each of the sets $N(x) - v + y$ and $N(y) - v + x$ does not dominate at least three vertices. So, we can choose distinct vertices x' not dominated by $N(x) - v + y$ and y' not dominated by $N(y) - v + x$. By definition, $d_{H_0}(x, x') \geq 3$ and $d_{H_0}(y, y') \geq 3$. Contracting the edges xx' and yy' in G_0 we get a $(3+t)$-vertex graph with all-adjacent vertices v, $x * x'$, and $y * y'$. □

A 2-vertex in H_0 is *weak* if at least one of its neighbors has degree at most 5.

Lemma 32. *If a vertex $v \in V(H_0)$ has at least five neighbors that are either* 1 *-vertices or weak* 2*-vertices, then it has at least five neighbors of degree at least* 3.

Proof. Suppose that $v \in V(H_0)$ is adjacent to ℓ 1-vertices $u_1, \ldots, u_\ell$, to s weak 2-vertices $z_1, \ldots, z_s$, and to k vertices $x_1, \ldots, x_k$ of degree at least 3, where $\ell + s \geq 5$ and $k \leq 4$. By Lemma 27, $\ell \leq 2$. In particular, $s \geq 3$. Vertices z_i and z_j form a *weak pair* if they are adjacent to each other.

Claim 1: There are no weak pairs.

Proof. Suppose (z_1, z_2) is a weak pair. Let y be the neighbor of z_3 other than v. We delete v and contract in $G_0 - v$ the edge z_1y. Now the vertices z_2, z_3, and $z_1 * y$ are all-adjacent ones in the graph obtained.

For $1 \le i \le s$, let the neighbor of z_i that is not v be y_i. By Claim 1, no y_j coincides with any z_i. Some y_j can coincide with some other $y_{j'}$ and with some x_i. Let $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_s\}$.

Claim 2: If $y_j \in Y - X$, then it has a neighbor in X.

Proof. Suppose that $y_{j'} \in Y - X$ and has no neighbor in X. Contract in G_0 edge $vy_{j'}$. In the resulting graph G_0', the neighbors of the vertex $v * y_{j'}$ are only some z_j, and these z_j have no other non-neighbors in G_0'. If there are at least three such z_j, then $G_0' - v * y_{j'}$ has at least three all-adjacent vertices. If there are exactly two of them, then contracting in G_0' vertex $v * y_{j'}$ with any vertex distinct from these two, we again get three all-adjacent vertices. Suppose that only $z_{j'}$ is a non-neighbor of $v * y_{j'}$ in G_0'. Then, since $s \ge 3$, there is some other $y_{j''}$. Contracting in G_0' vertex $v * y_{j'}$ with $y_{j''}$, we again get three all-adjacent vertices.

Claim 3: Every vertex in H_0 is at distance at most 2 from v.

Proof. Suppose that $d_{H_0}(w, v) \ge 3$. Contract the edge vw in G_0. If $\ell \ge 1$ or $y_j = y_{j'}$ for distinct j and j', then delete y_j and get a $K^*_{3,t}$. Suppose now that $\ell = 0$ and all y_j are distinct. Then $s \ge 5$. Since $z_1, \dots, z_s$ are weak, each of y_j has at most four neighbors in $H_0 - z_j$. Moreover, if $y_j \in X$, then $y_j v \in E(H_0)$ and if $y_j \notin X$, then y_j is adjacent to a vertex in X. So, some y_j has at most three neighbors in Y, and we may assume that $y_1y_2 \notin E(H_0)$. In this case, contract in G_0 edges vw and y_1y_2, and vertices $v * w$, z_1 and z_2 become all-adjacent ones in the graph obtained.

Claims 1, 2 and 3 together imply that the set $X + v$ of size at most 5 is dominating in H_0, a contradiction to Lemma 30. □

Lemma 33. *If a vertex $v \in V(H_0)$ of degree at most $\sqrt{n_0/2} - 4$ is adjacent to a vertex of degree* 1*, then there are no other vertices of degree* 1 *in H_0.*

Proof. Suppose that a vertex x of degree at most $\sqrt{n_0/2} - 4$ in H_0 is adjacent to vertex y of degree 1, and that z is another vertex of degree 1 in H_0. By Lemma 30, there are at least three vertices at distance at least 3 from x. Identify x with such a vertex u distinct from z and the neighbor of z and delete the neighbor of z. □

Consider the following discharging on the set of vertices of H_0. The initial charge $\phi(v)$ is the degree of v in H_0, and hence $\sum_{v \in V(H_0)} \phi(v) \le 3n_0 - 7$. The rules are:

(R1) If $d(v) > \sqrt{n_0/2} - 4$ and v has neighbors of degree 1, it gives 2 to one of them and nothing to the other neighbors of degree 1 (by Lemma 27, there could be only one "other neighbor" and only for one vertex v). If a vertex of degree 1 is adjacent to a vertex of degree at most $\sqrt{n_0/2} - 4$, it gets nothing. (By Lemma 33, in this case there are no other vertices of degree 1.)

(R2) If v is a weak 2-vertex, then it gets 1 from the neighbor of the larger degree.

(R3) If v is a non-weak 2-vertex, then it gets $1/2$ from each neighbor.

We claim that the new charge is at least 3 for all vertices, apart from at most one vertex of degree 1. That would imply that $e(H_0) \ge (3n_0 - 2)/2$, a contradiction to (26). To prove this, consider vertices of all possible degrees.

Case 1: $d(v) = 1$. By Lemmas 27 and 33, only one vertex of degree 1 may not receive the extra charge 2.

Case 2: $d(v) = 2$. By Rule (R1), v gives away nothing. By Rules (R2) and (R3), it gets 1 from the neighbors.

Case 3: $3 \le d(v) \le 5$. By Rules (R1) and (R2), v gives away nothing.

Case 4: $6 \le d(v) \le \sqrt{n_0/2} - 4$. By the rules, v gives away at most $d(v)/2$. So, it keeps at least $d(v)/2 \ge 3$.

Case 5: $d(v) > \sqrt{n_0/2} - 4$. If v has fewer than five adjacent weak 2-vertices, then it gives away at most $d(v)/2 + 4/2 + 1$ and hence retains at least $d(v)/2 - 3$. For $n_0 > 800$,

$$\frac{d(v)}{2} - 3 \ge \frac{\sqrt{n_0}}{2\sqrt{2}} - 7 \ge 3.$$

If v has at least five adjacent weak 2-vertices, then by Lemma 32, it gives away at most $1 + (d(v) - 5) = d(v) - 4$. This contradicts (26).

8. Concluding remarks

1. By elaborating Lemma 25, one can push the restriction $t \ge 6300$ in Theorem 4 down to about $t \ge 3000$. In the course of some proofs, we pointed out how large t needed to be for the proofs to go through. If the bounds on t were less than 500, then we did not try to obtain the best bounds.
2. It was shown in Section 2 that for infinitely many t, graphs $M(r, 4, t)$ and $M(r, 5, t)$ are not extremal for the existence of $K^*_{4,t}$-minors and $K^*_{5,t}$-minors, respectively. However, we do not know whether $M(r, 4, t)$ and $M(r, 5, t)$ are extremal for the existence of $K_{4,t}$-minors and $K_{5,t}$-minors, respectively.
3. It is annoying that the case $n_0 = t + 5$ took so much effort and space. We believe that if one could handle for $s = 4$ the cases $n_0 = t + 6$ and $n_0 = t + 7$, then we would be able to handle the remaining proof for $s = 4$ and large t using the technique of the present paper.

Acknowledgements

We thank the referees for helpful comments. The research of the first author is supported in part by NSF grant DMS-06-50784 and by grant 06-01-00694 of the Russian Foundation for Basic Research.

References

[1] M. Chudnovsky, B. Reed, P. Seymour, The edge density for $K_{2,t}$ minors (submitted for publication).
[2] P. Erdős, A. Rényi, V.T. Sós, On a problem of graph theory, Studia Sci. Math. Hungar. 1 (1966) 215–235.
[3] A.V. Kostochka, The minimum Hadwiger number for graphs with a given mean degree of vertices, Metody Diskret. Analiz. 38 (1982) 37–58.
[4] A.V. Kostochka, Lower bound of the Hadwiger number of graphs by their average degree, Combinatorica 4 (1984) 307–316.
[5] A.V. Kostochka, N. Prince, On $K_{s,t}$-minors in graphs with given average degree, Discrete Math. 308 (2008) 4435–4445.
[6] D. Kühn, D. Osthus, Forcing complete unbalanced bipartite minors, Euro. J. Comb. 26 (2005) 75–81.
[7] W. Mader, Homomorphieeigenschaften und mittlere Kantendichte von Graphen, Math. Annalen 174 (1967) 265–268.
[8] W.H. Mills, Covering designs. I. Coverings by a small number of subsets, Ars Combin. 8 (1979) 199–315.
[9] J.S. Myers, Graphs without large complete minors are quasi-random, Combin. Probab. Comput. 11 (2002) 571–585.
[10] J.S. Myers, Extremal theory of graph minors and directed graphs, Ph.D. Dissertation, University of Cambridge, 2003.
[11] J.S. Myers, The extremal function for unbalanced bipartite minors, Discrete Math. 271 (2003) 209–222.
[12] J.S. Myers, A. Thomason, The extremal function for noncomplete minors, Combinatorica 25 (2005) 725–753.
[13] N. Sauer, On the existence of regular n-graphs with given girth, J. Combin. Theory B 9 (1970) 144–147.
[14] J. Schönheim, On coverings, Pacific J. Math. 14 (1964) 1405–1411.
[15] P. Seymour, Oral communication.
[16] A. Thomason, An extremal function for contractions of graphs, Math. Proc. Cambridge Philos. Soc. 95 (1984) 261–265.
[17] A. Thomason, The extremal function for complete minors, J. Combin. Theory B 81 (2001) 318–338.
[18] D.R. Woodall, List colourings of graphs, in: Surveys in combinatorics, 2001 (Sussex), in: London Math. Soc. Lecture Note Ser., vol. 288, Cambridge Univ. Press, Cambridge, 2001, pp. 269–301.