JOURNAL OF COMBINATORIAL THEORY, Series A 25, 76 (1978)

Note

An Inequality for Pseudo-Subplanes of Sets of Orthogonal Latin Squares

E. T. PARKER

Department of Mathematics, University of Illinois, Urbana, Illinois 61801

Communicated by Marshall Hall, Jr.

Received December 28, 1976

Let $L_1, L_2, \ldots, L_t$ be a given set of t mutually orthogonal order-n latin squares defined on a symbol set S, $|S| = n$. The squares are equivalent to a $(t + 2)$—net N of order n which has n^2 points corresponding to the n^2 cells of the squares. A line of the net N defined by the latin square L_i comprises the n points of the net which are specified by a set of n cells of L_i all of which contain the same symbol x of S. If we pick out a particular $r \times r$ block B of cells, a line which contains points corresponding to r of the cells of B will be called an *r-cell* line. If there exist $r(r - 1)$ such lines among the tn lines of N, we shall say that they form a *pseudo-subplane* of order r—the "pseudo" means that these lines need not belong to only $r - 1$ of the latin squares. The purpose of the present note is to prove that the hypothesis that such a pseudo-plane exists in N implies that $r^3 - (t + 2)r^2 + r + nt \geqslant 0$.

The author thanks the referee for clarifying this abstract.

The definition, hypothesis, and conclusion are in the above abstract, and need not be repeated. For the $r \times r$ block of interest, let the nonnegative integer x_i, $0 \leqslant i \leqslant r$, be the number of lines from the latin squares incident with i cells. Under the hypothesis, each pair of these cells, on neither a common row or column, is joined by an r-cell line. Thus $x_i = 0$ for all i with $2 \leqslant i \leqslant r - 1$. Counting the lines, $x_r + x_1 + x_0 = nt$. Summing the incidence numbers in the block, $rx_r + x_1 = r^2 t$. Substituting $x_r = r(r - 1)$, solving the latter of these equations for x_1, and in turn the former for x_0, which cannot be negative, the inequality of the conclusion follows.

If t is chosen to be $n - 1$, corresponding to an affine plane of order n with a pseudo-subplane of order r, then either $r = n$ (an uninteresting case) or $n \geqslant r^2 - r + 1$. It would be interesting to know whether a plane can actually have a proper pseudo-subplane exceeding the square-root of its order—this cannot occur for a subplane; the question is probably difficult.

0097-3165/78/0251-0076$02.00/0