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Symmetrie–Gruppe–Dualität. By Erhard Scholz. Basel (Birkhäuser). 1989. 406 pp.

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In 1975, one of us (B.G.) gave a series of informal lectures at the University of Washington on two examples of "lost mathematics": tiling theory and the theory of rigidity. At that time, both subjects were indeed lost: they did not even exist as far as "mainstream" mathematics was concerned. But thanks in part to the interest generated by widely circulated mimeographed versions of these lectures, both subjects have been rediscovered and today are fields of active research. See, for example, Grünbaum and Shephard, [1980, 1983], Engel [1986], Crapo [1979], Connelly [1980], and Whitely [1984].

Independently of this renaissance, and evidently unaware of it, Erhard Scholz has undertaken the history of nineteenth-century developments in two subjects closely related to tilings and to rigidity: the refinement of the concept of symmetry in crystallography (and its impact on the development of group theory), and the application of projective geometry to graphical statics.

That pure mathematics was indeed influenced by these developments is well known to mathematicians working in these fields, but it is not appreciated by the larger mathematical community. At a time when mathematics is again becoming receptive to problems and methods of other fields, after a long period of self-involvement, Scholz's work is most welcome. Many mathematicians will be surprised to learn, in particular, that Camille Jordan's highly influential 1868 *Mémoire sur les groupes des mouvements* was inspired, at least in part, by the 1850 crystallographic work of August Bravais.

The first, "crystallographic," part of the book can be divided into three parts: before Bravais, Bravais, and after Bravais. In the first part, Scholz discusses two major schools of thought on the issue of crystal symmetry, the "atomistic" and the "dynamistic." Since Scholz restricts himself to the nineteenth century, the former is largely associated with the work of Haüy, but the atomistic view in crystallography can be traced to the work of Kepler, Hooke, and Steno in the seventeenth century. The dynamistic point of view is associated with the German school of dialectical natural philosophy, represented, for example, by Hessel and Weiss. Obviously, both schools reflected contemporary thinking about the structure of matter.

The atomists held that crystal growth and form can be explained on the assumption that a crystal is a modular structure; in the early nineteenth century, this hypothesis was further developed by Haüy and soon evolved into the theory of "regular systems of points." The dynamist school, as Scholz explains elsewhere [Scholz 1989], saw "the visible forms and laws of physics, chemistry, and crystallography as a result of dynamic patterns of forces in space" (p. 113); for them, crystal symmetry was a result of directional forces. In a sense, Bravais achieved a synthesis of the two viewpoints, placing the regular systems of points in a relatively abstract setting and giving a satisfactory classification of three-dimensional lattices. After Bravais, symmetry was linked with group theory through the work of Jordan, Schoenflies, and other mathematicians; eventually their work influenced Klein and Lie.

Scholz's aim, as we have already noted, is to show the impact of this work on the development of group theory in the nineteenth century. To do this, he treats the work of the crystallographers selectively. From his point of view, the dynamicist school, with its emphasis on the classification of crystals by symmetry, was the more influential of the two. Since it was Bravais who influenced the mathematicians, the question becomes which school had the greater influence on Bravais. Scholz evidently believes that it was the dynamicists. (In papers published after the book appeared, Scholz argues that Bravais was well acquainted with the work of the dynamicist crystallographers and appreciated it.) Thus, one is led by transitivity to the conclusion that German dynamicists influenced the development of group theory. Perhaps they did, but the influence of crystallographic ideas on mathematics was much richer than this.

It is an old story in the history of science that atomistic and dynamistic theories have different strengths, and one chooses between them partly on the basis of their ability to explain what one deems to be the crucial facts. In the case of crystals, one crucial fact is that crystals belong to only a small number of symmetry classes. Scholz seems to regard it as a telling argument against Haüy that Haüy could not explain the origin of crystal symmetry (he invoked an *a priori* law for this purpose). But Haüy could and did explain why the symmetries that they do have are so restricted. For the dynamicists, symmetry *per se* was not *ad hoc,* but it was necessary to invoke an *ad hoc* "principle of rationality" in order to explain the restriction. It is not so clear which theory is the more fundamental for the study of classification.

More importantly, classification by symmetry is not and never was the only goal of a mathematical theory of crystals. As Bravais showed, one also wants to explain growth and form. The atomist school, with its emphasis on modularity, played a major role in stimulating the theory of tilings in the plane and in space of any dimension. Today, with theoretical crystallography broadening its scope to include crystals which do not obey the classical restriction, the atomic view is more important than ever before.

Another drawback of a selective reading of the past is the temptation to recast the past in the language of the present. The reader of Scholz's book would be justified in concluding that nineteenth-century crystallographers thought of crystal structure in terms of exact sequences! But even the mathematicians of that time would have been bewildered by the framework in which their work is presented here. Recasting is fraught with danger: there is always the possibility that our contemporary language does not convey what the writer intended. Worse, the recasting may be inappropriate and even incorrect. Two examples will illustrate this point. Reporting on Sohncke's (1867) investigation of "unbounded regular systems of points in the plane," Scholz faults Sohncke (a crystallographer and physicist) for letting one see only those "symmetry systems" that can appear as the "complete symmetry group of a point pattern." However, Sohncke was not concerned with groups at all. He aimed at and was very successful in classifying point patterns and nothing else; groups are not even mentioned in his paper. While one might argue that groups were implicit in his classification scheme, this would miss the crucial point that, historically, group concepts were formulated to describe symmetry, not vice versa.

Even more misleading is the treatment of Fedorov, the late-nineteenth-century crystallographer of the atomist school who discovered the five parallelohedra (combinatorial types of convex polyhedra each of which can fill space by copies placed face-to-face in parallel positions). Scholz attributes to Fedorov the following theorem: "Es gibt genau 5 affine Klassen konvexer Paralleloeder . . ." (p. 117). ("There are precisely five affine classes of convex parallelohedra.") This is not what Fedorov proved, stated, or even mentioned: "combinatorial type" is not the same as "affine Klasse"; in fact, Scholz's assertion is wrong, since the number of affine classes of parallelohedra is infinite.

The second part of the book deals with the rigidity of frameworks, and, in particular, with the application of methods of projective geometry to graphical statics. Besides engineers (such as Culmann and Rankine), many well-known mathematicians—Cremona, Maxwell, and others—contributed to this field during the second half of the nineteenth century. Scholz gives a detailed account of many (but by no means all) of the developments and interactions in the mathematical and engineering directions. However, he does not overcome—or even face—a basic difficulty inherent in any presentation of this material: the absence, in the original literature, of precise and explicit statements of what is actually being established and discussed.

The authors whose work Scholz discusses operated very cavalierly with concepts such as polyhedra and duality, unaware of or unconcerned about the fact that their assertions had only limited validity. This vagueness led to, among other things, widespread confusion between rigidity and infinitesimal rigidity, a confusion that persisted well into the present century and permeated the literature from the simplest engineering texts to the prestigious *Encyklopädie der mathematischen Wissenschaften*. Scholz does not report on these difficulties, some of which are rooted in the fact that projective duality (in the three-dimensional case), interchanges points and planes, and not vertices and faces of general polyhedra—as the authors under consideration unhesitatingly professed. In fact, Scholz appears to be unaware of this kind of problem, which is inherent in his topic. It is unfortunate that he is unaware of the work of Crapo and others, mentioned above, which is aimed at developing Maxwell's ideas on a firm basis.

The third, brief, part of the book is devoted to a general discussion of the interactions between mathematical theories and their utilization in other fields. In particular, Scholz stresses the differences he perceives in the relation of crystallography to the theory of groups on the one hand, and the relation of graphical statics to projective geometry on the other. In the latter case, he finds that a body of preexisting mathematics formed the core of the applied development. In contrast, a large part of the geometric development in crystallography occurred within crystallography (and outside of mathematics); Scholz finds that this applied science, in fact, led later to the development and acceptance of group-theoretic approaches to geometry. But in fact the contrast is due in part to Scholz's selective reading of the past; in some cases, crystallographers, too, utilized preexisting mathematics. For example, Fedorov exploited Euler's relation $F - E + V = 2$ to derive the combinatorial types of parallelohedra; he appears to have been the first to use combinatorial methods in crystallography.

To conclude, while *Symmetrie—Gruppe—Dualität* presents a large amount of information that is otherwise not easily accessible, it is flawed in the ways that we have discussed. Still, it is an interesting case study in problems of mathematical historiography.

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