Discrete Mathematics 44 (1983) 317–318
North-Holland Publishing Company

NOTE

ON LINE DISJOINT PATHS OF BOUNDED LENGTH

Geoffrey EXOO

Bell Laboratories, Holmdel, NJ 07733, USA (Current address: Dept. of Computer Science, Indiana State University, Terre Haute, IN 47809, USA)

Received 20 May 1981
Revised 12 March 1982

In a recent paper Lovász, Neumann-Lara and Plummer proved some Mengerian theorems for paths of bounded length. In this note the line connectivity analogue of their problem is considered.

In [3] Lovász, Neumann-Lara and Plummer considered the problem of extending Menger's theorem to paths of bounded length. Given a graph G and nonadjacent points x and y of G they defined $S_k(x, y; G)$ to be the minimum number of points whose removal increases the distance from x to y to more than k, and defined $I_k(x, y; G)$ to be the maximum number of internally disjoint x-y paths whose length does not exceed k. They focused their study on the ratio $S_k(x, y; G)/I_k(x, y; G)$ and in particular on how large this ratio can be. If $f_0(k)$ is defined to be the supremum of the ratio taken over all graphs G and all nonadjacent points x and y, then they showed that $f_0(k) \leq [\frac{1}{2}k]$.

In [2] it was shown that $f_0(k) \geq [\frac{1}{4}(k+3)]$, settling a conjecture of Lovász [1, p. 248] that $f_0(k) \leq \sqrt{k}$. (The lower bound has recently been improved to $[\frac{1}{3}(k+1)]$ by F.R.K. Chung [personal communication].) In this note the analogous problem for line connectivity is considered.

Let $S'_k(x, y; G)$ be the minimum number of lines in a graph G whose removal increases the distance between distinct points x and y to more than k and let $I'_k(x, y; G)$ be the maximum number of line disjoint x-y paths in G whose length does not exceed k. Finally let $f_1(k)$ be the supremum of $S'_k(x_1, y_i, G)/I'_k(x_1, y_i, G)$ taken over all graphs G and points x and y in G.

Theorem. *For all* $k \geq 2$, $[\frac{1}{2}k] \geq f_1(k) \geq [\frac{1}{3}(k+2)]$.

Proof. The proof of the lower bound is by construction. A family of graphs $\{H_m\}$ is constructed containing points x and y such that $S'_{3m-2}(x, y; H_m) = m$ and $I'_{3m-2}(x, y; H_m) = 1$. We now describe H_m. Begin with a path of length $2m-1$ between points $u = w(1, 1)$ and $v = w(1, 2m)$ containing points $w(1, 1)$, $w(1, 2), \ldots, w(1, 2m)$. Call this path and its points layer 1. Layer 2 is added by

0012-365X/83/0000–0000/$03.00

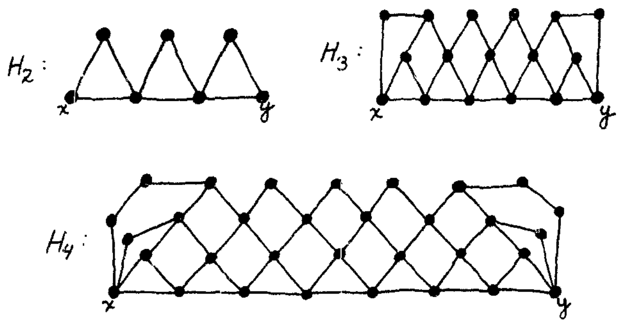

Fig. 1. The graphs H_2, H_3 and H_4.

including the $2m-1$ points $w(2, 1), w(2, 2), \ldots, w(2, 2m-1)$, so that, for $i = 1$ to $2m-1$, $w(2, i)$ is adjacent to $w(1, i)$ and $w(1, i+1)$ but to no other points in the first two layers. Suppose that layers 1 through $i-1$ have been constructed. Layer i contains points $w(i, 1), w(i, 2), \ldots, w(i, 2m-i+1)$, and $w(i, j)$ is adjacent to $w(i-1, j)$ and $w(i-1, j+1)$ but to no other points in the first i layers. Construct layers $1, 2, \ldots, m$. Finally to complete H_m, for each i, $2 \leq i \leq m$, add paths of length $i-1$ between $w(1, 1)$ and $w(i, 1)$ and between $w(1, 2m)$ and $w(1, 2m-1+1)$. The graphs H_2, H_3 and H_4 are shown in Fig. 1.

To see that $I'_{3m-2}(x, y; H_m) = 1$ observe that any x-y path with length $3m-2$ or less must contain m lines in layer 1. Since layer 1 has but $2m-1$ lines, any two such paths must share a line.

To show that $S'_{3m-2}(x, y; H_m) = m$ suppose to the contrary that there is a set A of $m-1$ lines in H_m whose removal increases the distance from x to y to more than $3m-2$. If A contains a line incident with a point of layer m then there are at most $m-1$ lines in A joining points in lower layers, and an easy induction argument gives the desired conclusion. If no line in A is incident with a point in layer m, then a path from x to y of length $3m-2$ or less can be found without much difficulty. (The reader interested in a more detailed exposition of a similar argument should consult [2].)

This establishes the lower bound for values of k of the form $3m-2$. For other values of k one can simply subdivide lines incident with x to obtain the appropriate examples.

For the upper bound, the proof given by Lovász, Neumann-Lara and Plummer [3] can be easily modified to the line version, giving the desired result.

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (American Elsevier, New York, 1976).
[2] S.M. Boyles and G. Exoo, A counterexample to a conjecture on paths of bounded length, J. Graph Theory, to appear.
[3] L. Lovász, V. Neumann-Lara and M.D. Plummer, Mengerian theorems for paths of bounded length, Period. Math. Hungar. 9 (1978) 269–276.