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A generalization of the binomial coefficients

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Dedicated to David and Maureen Loeb.

Abstract

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We pose the question of what is the best generalization of the factorial and the binomial coefficient. We give several examples, derive their combinatorial properties, and demonstrate their interrelationships.

1. Introduction

Despite being so fundamental to combinatorics, several authors have noticed that one is virtually unlimited in the choice of definition for the *factorial*—at least as far as umbral calculus is concerned. Indeed, one is presented with a bewildering number of alternatives each with its own notation.

We present a new definition of the factorial which generalizes the usual one, and study the binomial coefficients it induces. They are blessed with a variety of combinatorial properties. However, what we are most interested in is studying the interrelationship between this factorial and other famous ones.

2. The Roman factorial

We begin by presenting a generalization of the factorial $n!$ which makes sense for negative integral values of n as well as nonnegative called the *Roman factorial*

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Table 1
Roman factorials $\lfloor n \rceil !$

n	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$\lfloor n \rceil !$	$-\frac{1}{120}$	$\frac{1}{24}$	$-\frac{1}{6}$	$\frac{1}{2}$	-1	1	1	1	2	6	24	120	720

$\lfloor n \rceil !$ after its inventor Steve Roman. As usual for n a nonnegative integer the factorial is given by the product

$$\lfloor n \rceil ! = n! = 1 \times 2 \times 3 \times \cdots \times n.$$

However, for n a negative integer,

$$\lfloor n \rceil ! = \frac{(-1)^{n+1}}{(-n-1)!}, \text{ see Table 1.}$$

Proposition 2.1 (Knuth). *For any integer n,*

$$\lfloor n \rceil ! \lfloor -n \rceil ! = (-1)^n \lfloor n \rceil.$$

More generally, for every real number a, let

$$\lfloor a \rceil ! = \begin{cases} \Gamma(a+1) & \text{when } a \text{ is not a negative integer, and} \\ (-1)^{a-1}/(-a-1)! & \text{when } a \text{ is a negative integer} \end{cases}$$

where $\Gamma(a)$ is the analytic Gamma function.

Thus, for all a

$$\lfloor a \rceil ! / \lfloor a-1 \rceil ! = \lfloor a \rceil \tag{1}$$

where *Roman a* is defined to be

$$\lfloor a \rceil = \begin{cases} a & \text{for } a \neq 0, \\ 1 & \text{for } a = 0. \end{cases}$$

Note that equation (1) and the condition $\lfloor 0 \rceil = 1$ completely characterize the Roman factorial of integers.

3. The Roman coefficients

These extensions of the notion of factorial lead to a corresponding generalization of the definition of binomial coefficients.

Definition 3.1 (Roman coefficients). For all real numbers a and b, define the *Roman coefficient* (read: 'Roman a choose b') to be

$$\left\lfloor \begin{matrix} a \\ b \end{matrix} \right\rceil = \frac{\lfloor a \rceil !}{\lfloor b \rceil ! \lfloor a-b \rceil !}.$$

See Table 2.

Table 2
Roman coefficients, $\left\lfloor {n \atop k} \right\rceil$

$n\backslash k$	−4	−3	−2	−1	0	1	2	3	4	5	6
6	−1/840	1/252	−1/56	1/7	1	6	15	20	15	6	1
5	−1/504	1/168	−1/42	1/6	1	5	10	10	5	1	1/6
4	−1/280	1/105	−1/30	1/5	1	4	6	4	1	1/5	−1/30
3	−1/140	1/60	−1/20	1/4	1	3	3	1	1/4	−1/20	1/60
2	−1/80	1/30	−1/12	1/3	1	2	1	1/3	−1/12	1/30	−1/60
1	−1/20	1/12	−1/6	1/2	1	1	1/2	−1/6	1/12	−1/20	1/30
0	−1/4	1/3	−1/2	1	1	1	−1/2	1/3	−1/4	1/5	−1/6
−1	−1	1	−1	1	1	−1	1	−1	1	−1	1
−2	3	−2	1	−1	1	−2	3	−4	5	−6	7
−3	−3	1	−1/2	−1/2	1	−3	6	−10	15	−21	28
−4	1	−1/3	−1/6	−1/3	1	−4	10	−20	35	−56	84
−5	−1/4	−1/12	−1/12	−1/4	1	−5	15	−35	70	−126	210

When the two argument are both integers, the relationship between the Roman coefficients and the binomial coefficients is given by the following.

Proposition 3.2 (The six regions). *Let n and k be integers. Depending on what region of the Cartesian plane the point (n, k) is in, the following formulas apply:*

Region 1. *If $n \geqslant k \geqslant 0$, then*

$$\left\lfloor {n \atop k} \right\rceil = \binom{n}{k}.$$

See Table 3.

Region 2. *If $k \geqslant 0 > n$, then*

$$\left\lfloor {n \atop k} \right\rceil = (-1)^k \binom{-n+k-1}{k}.$$

See Table 4.

Region 3. *If $0 > n \geqslant k$, then*

$$\left\lfloor {n \atop k} \right\rceil = (-1)^{n+k} \binom{-k-1}{n-k}.$$

See Table 5.

Table 3
Region 1

$n\backslash k$	0	1	2	3	4	5	6	7
7	1	7	21	35	35	21	7	1
6	1	6	15	20	15	6	1	
5	1	5	10	10	5	1		
4	1	4	6	4	1			
3	1	3	3	1				
2	1	2	1					
1	1	1						
0	1							

Table 4
Region 2

$n\backslash k$	0	1	2	3	4	5	6
−1	1	−1	1	−1	1	−1	1
−2	1	−2	3	−4	5	−6	7
−3	1	−3	6	−10	15	−21	28
−4	1	−4	10	−20	35	−56	84
−5	1	−5	15	−35	70	−126	210

Table 5
Region 3

$n\backslash k$	−6	−5	−4	−3	−2	−1
−1	−1	1	−1	1	−1	1
−2	5	−4	3	−2	1	
−3	−10	6	−3	1		
−4	10	−4	1			
−5	−5	1				
−6	1					

Region 4. *If $k > n \geqslant 0$, then*

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^{n+k} \frac{1}{n-k} \binom{k}{n}^{-1} = (-1)^{n+k+1} \frac{1}{n+1} \binom{k}{n+1}^{-1}$$

$$= (-1)^{n+k+1} \frac{1}{k} \binom{k-1}{n}^{-1}$$

$$= (-1)^{n+k+1} n \sum_{j \geqslant 0} S(j, n)/k^{j+1} = (-1)^{n+k} \left[\Delta^n \frac{1}{x-k} \right]_{x=0},$$

where the $S(j, n)$ are the Stirling numbers of the second kind, and Δ is the forward difference operator $\Delta p(x) = p(x+1) - p(x)$. See Table 6.

Region 5. *If $n \geqslant 0 > k$, then*

$$\begin{bmatrix} n \\ k \end{bmatrix} = (-1)^k \frac{1}{k} \binom{n-k}{n}^{-1} = (-1)^k \frac{1}{k-n} \binom{n-k-1}{n}^{-1}$$

$$= (-1)^{k+1} \frac{1}{n+1} \binom{n-k-1}{n+1}^{-1}$$

$$= \begin{bmatrix} n \\ n-k \end{bmatrix} = (-1)^k \left[\Delta^n \frac{1}{x-n+k} \right]_{x=0}$$

$$= -B(k-n, -k),$$

where the pair $(n, n-k)$ lies in Region 4 (defined above), and $B(n, k)$ is the analytic Beta function. See Table 7.

Table 6
Region 4

$n\backslash k$	1	2	3	4	5	6	7
6							1/7
5						1/6	−1/42
4					1/5	−1/30	1/105
3				1/4	−1/20	1/60	−1/140
2			1/3	−1/12	1/30	−1/60	1/105
1		1/2	−1/6	1/12	−1/20	1/30	−1/42
0	1	−1/2	1/3	−1/4	1/5	−1/6	1/7

Region 6. *If* $0 > k > n$, *then*

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil = \frac{1}{n-k}\binom{-n-1}{-k-1}^{-1} = \frac{1}{k}\binom{-n-1}{-k}^{-1}$$

$$= \frac{1}{n+1}\binom{-n-2}{-k-1}^{-1} = \left[\Delta^{k-n-1}\frac{1}{x+n+1}\right]_{x=0}$$

$$= (-1)^{k+1}\left\lfloor \begin{matrix} k-n-1 \\ -n-1 \end{matrix} \right\rceil = (-1)^{k+1}\left\lfloor \begin{matrix} k-n-1 \\ k \end{matrix} \right\rceil,$$

where the pair $(k-n-1, -n-1)$ *lies in Region* 4 *(defined above), and the pair* $(k-n-1, k)$ *lies in Region* 5 *(defined above).*

Table 7
Region 5

$n\backslash k$	−4	−3	−2	−1
6	−1/840	1/252	−1/56	1/7
5	−1/504	1/168	−1/42	1/6
4	−1/280	1/105	−1/30	1/5
3	−1/140	1/60	−1/20	1/4
2	−1/80	1/30	−1/12	1/3
1	−1/20	1/12	−1/6	1/2
0	−1/4	1/3	−1/2	1

Note that in Regions 1, 2, and 3, the Roman coefficients equal binomial coefficients up to permutation and change of sign. In Regions 4, 5, and 6, the Roman coefficients are expressed simply in terms of the *reciprocals* of the binomial coefficients. Furthermore, regions 4, 5, and 6 are identical up to

Table 8
Region 6

$n\backslash k$	−6	−5	−4	−3	−2	−1
−2						−1
−3					−1/2	−1/2
−4				−1/3	−1/6	−1/3
−5			−1/4	−1/12	−1/12	−1/4
−6		−1/5	−1/20	−1/30	−1/20	−1/5
−7	−1/6	−1/30	−1/60	−1/60	−1/30	−1/6

permutation and change of sign. Thus, all of the Roman coefficients are related in a simple way to those in the first quadrant (Regions 1 and 4). In particular, the Roman coefficients always equal integers or the reciprocals of integers.

4. Properties of Roman coefficients

Several binomial coefficients identities extend to Roman coefficients.

Proposition 4.1 (Complementation Rule). *For all real numbers a and b,*

$$\left\lfloor\begin{matrix} a \\ b \end{matrix}\right\rceil = \left\lfloor\begin{matrix} a \\ a-b \end{matrix}\right\rceil.$$

Proposition 4.2 (Iterative Rule). *For all real numbers a, b, and c;*

$$\left\lfloor\begin{matrix} a \\ b \end{matrix}\right\rceil \left\lfloor\begin{matrix} b \\ c \end{matrix}\right\rceil = \left\lfloor\begin{matrix} a \\ c \end{matrix}\right\rceil \left\lfloor\begin{matrix} a-c \\ b-c \end{matrix}\right\rceil.$$

Proposition 4.3 (Pascal's Recursion). *If a and b are distinct and nonzero real numbers, then we have*

$$\left\lfloor\begin{matrix} a \\ b \end{matrix}\right\rceil = \left\lfloor\begin{matrix} a-1 \\ b \end{matrix}\right\rceil + \left\lfloor\begin{matrix} a-1 \\ b-1 \end{matrix}\right\rceil.$$

Proof. Since under these conditions $\lfloor a \rceil = a$, $\lfloor b \rceil = b$, and $\lfloor a-b \rceil = a-b$,

$$\begin{aligned}
\left\lfloor\begin{matrix} a-1 \\ b \end{matrix}\right\rceil + \left\lfloor\begin{matrix} a-1 \\ b-1 \end{matrix}\right\rceil &= \frac{\lfloor a-1 \rceil !}{\lfloor a-b-1 \rceil ! \, \lfloor b \rceil !} + \frac{\lfloor a-1 \rceil !}{\lfloor a-b \rceil ! \, \lfloor b-1 \rceil !} \\
&= \lfloor a-b \rceil \left(\frac{\lfloor a-1 \rceil !}{\lfloor a-b \rceil ! \, \lfloor b \rceil !} \right) + \lfloor b \rceil \left(\frac{\lfloor a-1 \rceil !}{\lfloor a-b \rceil ! \, \lfloor b \rceil !} \right) \\
&= \lfloor a \rceil \left(\frac{\lfloor a-1 \rceil !}{\lfloor a-b \rceil ! \, \lfloor b \rceil !} \right) \\
&= \frac{\lfloor a \rceil !}{\lfloor a-b \rceil ! \, \lfloor b \rceil !} \\
&= \left\lfloor\begin{matrix} a \\ b \end{matrix}\right\rceil. \qquad \square
\end{aligned}$$

Corollary 4.4. *If r is a nonnegative integer and the pairs of integers* (n, k), $(n+r, k)$, $(n, k+1)$, *and* $(n+r+1, k+1)$ *all lie in the same region (as defined in Theorem* 3.2), *we have*

$$\sum_{m=n}^{n+r} \left\lfloor\begin{matrix} m \\ k \end{matrix}\right\rceil = \left\lfloor\begin{matrix} n+r+1 \\ k+1 \end{matrix}\right\rceil - \left\lfloor\begin{matrix} n \\ k+1 \end{matrix}\right\rceil.$$

Proof. Induction on r. $\square$

Contrast this corollary with this classical result involving binomial coefficients in which for $n \geqslant k \geqslant 0$,

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

Analogous results hold more generally for real numbers.

If we adopt Iverson's notation, for the moment, writing logical expressions in parenthesis to mean 1 if true and 0 if false, then in the discrete case we have the following proposition.

Proposition 4.5 (Knuth's Rotation/Reflection law). *For any integer n and k,*

$$(-1)^{k+(k>0)} \left\lfloor {-n \atop k-1} \right\rceil = (-1)^{n+(n>0)} \left\lfloor {-k \atop n-1} \right\rceil.$$

Proof. By Proposition 2.1, we have

$$\left\lfloor {n \atop k} \right\rceil = (-1)^{n+k+(n<0)+(k<0)} \left\lfloor {-k-1 \atop -n-1} \right\rceil. \quad \square$$

Proposition 4.6 (Roman's Identity). *For all integers n and k,*

$$\left\lfloor {n \atop k} \right\rceil \left\lfloor {k \atop n} \right\rceil = \frac{(-1)^{n+k}}{|n-k|}.$$

Proof. Proposition 2.1. $\square$

5. Generalizations of the Roman coefficients

The Roman coefficients defined earlier were very useful. However, there are several other generalizations of binomial coefficients. For example, recall the classical definition of extended binomial coefficients.

Definition 5.1 (Classical extended binomial coefficient). Given a field element $x \in K$ in a field K of characteristic zero, and a nonnegative integer k, define the binomial coefficient 'x choose k' to be:

$$\binom{x}{k} = (x)_k / k!,$$

where $(x)_k$ denotes the lower factorial of x of degree k

$$(x)_k = \begin{cases} \prod_{i=0}^{k-1} (x-i) = x(x-1)\cdots(x-k+1) & \text{for } k \geq 0, \text{ and} \\ \prod_{i=k}^{-1} (x-i)^{-1} = 1/(x+1)(x+2)\cdots(x-k) & \text{for } k < 0. \end{cases}$$

What is the relationship between the Roman coefficients and the other generalizations of binomial coefficients? To fully answer this question, we must generalize our notation of harmonic factorial.

5.1. Knuth coefficients

Adopt the following convention independently discovered by Donald Knuth.

Definition 5.2 (Knuth factorial). Define $\lfloor a \rceil_!^{\varepsilon}$ for a real number to be the most significant term of $\Gamma(a+1+\varepsilon)$ where ε is an infinitesimal from the field of surreal numbers (a non-Euclidean field which contains the real numbers).

Thus, for a real,

$$\lfloor a \rceil_!^{\varepsilon} = \begin{cases} \Gamma(a+1) & \text{when } a \text{ is not a negative integer, and} \\ (-1)^{a-1}\omega/(-a-1)! & \text{when } a \text{ is a negative integer,} \end{cases} \tag{2}$$

where $\omega = 1/\varepsilon$. This choice of factorial would have led to 'tags' of ε or ω in appropriate places in results of this paper.

For instance, again for a real,

$$\lfloor a \rceil^{\varepsilon} = \begin{cases} a & \text{if } a \neq 0, \text{ and} \\ \varepsilon & \text{if } a = 0. \end{cases}$$

This is perhaps more natural since then $\lfloor a \rceil^{\varepsilon}$ only differs from a by at most an infinitesimal.

If we adopt equation (2) as our definition where ε can be any arbitrary constant, then the Roman factorial can be seen as a special case of the Knuth factorial where $\varepsilon = 1$. That is, $\lfloor a \rceil ! = \lfloor a \rceil_!^{1}$. Thus, the motivation for our notation.

Let us proceed to generalize the Roman coefficients.

Definition 5.3 (Knuth coefficient). For all a and b, define the *Knuth coefficient* $\left\lfloor {a \atop b} \right\rceil_!^{\varepsilon}$ by the fraction

$$\left\lfloor {a \atop b} \right\rceil_!^{\varepsilon} = \frac{\lfloor a \rceil_!^{\varepsilon}}{\lfloor b \rceil_!^{\varepsilon} \lfloor a-b \rceil_!^{\varepsilon}}.$$

Clearly, $\left\lfloor {a \atop b} \right\rceil_!^{1} = \left\lfloor {a \atop b} \right\rceil$.

Let us calculate $\left\lfloor {n \atop k} \right\rceil_!^{\varepsilon}$ for each of the six regions mentioned in Theorem 3.2.

Proposition 5.4 (The six regions). *Let ε be a nonzero complex number or surreal number, and n, k be integers. Depending on what region of the Cartesian plane the pair (n, k) is in, the following formulas apply*:

Region 1. *If $n \geq k \geq 0$, then*

$$\left\lfloor {n \atop k} \right\rceil^{\varepsilon}_{\bullet} = \binom{n}{k} = \left\lfloor {n \atop k} \right\rceil.$$

Region 2. *If $k \geq 0 > n$, then*

$$\left\lfloor {n \atop k} \right\rceil^{\varepsilon}_{\bullet} = (-1)^{k}\binom{-n+k-1}{k} = \left\lfloor {n \atop k} \right\rceil.$$

Region 3. *If $0 > n \geq k$, then*

$$\left\lfloor {n \atop k} \right\rceil^{\varepsilon}_{\bullet} = (-1)^{n+k}\binom{-k-1}{n-k} = \left\lfloor {n \atop k} \right\rceil.$$

Region 4. *If $k > n \geq 0$, then*

$$\left\lfloor {n \atop k} \right\rceil^{\varepsilon}_{\bullet} = \frac{(-1)^{n+k}\varepsilon}{(n-k)\binom{k}{n}} = \varepsilon \left\lfloor {n \atop k} \right\rceil.$$

Region 5. *If $n \geq 0 > k$, then*

$$\left\lfloor {n \atop k} \right\rceil^{\varepsilon}_{\bullet} = \frac{(-1)^{n+k}\varepsilon}{k\binom{n-k}{n}} = \varepsilon \left\lfloor {n \atop k} \right\rceil.$$

Region 6. *If $0 > k > n$, then*

$$\left\lfloor {n \atop k} \right\rceil^{\varepsilon}_{\bullet} = \frac{\varepsilon}{k\binom{-n-1}{-k}} = \varepsilon \left\lfloor {n \atop k} \right\rceil.$$

5.2. Gamma-coefficients

A limiting case of the Knuth coefficient is of special interest.

Definition 5.5 (Gamma-coefficient). Let n and k be arbitrary integers. Define the Gamma-coefficient

$$\left\lfloor {n \atop k} \right\rceil^{0}_{\bullet} = \lim_{\varepsilon \to 0} \left\lfloor {n \atop k} \right\rceil^{\varepsilon}_{\bullet} = \lim_{\varepsilon \to 0} \frac{\Gamma(n+1+\varepsilon)}{\Gamma(k+1+\varepsilon)\Gamma(n-k+1+\varepsilon)}.$$

See Table 9. Note, however, that $\left\lfloor {-1 \atop 1/2} \right\rceil^{\varepsilon}_{\bullet}$ diverges as ε tends to zero, so it is impossible to define a Gamma-coefficient $\left\lfloor {a \atop b} \right\rceil^{0}_{\bullet}$ for a and b real.

In Regions 1, 2, and 3, the Gamma-coefficients are equal to the Roman coefficients. In Region 4, 5, and 6, the Gamma-coefficients are identically zero whereas the Roman coefficients are never zero. Nevertheless, one should note that even when the Classical binomial coefficient and the Roman coefficient differ, the difference is at most one.

Table 9
Gamma-coefficient $\left\lfloor {n \atop k} \right]^{0}_{\bullet}$

$n\backslash k$	-4	-3	-2	-1	0	1	2	3	4	5	6
6	0	0	0	0	1	6	15	20	15	6	1
5	0	0	0	0	1	5	10	10	5	1	0
4	0	0	0	0	1	4	6	4	1	0	0
3	0	0	0	0	1	3	3	1	0	0	0
2	0	0	0	0	1	2	1	0	0	0	0
1	0	0	0	0	1	1	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0
−1	−1	1	−1	1	1	−1	1	−1	1	−1	1
−2	3	−2	1	0	1	−2	3	−4	5	−6	7
−3	−3	1	0	0	1	−3	6	−10	15	−21	28
−4	1	0	0	0	1	−4	10	−20	35	−56	84
−5	0	0	0	0	1	−5	15	−35	70	−126	210

Also, notice that the Gamma-coefficients are always integers. In particular, for $k \geqslant 0$ (i.e. Regions 1, 2, and 3), the Gamma-coefficients agree with the classical extended binomial coefficients.

The identities mentioned in Section 4 generalize to Gamma-coefficients. However, we defer any discussion of the combinatorial significance to [7].

Proposition 5.6 (Complementation Rule). *For all real numbers a, b, and ε, $\left\lfloor {a \atop b} \right]^{\varepsilon}_{\bullet} = \left\lfloor {a \atop a-b} \right]^{\varepsilon}_{\bullet}$. In particular, for all integers n and k, $\left\lfloor {n \atop k} \right]^{0}_{\bullet} = \left\lfloor {n \atop n-k} \right]^{0}_{\bullet}$.*

Proposition 5.7 (Iterative Rule). *For all real numbers a, b, c, and ε,*

$$\left\lfloor {a \atop b} \right]^{\varepsilon}_{\bullet} \left\lfloor {b \atop c} \right]^{\varepsilon}_{\bullet} = \left\lfloor {a \atop c} \right]^{\varepsilon}_{\bullet} \left\lfloor {a-c \atop b-c} \right]^{\varepsilon}_{\bullet}.$$

In particular, for all integers m, n, and k,

$$\left\lfloor {m \atop n} \right]^{0}_{\bullet} \left\lfloor {n \atop k} \right]^{\varepsilon}_{\bullet} = \left\lfloor {m \atop k} \right]^{0}_{\bullet} \left\lfloor {m-k \atop n-k} \right]^{0}_{\bullet}.$$

Proposition 5.8 (Pascal's Recursion). (1) *Let a and b be distinct nonzero real numbers, and let ε be a nonzero complex number. Then*

$$\left\lfloor {a \atop b} \right]^{\varepsilon}_{\bullet} = \left\lfloor {a-1 \atop b} \right]^{\varepsilon}_{\bullet} + \left\lfloor {a-1 \atop b-1} \right]^{\varepsilon}_{\bullet}.$$

(2) *For all n and k,*

$$\left\lfloor {n \atop k} \right]^{0}_{\bullet} = \left\lfloor {n-1 \atop k} \right]^{0}_{\bullet} + \left\lfloor {n-1 \atop k-1} \right]^{0}_{\bullet}$$

unless $n = k = 0$.

Nevertheless, $\left\lfloor {0 \atop 0} \right]^{0}_{\bullet} = 1$ whereas $\left\lfloor {-1 \atop -1} \right]^{0}_{\bullet} + \left\lfloor {-1 \atop 0} \right]^{0}_{\bullet} = 1 + 1 = 2$.

5.3. Other factorials

Actually as noted by Ueno [12] and Roman [8, 9], any choice of $\lfloor a \rceil !$ could be used for computations involving an umbral calculus. The only restrictions are that $\lfloor 0 \rceil !$ must equal one, and for the so called continuous iterated logarithmic algebra of [5], the function $a \mapsto \lfloor a \rceil !$ must be continuous.

For example, if we choose $\lfloor a \rceil ! = 1$ as in [1], then we have the theory of convolution sequences. Whereas, if for n an integer, we set as in [11]

$$\lfloor \lfloor n \rceil \rceil = \frac{q^{\lfloor n \rceil} - 1}{q - 1},$$

then we achieve a q-analog of the '$\lfloor n \rceil$-logarithmic theory'.

5.4. Multinomial coefficients

Recall the usual definition of a multinomial coefficient.

Definition 5.9 (Classical multinomial coefficient). Let n be a nonnegative integer, and let β be a vector with finite support of nonnegative integers. Then define the multinomial coefficients n choose β to be

$$\binom{n}{\beta} = \begin{cases} n!\left(\prod_k \beta_k!\right)^{-1} & \text{if } |\beta| = n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\binom{n}{\beta}$ is the number of ordered partitions of type β of a given n-set.

By analogy, for all reals a, and all real vectors β with finite support, define the *multinomial Roman coefficient* a choose β to be

$$\left\lfloor \begin{matrix} a \\ \beta \end{matrix} \right\rceil = \begin{cases} \lfloor a \rceil ! \left(\prod_k \lfloor \beta_k \rceil !\right)^{-1} & \text{if } |\beta| = a, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

Define the multinomial Knuth coefficients and Gamma-coefficients similarly. The multinomial Gamma-coefficients are well defined since they would only diverge if some denominator has an excess of factors of ε. However, that could only happen if $n < 0$ and $k_i \geqslant 0$ for all i, but in that case, $n \neq \sum_{i=1}^{i} k_i$, so the multinomial ε-coefficient, $\left\lfloor \begin{smallmatrix} n \\ (k_i)_{i=1}^{i} \end{smallmatrix} \right\rceil_{\varepsilon}^{\varepsilon}$ is zero by definition. Contradiction! Thus, the Gamma-coefficients are well defined.

In terms of multinomial coefficients, Proposition 4.2 becomes

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil \left\lfloor \begin{matrix} k \\ r \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n \\ n-k, k-r, r \end{matrix} \right\rceil = \left\lfloor \begin{matrix} n \\ r \end{matrix} \right\rceil \left\lfloor \begin{matrix} n-r \\ k-r \end{matrix} \right\rceil,$$

and Proposition 5.7 becomes

$$\left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rceil_{\bullet}^{0} \left\lfloor \begin{matrix} k \\ r \end{matrix} \right\rceil_{\bullet}^{0} = \left\lfloor \begin{matrix} n \\ n-k,\ k-r,\ r \end{matrix} \right\rceil_{\bullet}^{0} = \left\lfloor \begin{matrix} n \\ r \end{matrix} \right\rceil_{\bullet}^{0} \left\lfloor \begin{matrix} n-r \\ k-r \end{matrix} \right\rceil_{\bullet}^{0},$$

More generally, we have the following theorem.

Proposition 5.10 (Iterative Rule). *Let $(k_i)_{i=1}^{j}$ be a finite sequence of integers with sum n. Then*

$$\left\lfloor \begin{matrix} n \\ (k_i)_{i=1}^{j} \end{matrix} \right\rceil = \prod_{m=2}^{j} \left\lfloor \begin{matrix} \sum_{i=1}^{m} k_i \\ k_m \end{matrix} \right\rceil,$$

$$\left\lfloor \begin{matrix} n \\ (k_i)_{i=1}^{j} \end{matrix} \right\rceil_{\bullet}^{\varepsilon} = \prod_{m=2}^{j} \left\lfloor \begin{matrix} \sum_{i=1}^{m} k_i \\ k_m \end{matrix} \right\rceil_{\bullet}^{\varepsilon}, \quad \text{and}$$

$$\left\lfloor \begin{matrix} n \\ (k_i)_{i=1}^{j} \end{matrix} \right\rceil_{\bullet}^{0} = \prod_{m=2}^{j} \left\lfloor \begin{matrix} \sum_{i=1}^{m} k_i \\ k_m \end{matrix} \right\rceil_{\bullet}^{0}.$$

As opposed to ordinary Roman coefficients, these multinomial Roman coefficients are not always integers or reciprocals of integers—even when all of the arguments are integers. For example, $\left\lfloor \begin{smallmatrix} 3 \\ 2,2,-1 \end{smallmatrix} \right\rceil = \frac{3}{2}$.

However, the multinomial Gamma-coefficients are always integers, for if $\left\lfloor \begin{smallmatrix} n \\ (k_i)_{i=1}^{j} \end{smallmatrix} \right\rceil_{\bullet}^{0}$ is nonzero, then we are in one of the following two cases. Either $n \geqslant 0$, and $k_i \geqslant 0$ for all i, or $n < 0$ and there is a unique i such that $k_i < 0$. In the first case, these are ordinary multinomial coefficients. It suffices to consider the other case. Thus, $n < 0$. Without loss of generality, let $k_1 < 0$. now,

$$\left\lfloor \begin{matrix} n \\ (k_i)_{i=1}^{j} \end{matrix} \right\rceil_{\bullet}^{0} = (-1)^{n+k_1} \left\lfloor \begin{matrix} -k_1-1 \\ -n-1,\ k_2, \ldots, k_j \end{matrix} \right\rceil_{\bullet}^{0}$$
$$= (-1)^{n+k_1} \binom{-k_1-1}{-n-1,\ k_2, \ldots, k_j},$$

where $-k_1-1,\ -n-1,\ k_2, k_3, \ldots, k_{j-1},\ k_j \geqslant 0$. Hence, all the nonzero multinomial Gamma-coefficients are (up to sign) ordinary multinomial coefficients, and thus integers.

6. Resistance of the *n*-cube

Via the Gamma-coefficients and the theory of sets with a negative number of elements [7], we have a simple combinatorial interpretation for the Roman

Table 10
Resistance of the n-cube

n	0	1	2	3	4	5	6	7
R_n	0	1	1	5/6	2/3	8/15	13/30	151/340

coefficients in Regions 1, 2, and 3. However, what is the significance of the Roman coefficients in Regions 4, 5, and 6? In these regions, the Roman coefficients are the reciprocals of integers, so they do not enumerate any set. However, the following application illustrates their combinatorial significance.

Proposition 6.1. *Consider an n-cube in which each edge is represented by a wire of resistance* 1Ω *(one Ohm). The resistance between two opposing vertices of the cube is (see Table* 10)

$$R_n = 2^{-n} \sum_{i=1}^{n} i^{-1} 2^i \Omega.$$

Proof. The cube is isomorphic to the Hasse diagram of the boolean lattice of subsets of $\{1, 2, \ldots, n\}$. Without loss of generality, the two opposing vertices are $\emptyset$, and $\{1, 2, \ldots, n\}$. To compute the resistance, connect these two vertices to a 1V battery. The resulting current (in Ampères) is equal to the resistance (in Ohms).

By symmetry, each vertex on level i of the lattice has the same potential. Hence, we can consider each level as a single node without effecting the resistance. Any two adjacent levels i and $i+1$ are connected by $(n-i)\binom{n}{i}$ edges. Thus, the resistance between levels i and $i+1$ is $(1/n-i)\binom{n}{i}^{-1}\Omega$, or in the notation of Roman coefficients, the resistance between levels i and $i+1$ is

$$(-1)^{n+i} \left\lfloor \begin{matrix} i \\ n \end{matrix} \right\rceil \Omega = - \left\lfloor \begin{matrix} -n-1 \\ 1-i \end{matrix} \right\rceil \Omega .$$

The total resistance R_n is the sum of the resistances between the adjacent levels,

$$R_n = - \sum_{i=-n}^{-1} \left\lfloor \begin{matrix} -n-1 \\ i \end{matrix} \right\rceil \Omega .$$

By Theorem 4.3,

$$2R_n = R_{n-1} + \frac{2}{n} \Omega .$$

We conclude by induction noting that $R_0 = 0$ and $R_1 = 1\Omega$. □

Note that as n tends towards infinity, R_n tends towards zero as $(2/n)\Omega$.

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