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A note on Browder spectrum of operator matrices ☆

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ABSTRACT

When $A \in B(H)$ and $B \in B(K)$ are given, we denote by M_C the operator acting on the Hilbert space $H \oplus K$ of the form $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$. In this note, it is shown that the following results in [Hai-Yan Zhang, Hong-Ke Du, Browder spectra of upper-triangular operator matrices, J. Math. Anal. Appl. 323 (2006) 700–707]

$$W_3(A,B,C) = W_1(A,B,C) \quad \text{(in line 17 on p. 705)}$$

and

$$\bigcap_{C \in B(K,H)} \sigma_b(M_C) = \left(\bigcap_{C \in B(K,H)} \sigma(M_C)\right) \setminus \left[\rho_b(A) \cap \rho_b(B)\right]$$

are not always true, although the authors tried to fill the gap in their proofs by proposing an additional condition in [H.-Y. Zhang, H.-K Du, Corrigendum to "Browder spectra of upper-triangular operator matrices" [J. Math. Anal. Appl. 323 (2006) 700–707], J. Math. Anal. Appl. 337 (2007) 751–752]. A counterexample is given and then we show that under one of the following conditions:

(i) $\sigma_{su}(B) = \sigma(B)$;
(ii) $int\,\sigma_p(B) = \phi$;
(iii) $\sigma(A) \cap \sigma(B) = \phi$;
(iv) $\sigma_a(A) = \sigma(A)$,

we have

$$\bigcap_{C \in B(K,H)} \sigma_b(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup W(A,B) \cup \sigma_D(A) \cup \sigma_D(B),$$

where $W(A,B) = \{\lambda \in \mathbb{C}\colon\ N(B-\lambda)$ and $H/\overline{R(A-\lambda)}$ are not isomorphic up to a finite dimensional subspace$\}$.

1. Introduction

Throughout this note, let H and K be Hilbert spaces, $B(H,K)$ be the set of all bounded linear operators from H into K. For simplicity, we also write $B(H,H)$ as $B(H)$. If $T \in B(H,K)$, we use $R(T)$ and $N(T)$ to denote the range and kernel of T,

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respectively. And let $\alpha(T) = \dim N(T)$ and $\beta(T) = \dim K/R(T)$. Denote $asc(T)$ and $des(T)$ for the ascent and the descent of T, respectively. If the ascent and the descent of T are finite, then they are equal (see [3]). For a compact subset M of $\mathbb{C}$, we write $acc\, M$, $int\, M$, $iso\, M$ and ∂M, respectively, to denote the set of all points of accumulation of M, the interior of M, the isolated points of M and the boundary of M. The sets of left and right Fredholm operators on H, respectively, are defined as $\Phi_l(H) = \{T \in B(H)\colon\ R(T) \text{ is closed and } \alpha(T) < \infty\}$ and $\Phi_r(H) = \{T \in \Phi(H)\colon\ \beta(T) < \infty\}$. The sets of Fredholm operators and Browder operators on H, respectively, are defined by $\Phi(H) = \{T \in B(H)\colon\ \alpha(T) < \infty \text{ and } \beta(T) < \infty\}$ and $\Phi_b(H) = \{T \in \Phi(H)\colon\ asc(T) = des(T) < \infty\}$. The spectrum, the approximate point spectrum, the defect spectrum, the point spectrum, the essential spectrum, the Browder spectrum, the Browder resolvent of $T \in B(H)$ are, respectively, defined by $\sigma(T) = \{\lambda \in \mathbb{C}\colon\ \lambda - T \text{ is not invertible}\}$, $\sigma_a(T) = \{\lambda \in \mathbb{C}\colon\ \lambda - T \text{ is not bounded below}\}$, $\sigma_{su}(T) = \{\lambda \in \mathbb{C}\colon\ R(\lambda - T) \neq H\}$, $\sigma_p(T) = \{\lambda \in \mathbb{C}\colon\ \lambda - T \text{ is not injective}\}$, $\sigma_e(T) = \{\lambda \in \mathbb{C}\colon\ \lambda - T \notin \Phi(H)\}$, $\sigma_b(T) = \{\lambda \in \mathbb{C}\colon\ \lambda - T \notin \Phi_b(H)\}$ and $\rho_b(T) = \mathbb{C} \setminus \sigma_b(T)$. The Drazin inverse of $T \in B(H)$ is the unique operator $T^D \in B(H)$ satisfying

$$TT^D = T^D T, \qquad T^D T T^D = T^D, \qquad T^{k+1} T^D = T^k,$$

for some nonnegative integer k (see [3]). It is well known that T is Drazin invertible if and only if it has finite ascent and descent, which is equivalent to the fact that $T = T_1 \oplus T_2$, where T_1 is an invertible operator and T_2 is nilpotent (see [6]). The Drazin spectrum and the Drazin resolvent of T are defined by $\sigma_D(T) = \{\lambda \in C\colon\ \lambda - T \text{ is not Drazin invertible}\}$ and $\rho_D(T) = C \setminus \sigma_D(T)$, respectively.

For given $A \in B(H)$, $B \in B(K)$ and $C \in B(K, H)$, we have $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right) \in B(H \oplus K)$.

2. Main results and an example

We begin with some notations and terminology in [1].

$$\sigma(A) \cup \sigma(B) = \sigma(M_C) \cup W_1(A, B, C),$$

where $W_1(A, B, C)$ is the union of certain of the holes in $\sigma(M_C)$, which happen to be subsets of $\sigma(A) \cap \sigma(B)$.

$$\sigma_b(A) \cup \sigma_b(B) = \sigma_b(M_C) \cup W_3(A, B, C),$$

where $W_3(A, B, C)$ is the union of certain of the holes in $\sigma_b(M_C)$, which happen to be subsets of $\sigma_b(A) \cap \sigma_b(B)$.

Before explaining the main theorem, we give a counterexample to say that there exist $A \in B(H)$, $B \in B(K)$, $C \in B(K, H)$, such that neither

$$\bigcap_{C \in B(K,H)} \sigma_b(M_C) = \left(\bigcap_{C \in B(K,H)} \sigma(M_C) \right) \setminus \left[\rho_b(A) \cap \rho_b(B)\right],$$

nor

$$W_3(A, B, C) = W_1(A, B, C)^1$$

holds.

Example 2.1. Let l_2 be a complex separable infinite dimensional Hilbert space and X_n be a complex finite dimensional Hilbert space with $\dim X_n = n < \infty$. Define operators $T, S, C \in B(l_2)$ by

$$T(x_1, x_2, x_3, \ldots) = (0, x_1, x_2, x_3, \ldots) \quad \text{for any } (x_1, x_2, x_3, \ldots) \in l_2$$

and

$$S = T^*, \qquad C = I - TS.$$

Let

$$M_{C_1} = \begin{pmatrix} T & C_1 \\ 0 & S' \end{pmatrix} : l_2 \oplus (l_2 \oplus X_n) \to l_2 \oplus (l_2 \oplus X_n) = \begin{pmatrix} T & C & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{pmatrix} : l_2 \oplus l_2 \oplus X_n \to l_2 \oplus l_2 \oplus X_n,$$

where $C_1 = (C\ \ 0) : l_2 \oplus X_n \to l_2$, $S' = \left(\begin{smallmatrix} S & 0 \\ 0 & 0 \end{smallmatrix}\right) : l_2 \oplus X_n \to l_2 \oplus X_n$.

[1] In [2], Zhang and Du found that formula (12) used in their proofs of Theorem 4 in [1] is not always true and proposed an additional condition to fill this gap. However, there is another gap in the proof of Theorem 4 in [1], that is, the claim in line 17 on p. 705 is not always true. It seems that this claim had not been corrected by the authors, but the additional condition in [2] can also fill this gap, in fact, if $iso(\partial \Lambda_{(A,B)}) = \phi$, then $W_3(A, B, C) = W_1(A, B, C)$.

Then we can claim that

(1)

$$0 \in \Bigg(\bigcap_{Q \in B(l_2 \oplus X_n, l_2)} \sigma(M_Q)\Bigg) \setminus \big(\rho_b(T) \cap \rho_b(S')\big), \quad \text{but} \quad 0 \notin \bigcap_{Q \in B(l_2 \oplus X_n, l_2)} \sigma_b(M_Q).$$

Thus,

$$\bigcap_{Q \in B(l_2 \oplus X_n, l_2)} \sigma_b(M_Q) \neq \Bigg(\bigcap_{Q \in B(l_2 \oplus X_n, l_2)} \sigma(M_Q)\Bigg) \setminus \big(\rho_b(T) \cap \rho_b(S')\big).$$

In fact:

(i) Since $M_Q = \begin{pmatrix} T & Q \\ 0 & S' \end{pmatrix}$ cannot be surjective for any $Q \in B(l_2 \oplus X_n, l_2)$, then M_Q is not invertible. Thus $0 \in \bigcap_{Q \in B(l_2 \oplus X_n, l_2)} \sigma(M_Q)$.
(ii) Since $des(T) = asc(S') = \infty$, then neither S' nor T is Browder operator. Hence $0 \notin (\rho_b(T) \cap \rho_b(S'))$.
(iii) It is well known that $\begin{pmatrix} T & C \\ 0 & S \end{pmatrix}$ is unitary and then $\begin{pmatrix} T & C \\ 0 & S \end{pmatrix}$ is invertible. Moreover, since $\dim X_n = n < \infty$, then M_{C_1} is Drazin invertible Fredholm operator. Therefore $0 \notin \bigcap_{Q \in B(l_2 \oplus X_n, l_2)} \sigma_b(M_Q)$.

(2) From (iii), we can get that $\sigma(M_{C_1}) = \{\lambda \in \mathbb{C}\colon\ |\lambda| = 1\} \cup \{0\}$ and $\sigma_b(M_{C_1}) = \{\lambda \in \mathbb{C}\colon\ |\lambda| = 1\}$. Meantime, it is easy to prove that

$$\sigma(T) = \sigma_b(T) = \sigma(S') = \sigma_b(S') = \big\{\lambda \in \mathbb{C}\colon\ |\lambda| \leqslant 1\big\}.$$

Thus

$$\big\{\lambda \in \mathbb{C}\colon\ 0 < |\lambda| < 1\big\} = W_1(T, S', C_1) \neq W_3(T, S', C_1) = \big\{\lambda \in \mathbb{C}\colon\ |\lambda| < 1\big\}$$

and

$$\{0\} = iso\,\sigma(M_{C_1}) \neq iso\big(\sigma(T) \cup \sigma(S')\big) = \phi.$$

In order to prove our main result, we need the following lemmas.

Lemma 2.2.

(i) *Suppose that $\lambda_0 \in \sigma(T)$ and suppose that each neighborhood of λ_0 contains a point that is not an eigenvalue of T. Then λ_0 is a pole of the resolvent operator $R_\lambda = (\lambda - T)^{-1}$ if and only if $\lambda_0 - T$ has finite descent.*
(ii) *If $0 \in \sigma(T)$, then 0 is a finite order pole of the resolvent operator R_λ if and only if $des(T) < \infty$ and $asc(T) < \infty$.*

The result of (i) *can be found from Theorem* 10.5 *in* [5] *and the result of* (ii) *can also be obtained easily from Theorems* 10.1 *and* 10.2 *in* [5].

Lemma 2.3. *(See [4].) For a given pair (A, B) of operators, if M_C is Drazin invertible for some $C \in B(K, H)$, then A is Drazin invertible if and only if B is Drazin invertible.*

Lemma 2.4. *(See [4].) For a given pair (A, B) of operators, if M_C is Drazin invertible for some $C \in B(K, H)$, then*

(a) *$des(A^*) < \infty$ and $asc(A) < \infty$;*
(b) *$des(B) < \infty$ and $asc(B^*) < \infty$.*

Lemma 2.5. *(See [3].) For a given pair (A, B) of operators, then*

$$\bigcap_{C \in B(K,H)} \sigma_e(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup W(A, B),$$

where $\sigma_{le}(A) = \{\lambda \in \mathbb{C}\colon\ A - \lambda \notin \Phi_l(H)\}$, $\sigma_{re}(A) = \{\lambda \in \mathbb{C}\colon\ A - \lambda \notin \Phi_r(H)\}$, $W(A, B) = \{\lambda \in \mathbb{C}\colon\ N(B - \lambda)$ and $H/\overline{R(A - \lambda)}$ are not isomorphic up to a finite dimensional subspace$\}$.

Theorem 2.6. *For a given pair (A, B) of operator, we have*

$$\bigcap_{C \in B(K,H)} \sigma_b(M_C) \subseteq \big(\sigma_{le}(A) \cup \sigma_{re}(B) \cup W(A, B) \cup \sigma_D(A) \cup \sigma_D(B)\big).$$

In particular, under one of the following conditions:

(i) $\sigma_{su}(B)=\sigma(B)$;
(ii) $int\,\sigma_p(B)=\phi$;
(iii) $\sigma(A)\cap\sigma(B)=\phi$;
(iv) $\sigma_a(A)=\sigma(A)$,

we have

$$\bigcap_{C\in B(K,H)}\sigma_b(M_C)=\sigma_{le}(A)\cup\sigma_{re}(B)\cup W(A,B)\cup\sigma_D(A)\cup\sigma_D(B),$$

where $W(A,B)=\{\lambda\in\mathbb{C}\colon\ N(B-\lambda)$ *and* $H/\overline{R(A-\lambda)}$ *are not isomorphic up to a finite dimensional subspace*$\}$.

Proof. (1) The proof of the inclusion can be found in [3].

(2) In order to prove

$$\bigcap_{C\in B(K,H)}\sigma_b(M_C)=\sigma_{le}(A)\cup\sigma_{re}(B)\cup W(A,B)\cup\sigma_D(A)\cup\sigma_D(B)$$

under certain conditions, we only need to prove

$$\bigcap_{C\in B(K,H)}\sigma_b(M_C)\supseteq\left(\bigcap_{C\in B(K,H)}\sigma_e(M_C)\right)\cup\sigma_D(A)\cup\sigma_D(B)$$

from (1) and Lemma 2.5. That is, it suffices to show that

$$\lambda\notin\left(\bigcap_{C\in B(K,H)}\sigma_e(M_C)\right)\cup\sigma_D(A)\cup\sigma_D(B)$$

for any $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_b(M_C)$.

(i) If $\sigma_{su}(B)=\sigma(B)$ and $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_b(M_C)$, then there exists $C\in B(K,H)$ such that $M_C-\lambda$ is Browder operator. That is, $M_C-\lambda$ is Drazin invertible Fredholm operator. This implies that $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_e(M_C)$ and $0\in\rho(M_C)\cup iso(\sigma(M_C))$. Thus, there exists $\varepsilon>0$, such that for any $0<|\lambda_0|<\varepsilon$, $M_C-\lambda-\lambda_0$ is invertible and it is easy to prove that $B-\lambda-\lambda_0$ is right invertible. It follows $\lambda\notin acc\,\sigma_{su}(B)=acc\,\sigma(B)$. Since $M_C-\lambda$ is Drazin invertible, from Lemma 2.4 we have $des(B-\lambda)<\infty$. If $\lambda\notin\sigma(B)$, then $B-\lambda$ is Drazin invertible. Otherwise, if $\lambda\in\sigma(B)$, by Lemma 2.2, we also have that $B-\lambda$ is Drazin invertible. So $B-\lambda$ is Drazin invertible. Furthermore, using Lemma 2.3, we get $A-\lambda$ is Drazin invertible. Thus, $\lambda\in[\rho_D(A)\cap\rho_D(B)]$. Therefore

$$\lambda\notin\left(\bigcap_{C\in B(K,H)}\sigma_e(M_C)\right)\cup\sigma_D(A)\cup\sigma_D(B)$$

for any $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_b(M_C)$.

(ii) If $int\,\sigma_p(B)=\phi$ and $\lambda\notin\bigcap_{C\in B(Y,X)}\sigma_b(M_C)$, similar to (i), we can also obtain that $\lambda\in[\rho_D(A)\cap\rho_D(B)]$ and $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_e(M_C)$.

It follows $\lambda\notin(\bigcap_{C\in B(K,H)}\sigma_e(M_C))\cup\sigma_D(A)\cup\sigma_D(B)$ for any $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_b(M_C)$.

(iii) If $\sigma(A)\cap\sigma(B)=\phi$ and $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_b(M_C)$, then there exists $C\in B(K,H)$ such that $M_C-\lambda$ is Browder operator. That is, $M_C-\lambda$ is Drazin invertible Fredholm operator. Thus $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_e(M_C)$. Since $\sigma(A)\cap\sigma(B)=\phi$, then for any $\lambda\in\mathbb{C}$, either $\lambda\in\rho(A)$ or $\lambda\in\rho(B)$. If $\lambda\in\rho(A)$, then $A-\lambda$ is Drazin invertible. By Lemma 2.3, $B-\lambda$ is Drazin invertible. Thus $\lambda\in[\rho_D(A)\cap\rho_D(B)]$. If $\lambda\in\rho(B)$, similarly, we can obtain $\lambda\in[\rho_D(A)\cap\rho_D(B)]$. It shows that

$$\lambda\notin\left(\bigcap_{C\in B(K,H)}\sigma_e(M_C)\right)\cup\sigma_D(A)\cup\sigma_D(B)\quad\text{for any }\lambda\notin\bigcap_{C\in B(K,H)}\sigma_b(M_C).$$

(iv) If $\sigma_a(A)=\sigma(A)$ and $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_b(M_C)$, then there exists $C\in B(K,H)$ such that $M_C-\lambda$ is Drazin invertible Fredholm operator. This implies that $\lambda\in\rho(M_C)\cup iso\,\sigma(M_C)$. Thus there exists $\varepsilon>0$ such that for any $0<|\lambda_0|<\varepsilon$, $M_C-\lambda-\lambda_0$ is invertible and it is easy to prove that $A-\lambda-\lambda_0$ is left invertible. Thus $\lambda\notin acc\,\sigma_a(A)=acc\,\sigma(A)$, that is $\bar{\lambda}\notin acc\,\sigma(A^*)$. Since $M_C-\lambda$ is Drazin invertible, then $M_C^*-\bar{\lambda}$ is Drazin invertible. Furthermore, by Lemma 2.4, we have $des(A^*-\bar{\lambda})<\infty$. From Lemma 2.2, it follows that $A^*-\bar{\lambda}$ is Drazin invertible and then $A-\lambda$ is Drazin invertible. Then we can get that $B-\lambda$ is Drazin invertible from Lemma 2.3. So $\lambda\in[\rho_D(A)\cap\rho_D(B)]$. Moreover, $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_b(M_C)$ implies that $\lambda\notin\bigcap_{C\in B(K,H)}\sigma_e(M_C)$. It follows

$$\lambda\notin\left(\bigcap_{C\in B(K,H)}\sigma_e(M_C)\right)\cup\sigma_D(A)\cup\sigma_D(B)\quad\text{for any }\lambda\notin\bigcap_{C\in B(K,H)}\sigma_b(M_C).\qquad\Box$$

Corollary 2.7. *For a given pair (A, B) of operator, if any one of the following conditions is met*:

(i) $acc\,\sigma(B) = \phi$;
(ii) $int\,\sigma(B) = \phi$;
(iii) $acc\,\sigma(A) = \phi$;
(iv) $int\,\sigma(A) = \phi$,

then we have $\bigcap_{C\in B(K,H)} \sigma_b(M_C) = \sigma_{le}(A) \cup \sigma_{re}(B) \cup W(A, B) \cup \sigma_D(A) \cup \sigma_D(B)$.

Proof. Notice that if either $acc\,\sigma(B) = \phi$ or $int\,\sigma(B) = \phi$ holds, then $int\,\sigma_p(B) = \phi$ and if either $acc\,\sigma(A) = \phi$ or $int\,\sigma(A) = \phi$, then $\sigma_a(A) = \sigma(A)$. By these two facts and Theorem 2.6, the result is proved. □

The results of Theorem 2.6 and Corollary 2.7 are similar to that of Theorem 3.9 in [3].

Proposition 2.8. *If $\sigma_b(M_C) = \sigma_b(A) \cup \sigma_b(B)$, then $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$ holds.*

Proof. Suppose that $\sigma_b(M_C) = \sigma_b(A) \cup \sigma_b(B)$, we prove that $\sigma_D(M_C) \supseteq \sigma_D(A) \cup \sigma_D(B)$ firstly. Without loss of generality, we suppose that $0 \notin \sigma_D(M_C)$, which implies $0 \in \rho(M_C) \cup iso(\sigma(M_C))$. Therefore there exists $\varepsilon > 0$ such that for any $0 < |\lambda| < \varepsilon$, $M_C - \lambda$ is invertible and it is easy to prove that $B - \lambda$ is right invertible. Hence we have $\beta(B - \lambda) = 0$. Moreover, since $M_C - \lambda$ is invertible for every $0 < |\lambda| < \varepsilon$, then $\lambda \notin \sigma_b(M_C) = \sigma_b(A) \cup \sigma_b(B)$, so $B - \lambda \in \Phi_b(Y)$. Therefore $\alpha(B - \lambda) = \beta(B - \lambda) = 0$, that is, $B - \lambda$ is invertible for every $0 < |\lambda| < \varepsilon$. Since $0 \notin \sigma_D(M_C)$, from Lemma 2.4 we have $des(B) < \infty$. By Lemma 2.2 we know that B is Drazin invertible. And hence A is also Drazin invertible from Lemma 2.3. Thus $0 \notin \sigma_D(A) \cup \sigma_D(B)$. It proves that $\sigma_D(M_C) \supseteq \sigma_D(A) \cup \sigma_D(B)$. In [4], Cao et al. have proved that $\sigma_D(M_C) \subseteq \sigma_D(A) \cup \sigma_D(B)$ for every $C \in B(K, H)$. It is clear that $\sigma_D(M_C) = \sigma_D(A) \cup \sigma_D(B)$. □

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