JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **125**, 568–588 (1987)

Stable Dynamical Systems under Small Perturbations

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Submitted by H. Kushner

Received June 11, 1986

THE PAPER EXTENDS EXIT THEOREMS OF FREIDLIN AND WENTZELL TO DEGENERATE DIFFUSIONS. IT USES EXTENSIVELY GENERAL, EXPONENTIAL ESTIMATES DUE TO AZENCOTT

1. INTRODUCTION

It is of everyday experience that stable dynamical systems deteriorate or destabilize after operating for some, usually rather long, time. This is often due to the persistent influence of small random perturbations. The question of describing accurately the process of destabilization has been studied in the scientific literature, mainly probabilistic, as the so-called exit problem. Basic results on the exit problem belong to Freidlin and Wentzell and can be found in their monograph [6]. As was pointed out in [12] and [3] some of the conditions imposed in [6] on the studied model limited the number of applications considerably. An attempt to remove some of the restrictions was made in [12]. In the present paper we go much further than in [12] and we also make the control theoretic aspect of the subject more apparent. More specifically we are concerned here with the exit problem for solutions $X^{x,\varepsilon}$ of the general stochastic equation

$$dX = f(X)\,dt + \varepsilon\sigma(X)\,dW_t, \qquad X(0) = x,\ \varepsilon \geqslant 0 \tag{1}$$

without making any non-degeneracy assumptions on the diffusion term in (1). The assumption that the matrix $a(x) = \sigma(x)\,\sigma^*(x)$, $x \in R^n$ is positive definite has been made in all derivations of the exit theorems known to the author see [6, 10, 11]. Moreover we do not impose neither smoothness assumptions on the boundary ∂D of the basic reference set D nor invariance properties of D. The main technical tool of the paper is the Azencott's generalization, see [1], of the exponential estimates for

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solutions of (1). In our proofs we impose some conditions on the control system

$$dy = f(y)\,dt + \sigma(y)u\,dt, \qquad y(0) = x \tag{2}$$

As a rule they are weak and easy to check although it is possible that even weaker conditions suffice for the results to hold true, see Section 6 for comments. Contrary to the non-degenerate case [6, 11] the quasi-potential associated with (2) is no longer continuous.

Probabilistic considerations of Section 3 to Section 5 follow those of the book by Freidlin and Wentzell [6] and CBMS Notes of Varadhan [11]. They are included in the paper to make it self-sufficient and also because it was not always clear that the considerations of [6, 11] do generalize to the situation studied here. Similar extensions should be possible for systems with more equilibria. The case of invariant measures for (1) as $\varepsilon \downarrow 0$ is treated in [9]. Some results for the infinite dimensional systems have been obtained in [4] and [13].

2. Formulation of the Results

We assume that f and σ are C^1 functions defined on R^n with values respectively in R^n and in the space $L(R^n, R^m)$ of $n \times m$ matrices. Then Eq. (1) has a unique solution $X^{x,\varepsilon}$ defined up to the explosion time $\tau(X^{x,\varepsilon})$. In a similar way, for arbitrary x and a square integrable function $u(\cdot)$, Eq. (2) has a unique solution $y^{x,u}$ defined up to the explosion time $\tau(y^{x,u})$.

Define, for arbitrary $x \in R^n$ and number $\eta > 0$ following subset $\gamma^x(\eta)$ of R^n.

$$\gamma^x(\eta) = \text{closure}\left\{ y \in R^n;\ y = y^{x,u}(t) \text{ for some } t < t(y^{x,u}) \right.$$
$$\left. \text{and control } u \text{ such that } \frac{1}{2}\int_0^t |u(s)|^2\,ds \leqslant \eta \right\}.$$

Thus $\gamma^x(\eta)$ is the ball with the centre at x and of radius $\sqrt{\eta}$ with respect to the following quasi-distance Δ:

$$\Delta(a, b) = \inf\left\{ \sqrt{\eta};\ \eta = \frac{1}{2}\int_0^t |u(s)|^2\,ds,\ t \geqslant 0 \text{ and } y^{a,u}(0) = a,\ y^{a,u}(t) = b \right\}.$$

The distance is not in general finite and commutative but it satisfies the triangle inequality, see [2, 7] for more details.

We impose the following crucial hypothesis on system (2):

(H1) For arbitrary $\eta > \tilde{\eta} \geqslant 0$ and $\delta > 0$ there exists $r > 0$ such that

$$\gamma^x(\tilde{\eta}) \cap B(\partial D, r) \subset B(\gamma^0(\eta), \delta), \qquad \text{for all} \quad x, |x| \leqslant r.$$

In the above definition, $B(K, \delta)$ stands for the δ-neighbourhood of a set K.

Before going further we give sufficient, easy to check, conditions which imply (H1). The proofs are relegated to Section 3. In the formulation of the condition (iii), V will stand for the so-called quasipotential defined as follows

$$V(a, b) = \frac{1}{2} \inf \left\{ \int_0^T |u(s)|^2 \, ds; \; y^{a,u}(0) = a, \; y^{a,u}(T) = b, \; T \geqslant 0 \right\} a, b \in R^n.$$

with the $\inf \varnothing = +\infty$.

PROPOSITION 1. *Hypothesis* (H1) *is satisfied if one of the fillowing conditions hold*:

(i) *For arbitrary* $\eta > 0$ *there exists* $r > 0$ *such that*

$$\gamma^0(\eta) \supset B(0, r)$$

(ii) $\operatorname{Rank}[B, AB, ..., A^{n-1}B] = n$ *where*

$$A = f_x(0) \qquad \textit{and} \qquad B = \sigma(0)$$

(iii) *There exists* $r > 0$ *such that the quasipotential* $V(\cdot, \cdot)$ *is continuous on*

$$B(0, r) \times B(\partial D, r).$$

(iv) *System* (2) *is linear*: $f(x) = Ax$, $\sigma(x) \equiv \sigma$, $x \in R^n$ *and* A *is a stable matrix.*

Let $D \subset R^n$ be an open, bounded set containing the origin. Set D is our basic reference set which is assumed to be uniformly attracted by the system

$$\dot{z} = f(z), \qquad z(0) = x \tag{3}$$

to 0 in the sense that:

(H2) For arbitrary $r > 0$ there exists $T > 0$ such that for all $x \in \bar{D}$ and $t \geqslant T$

$$|z^x(t)| \leqslant r. \tag{4}$$

In (4), $z^x(\cdot)$ stands for the solution of problem (3). It is clear that the set

$$D_0 = \{x \in D;\, z^x(t) \in D \text{ for all } t \geq 0\}$$

contains an open neighbourhood of 0 provided that (H2) is satisfied.

Finally let us define the exit time $\tau^{x,\varepsilon}$

$$\tau^{x,\varepsilon} = \inf\{t \geq 0;\, X^{x,\varepsilon}(t) \in \partial D\}.$$

Our main results are given in the following two theorems:

THEOREM 1. *Under hypotheses* (H1) *and* (H2) *the following estimates hold*:

(i) *For all* $x \in D$

$$\limsup_{\varepsilon \downarrow 0} \varepsilon^2 \ln \mathbb{E}(\tau^{x,\varepsilon}) \leq \sup\{\eta;\, \gamma^0(\eta) \subseteq \bar{D}\}$$

(ii) *For all* $x \in D_0$.

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^2 \ln \mathbb{E}(\tau^{x,\varepsilon}) \geq \sup\{\eta;\, \gamma^0(\eta) \subseteq D\}$$

Let us define

$$\bar{\eta} = \sup\{\eta;\, \gamma^0(\eta) \subseteq \bar{D}\}$$
$$\underline{\eta} = \sup\{\eta;\, \gamma^0(\eta) \subseteq D\}.$$

It is clear that $\bar{\eta} < +\infty$ if and only if the following controllability assumption is satisfied:

(H3) There exists a square integrable control $u(\cdot)$ which transfers 0 to a point in $\bar{D}^c$; (for some $t > 0$, $y^{0,u}(t) \in \bar{D}^c$).

If $\bar{\eta} < +\infty$ then the following set E is well defined and as we will see later non-empty

$$E = \partial D \cap \gamma^0(\bar{\eta}).$$

THEOREM 2. *Assume that* (H1)–(H3) *hold then for arbitrary* $x \in D_0$ *and* $\delta > 0$,

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\rho(X^{x,\varepsilon}(\tau^{x,\varepsilon}), E) > \delta) = 0.$$

In the formulation of Theorem 2 the letter ρ stands for the Euclidian distance between a point and a set. The symbol ρ_T will be used in the sequel as the standard metric in the space $C[0, T; R^n]$ of continuous functions from $[0, T]$ into R^n.

Remark 1. It will follow from the proofs that hypothesis (H1) has to be required only for all $\eta \leqslant \bar{\eta} + \delta$ where δ is a positive number. Also the proof of the estimate (i) in Theorem 1 does not require (H1).

3. Some Deterministic Results

In this section we gather some results on the deterministic system (2) needed in the sequel. We will prove also Proposition 1.

3.1. *Properties of the distance Δ*

Let us remark that sets $\gamma^x(\eta)$ defined as in Section 2 but without the closure operation are not in general closed. To see this it is enough to consider the example of system (2) with $\sigma(x) = 0$ for all $x \in R^n$. It is also important to note that all sets $\gamma^x(\eta)$, $x \in R^n$, $\eta > 0$ are invariant for (3). The following two propositions are less obvious.

Proposition 2. *For arbitrary compact set $F \subset R^n$ separated from* 0 *and uniformly attracted by* (3) *to* 0, *there exist $M > 0$ and $T_0 > 0$ such that if $T > T_0$ and $y^{x,u}(t) \in F$ for all $t \leqslant T$ then*

$$\frac{1}{2}\int_0^t |u(s)|^2\, ds \geqslant M(t - T_0), \qquad \textit{for all} \quad t \in [T_0, T].$$

Proof. Since F is a bounded set and f if of class C^1 one can assume that f satisfies Lipschitz condition:

$$|f(x) - f(y)| \leqslant k\, |x - y|,$$

for some $k > 0$ and all $x, y \in R^n$. Then for all $t \leqslant T$:

$$\begin{aligned} &|\, y^{x,u}(t) - z^x(t)| \\ &\quad \leqslant k \int_0^t |\, y^{x,u}(s) - z^x(s)|\, ds + \sup_{s \leqslant t} \left| \int_0^s \sigma(y^{x,u}(r))\, u(r)\, dr \right|. \end{aligned}$$

Consequently,

$$\sup_{s \leqslant t} |\, y^{x,u}(s) - z^x(s)| \leqslant e^{kt} \sup_{s \leqslant t} \left| \int_0^s \sigma(y^{x,u}(r))\, u(r)\, dr \right|.$$

Since $|\sigma(y)| \leqslant l$ for some $l \geqslant 0$ and all $y \in F$ therefore

$$\sup_{s \leqslant t} |y^{x,u}(s) - z^x(s)| \leqslant e^{kt} l \sqrt{t} \left(\int_0^t |u(s)|^2 \, ds \right)^{1/2}.$$

Let $T_0 > 0$ be a number such that

$$|z^x(t)| \leqslant \delta/2 \qquad \text{for all} \qquad x \in F \qquad \text{and} \qquad t \geqslant T_0$$

where $\delta = \rho(F, 0)$. Then for all $t \in [T_0, T]$ and $x \in F$

$$\delta/2 \leqslant e^{kt} l \sqrt{t} \left(\int_0^t |u(s)|^2 \, ds \right)^{1/2}.$$

In particular taking $t = T_0$,

$$\int_0^{T_0} |u(s)|^2 \, ds \geqslant \left(\frac{\delta}{2l\sqrt{T_0}} \right)^2 e^{-kT_0} = M.$$

By a simple induction, for arbitrary $j = 1, 2,...$ such that $jT_0 \leqslant T$ one has

$$\int_0^{jT_0} |u(s)|^2 \, ds \geqslant jM$$

and this implies the result.

Proposition 3. *Let F be a compact set invariant for* (3) *contained in an open set G such that $\bar{G}$ is uniformly attracted by* (3) *to* 0. *Then there exists $\delta > 0$ such that for arbitrary $a \in F$ and $b \in G^c$ and a control $u(\cdot)$ transferring a to b in time $T \geqslant 0$ one has*

$$\int_0^T |u(s)|^2 \, ds \geqslant \delta$$

Proof. Without any loss of generality one cas assume that F contains a neighbourhood of 0.

Define $H = \bar{G} - F$ and let us assume that the proposition is not true. Then one can construct sequences of points (a_l), controls (u_l) and positive numbers (T_l) with the properties:

$$a_l \in H \cap F, \qquad l = 1, 2,... \qquad \text{and} \qquad a_l \to a \in F \tag{5}$$

$$y^{a_l, u_l}(t) \in H \text{ for } t \leqslant T_l, \qquad \text{and} \qquad y^{a_l, u_l}(T_l) \in G^c, \; l = 1, 2,... \tag{6}$$

$$\int_0^{T_l} |u_l(s)|^2 \, ds \to 0 \qquad \text{as} \quad l \to +\infty. \tag{7}$$

It follows from Proposition 2 that the sequence (T_l) is bounded. So suitably constructing a subsequence one can claim that $T_l \to T_\infty < +\infty$. Let α and β be positive numbers such that for all $x \in G^c$, and $t \leqslant \alpha$, $\rho(z^x(t), F) \geqslant \beta$. Let, in addition, l_0 be a natural number such that for $l \geqslant l_0$, $|T_l - T_\infty| < \alpha$. Extending all $u_l(\cdot)$, $l = l_0, \ldots$ to the interval $[0, T_\infty + \alpha]$ by setting $u_l(t) = 0$ for $t \in [T_l, T_\infty + \alpha]$ one has that $\int_0^{T_\infty + \alpha} |u_l(s)|^2\, ds \to 0$ and consequently $y^{a_l, u_l}(\cdot) \to z^a(\cdot)$ uniformly on $[0, T_\infty + \alpha]$. Since $a \in F$ and the set F is invariant for (3) therefore $z^a(T_\infty + \alpha) \in F$. However for all $l = 1, \ldots$, $y^{a_l, u_l}(T_l) \in G^c$ and therefore $\rho(y^{a_l, u_l}(T_\infty + \alpha), F) \geqslant \beta$. Thus $\rho(z^a(T_\infty + \alpha), F) \geqslant \beta > 0$ a contradiction.

We will need also the following lemma:

LEMMA 1. *If a compact set F is uniformly attracted to* 0 *by* (3) *then there exists $\delta > 0$ such that the δ-neighbourhood $B(F, \delta)$ of F is also uniformly attracted to* 0.

Proof. Let $r > 0$ be a fixed number. Since F is uniformly attracted to 0 and solutions of (3) depend continuously on the initial condition therefore, for arbitrary $x \in F$ there exists a neigbourhood $B(x)$ of x and a number $T(x) > 0$ such that for all $y \in B(x)$ and $t \geqslant T(x)$, $|z^y(t)| \leqslant r$. Since the set F is compact, it can be covered by a finite number of $B(x)$ say $F \supset \bigcup_{i=l}^N B(x_i)$. If now a δ-neighbourhood of F is contained in $\bigcup_{i=l}^N B(x_i)$ and $T \geqslant \max_{i \leqslant N} T(x_i)$ then for $y \in B(F, \delta)$ and $t \geqslant T$, $|z^y(t)| \leqslant r$. This implies the result.

The following theorem summarizes some basic properties of $\gamma^x(\eta)$, $x \in R^n$, $\eta > 0$.

THEOREM 3. (i) *For arbitrary $x \in R^n$ and $\eta > 0$*

$$\gamma^x(\eta) = \overline{\bigcup_{\tilde{\eta} < \eta} \gamma^x(\tilde{\eta})}$$

(ii) *If for some $x \in R^n$ and $\eta > 0$ the set $\gamma^x(\eta)$ is compact and uniformly attracted to* 0, *then*

$$\gamma^x(\eta) = \bigcap_{\tilde{\eta} > \eta} \gamma^x(\tilde{\eta})$$

(iii) *Assume* (H2). *Then for arbitrary $r > 0$ there exist $r_0 > 0$ and $\eta_0 > 0$ such that if $|x| \leqslant r_0$ and $\eta < \eta_0$, then*

$$\gamma^x(\eta) \subset B(0, r).$$

Proof. (i) Assume that for a control $u(\cdot)$, $\frac{1}{2}\int_0^T |u(s)|^2\, ds \leqslant \eta$, $y^{x,u}(T) = y$. If $\frac{1}{2}\int_0^T |u(s)|^2\, ds < \eta$ then $y \in \bigcup_{\tilde{\eta} < \eta} \gamma^x(\tilde{\eta})$. If $\frac{1}{2}\int_0^T |u(s)|^2\, ds = \eta$ and

$\frac{1}{2}\int_0^t |u(s)|^2\,ds<\eta$ for all $t<T$ then $y^{x,u}(t)\in\bigcup_{\tilde\eta<\eta}\gamma^x(\eta)$ and as $t\uparrow T$, $y^{x,u}(t)\to y$. If F denotes $\overline{\bigcup_{\tilde\eta<\eta}\gamma^x(\tilde\eta)}$ then $y\in F$. Let $T'=\inf\{t\leqslant T, \frac{1}{2}\int_0^t |u(s)|^2\,ds=\eta\}<T$. Then $y^{x,u}(T')=y'\in F$ by the revious argument and since the set F is invariant for (3) therefore also $y=z^{y'}(T-T')\in F$ as well.

(ii) By Lemma 1 for sufficiently small $\delta>0$ the set $\bar B(\gamma^x(\eta),\delta)$ is uniformly attracted to 0. Consequently, Proposition 3 implies that for $\tilde\eta$ sufficiently close to η, $\gamma^x(\tilde\eta)\subset B(\gamma^x(\eta),\delta)$. Since $\gamma^x(\tilde\eta)\supset\gamma^x(\eta)$ the result follows.

(iii) The hypotheses (H2) implies that for $r>0$ there exists $r_0>0$ such that for all x, $|x|\leqslant r_0$ and $t\geqslant 0$, $|z^x(t)|\leqslant r$. Let $F=\{z\in R^n;\ z=z^x(t)$ for some $t\geqslant 0$ and $|x|\leqslant r_0\}$. It is easy to see that F is a compact set invariant for (3). Without any loss of generality one can assume that $B(0,2r)\subset D$. Proposition 3 gives that there exists $\alpha>0$ such that if for some control $u(\cdot)$, $T>0$, $x\in F$ and $|b|\geqslant 2r$, $y^{x,u}(T)=b$, then $\frac{1}{2}\int_0^T |u(s)|^2\,ds\geqslant\alpha$. Thus for all x, $|x|\leqslant r_0$, $\gamma^x(\alpha/2)\subset\bar B(0,2r)$ the desired result.

COROLLARY 1. *For arbitrary* $x\in D$, $\gamma^x(\bar\eta)\subseteq\bar D$ *and* $E=\partial D\cap\gamma^0(\bar\eta)=\partial D\cap\bigcap_{\tilde\eta>\eta}\gamma^0(\tilde\eta)$. This identity will be useful in the proof of Theorem 2.

COROLLARY 2. *Always* $\underline{\eta}>0$.

3.2. *Proof of Proposition* 1

(i) Let $\eta>\tilde\eta\geqslant 0$ be given. There exists $r>0$ such that $\gamma^0(\eta-\tilde\eta)\supset B(0,r)$. If $|x|<r$ and $y\in\gamma^x(\tilde\eta)$ then there exist sequences of points (x_k), (y_k) and controls (u_k), (v_k) and positive numbers (T_k), (S_k) such that $x_k\to x$ and $y_k\to y$ as $k\to+\infty$ and for $k=1,2,\ldots$

$$x_k=y^{0,u_k}(T_k),\ \frac{1}{2}\int_0^{T_k}|u_k(s)|^2\,ds\leqslant\eta-\tilde\eta$$

$$y_k=y^{x_k,v_k}(s_k),\ \frac{1}{2}\int_0^{s_k}|v_k(s)|^2\,ds\leqslant\tilde\eta.$$

If a control $w_k(\cdot)$ is defined as

$$\begin{aligned} w_k(t)&=u_k(t)\text{ for }t\leqslant T_k\text{ and}\\ &=v_k(t-T_k)\text{ for }T_k<t\leqslant T_k+S_k\text{ then}\\ &\qquad y^{0,w_k}(T_k+S_k)=y_k.\end{aligned}$$

Consequently $y \in \gamma^0(\eta)$ and therefore for $|x| < r$

$$\gamma^x(\tilde{\eta}) \subset \gamma^0(\eta)$$

which is stronger property than (H1).

(ii) See also [8, 12]. Let $y_1,..., y_n$ be solutions of the linear equations $\dot{y}_i = f_x(0)\, y_i + \sigma(0) u_i$, $i = 1,..., n$ such that $y_i(0) = 0$ $i = 1,..., n$, vectors $y_1(T),..., y_n(T)$ are linearly independent and functions $u_1(\cdot),..., u_n(\cdot)$ are continuous on $[0, T]$. Existence of functions $u_1,..., u_n$ such that the corresponding $y_1,..., y_n$ have the desired properties for some $T > 0$ follows from the controllability condition given in (ii). For arbitrary $\xi = (\xi',..., \xi^n) \in R^n$ define $u(\xi, t) = \xi' u_1(t) + \cdots + \xi^n u_n(t)$ and let $y(\xi_j)$ be the unique solution of

$$\dot{y}(\xi, t) = f(y(\xi, t)) + \sigma(y(\xi, t))\, u(\xi, t), \qquad t \geq 0.$$

If $X(t)$ is the Jacobian of the transformation $\xi \to y(\xi, t)$ at $\xi = 0$, then

$$\dot{X}(t) = f_x(0)\, X(t) + \sigma(0)[u_1(t),..., u_n(t)].$$

Therefore the matrix $X(T)$ consists of linearly independent columns $y_1(T),..., y_n(T)$ and therefore det $X(T) \neq 0$. The implicit function theorem implies that the transformation $\xi \to y(\xi, T)$ transforms an arbitrary neighbourhood of 0 onto a neighbourhood of 0. Since for some $\beta > 0$ and all $\xi \in R^n$ $\frac{1}{2}\int_0^T |u(\xi, t)|^2\, dt \leq \beta\, |\xi|^2$ therefore, for all x of the form $x = y(\xi, T)$ where $|\xi| \leq (\eta - \tilde{\eta}/\beta)^{1/2}$, one has

$$\gamma^x(\tilde{\eta}) \subset \gamma(\eta)$$

which implies the desired inquality even with $\delta = 0$.

(iii) Continuity of the function V implies that for $\delta > 0$, $\delta < \eta - \tilde{\eta}$ there exists $r_1 < r$ such that $V(0, y) - \delta \leq V(x, y)$ for $|x| < r_1$ and $y \in B(\partial D, r)$. Now

$$\begin{aligned}
\gamma^x(\tilde{\eta}) \cap B(\partial D, r) &\subseteq \{ y \in B(\partial D, r);\ V(x, y) \leq \tilde{\eta} \} \\
&\subseteq \{ y \in B(\partial D, r);\ V(0, y) \leq \tilde{\eta} + \delta \} \\
&\subseteq \gamma^0(\tilde{\eta} + \delta) \cap B(\delta D, r) \\
&\subseteq \gamma^0(\eta) \cap B(\partial D, r)
\end{aligned}$$

which proves (iii).

(iv) Solution to the Eq. (2) is of the form

$$y^{x,u}(t) = S(t)x + \int_0^t s(t-s)\, \sigma u(s)\, ds$$

where $S(t) = \exp tA$ and for some $M > 0$ and $\alpha > 0$, $|S(t)| \leqslant M \exp(-\alpha t)$, $t \geqslant 0$. Consequently for arbitrary $t \geqslant 0$ and a square integrable u

$$|y^{x,u}(t) - y^{0,u}(t)| \leqslant \delta$$

provided that $|x| \leqslant \partial/M$ and this proves the result.

3. Some Auxiliary Probabilistic Results

From now on we make the simplifying assumption that f and σ vanish for sufficiently large arguments. This assumption implies that solutions $X^{x,\varepsilon}$ and $y^{x,u}$ exist globally. We first quote the following uniform estimates due to Azencott [1]. For their formulation we need however some more notation:

$$\Gamma_T^x(\eta) = \left\{ y^{x,u} \in C[0, T; R^n]; \frac{1}{2}\int_0^T |u(s)|^2 \, ds \leqslant \eta \right\}$$

$$\gamma_T^x(\eta) = \text{closure of } \{ y \in R^n;\ y = y^{x,u}(t) \text{ for some } t \leqslant T \text{ and } y^{x,u} \in \Gamma_T^x(\eta) \}$$

Note that $\gamma^x(\eta) = \text{closure } \bigcup_{T \geqslant 0} \gamma_T^x(\eta)$.

Proposition 4 [1]. *For arbitrary compact set $K \subset R^n$ and constants $T > 0$, $\eta > 0$, $\alpha > 0$, and $\beta > 0$ there exists $\varepsilon_0 > 0$ such that for all $x \in K$ and $u \in L^2[0, T; R^m]$, $\frac{1}{2}\int_0^T |u(s)|^2 \, ds \leqslant \eta$ one has*

$$\mathbb{P}(\rho_T(X^{x,\varepsilon}, y^{x,u}) \leqslant \alpha) \geqslant e^{-(1/\varepsilon^2)(1/2 \int_0^T |u(s)|^2 ds + \beta)}$$

provided $x \in K$ and $\varepsilon < \varepsilon_0$.

Proposition 5 [1]. *For arbitrary compact set $K \subset R^n$ and constants $T > 0$, $\eta > 0$, $\alpha > 0$, and $\beta > 0$ there exists $\varepsilon_0 > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ and $x \in K$*

$$\mathbb{P}(\rho_T(X^{x,\varepsilon}, \Gamma_T^x(\eta)) > \alpha) \leqslant e^{-(1/\varepsilon^2)(\eta - \beta)}$$

We will need also the following result:

Proposition 6. *For arbitrary $\delta > 0$, $T > 0$*

$$\mathbb{P}(\sup_{s \leqslant T} |X^{x,\varepsilon}(s) - z^x(s)| \leqslant \delta) \to 1$$

as $\varepsilon \downarrow 0$ uniformly on compact sets.

Proof. One can assume that f satisfies the Lipschitz condition with a constant $L>0$ and that $|\sigma(\cdot)|$ is bounded by M. Therefore

$$|X^{x,\varepsilon}(t)-z^x(t)| \leqslant L\int_0^t |X^{x,\varepsilon}(s)-z^x(s)|\,ds+\sup_{s\leqslant t}|m(s)|$$

where $m(t)=\int_0^t \sigma(X^{x,\varepsilon}(s))\,dW(s)$. From Gronwall's inequality it follows that

$$\sup_{s\leqslant t}|X^{x,\varepsilon}(s)-z^x(s)| \leqslant \varepsilon\, e^{Lt}\sup_{s\leqslant t}|m(s)|.$$

Doob's inequality applied to the continuous martingale $m(\cdot)$ implies

$$\begin{aligned}&\mathbb{P}(\sup_{s\leqslant t}|X^{x,\varepsilon}(s)-z^x(s)|\geqslant\delta)\\&\qquad\leqslant \mathbb{P}(\sup_{s\leqslant t}|m(s)|\geqslant\frac{\delta}{\varepsilon}e^{-Lt})\leqslant\frac{\varepsilon}{\delta}e^{Lt}\mathbb{E}\,|m(t)|.\end{aligned}$$

But $\mathbb{E}\,|m(t)|\leqslant(\mathbb{E}(\int_0^t\sigma^2(X^{x,\varepsilon}(s))\,ds))^{1/2}\leqslant M\sqrt{t}$. The result follows.

The following result is of independent interest. Its proof uses Prop. 2 and Prop. 5 in the same way as in [6].

PROPOSITION 7. *Let F be an arbitrary compact set separated from* 0 *and uniformly attracted to* 0. *Then for arbitrary $K>0$ there exist $T>0$ and $\varepsilon_0>0$ such that for all $x\in F$ and $0<\varepsilon<\varepsilon_0$*

$$\mathbb{P}(X^{x,\varepsilon}(t)\in F \textit{ for all } t\leqslant T)\leqslant e^{-(1/\varepsilon^2)K}$$

Proof. Let F_1 be a compact set also separated from 0 and uniformly attracted to 0 and such that $F_1\supset B(F,\delta)$ for some positive $\delta>0$. By Proposition 2 for arbitrary $K_1>0$ one can find $T>0$ such that if $\int_0^T|u(s)|^2\,ds\leqslant K_1$ and $x\in F_1$ then for some $t\leqslant T$, $y^{x,u}(t)\in F_1^c$. Applying Proposition 5 one gets in turn that for $\beta>0$ there exists $\varepsilon_0>0$ such that

$$\mathbb{P}\left(\rho_T(X^{x,\varepsilon},\Gamma_T^x(K_1))>\frac{\delta}{2}\right)\leqslant e^{-(1/\varepsilon^2)(K_1-\beta)}\qquad\text{for all}\quad x\in F\text{ and }\varepsilon<\varepsilon_0.$$

Since the event

$$\{X^{x,\varepsilon}(t)\in F\text{ for all } t\leqslant T\}$$

is contained in

$$\left\{\rho_T X^{x,\varepsilon},\Gamma_T^x(K_1))>\frac{\delta}{2}\right\}$$

therefore for all $x \in F$ and $\varepsilon < \varepsilon_0$

$$\mathbb{P}(X^{x,\varepsilon}(t) \in F \text{ for all } t \leqslant T)\, e^{-(1/\varepsilon^2)(K_1 - \beta)}.$$

It is enough to put $K_1 - \beta = K$.

4. Proof of Theorem 1

With the results of Section 2 and 3 we can prove Theorem 1 similarly as in [6].

(i) Without any loss of generality one can assume that $\bar{\eta} + \infty$. If $\eta > \bar{\eta}$ then there exist $T > 0$ and control $u(\cdot)$ such that $\frac{1}{2}\int_0^T |u(s)|^2\, ds \leqslant \eta$ and the trajectory $y^{0,u(\cdot)}(t)$, $t \leqslant T$ is not contained in $\bar{D}$. One can therefore require that $y^{0,u(\cdot)}(T) = b \in \bar{D}^c$. Since solutions to Eq. (2) depend continuously on initial data therefore there exist $r_1 > 0$, $r_2 > 0$, such that $\overline{B(b, r_2)} \subset \bar{D}^c$ and for all x, $|x| \leqslant r_1$ $y^{x,u(\cdot)}(T) \in B(b, r_2/2)$. Let $\delta < r_2/2$ and $\beta > 0$. By Proposition 4 there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$ and $|x| \leqslant r_1$

$$\mathbb{P}(\sup_{t \leqslant T} |X^{x,\varepsilon}(t) - y^{x,u}(t)| < \delta) \geqslant e^{-(1/\varepsilon^2)((1/2)\int_0^T |u(s)|^2 ds + \beta)} \geqslant e^{-(1/\varepsilon^2)(\eta + \beta)}.$$

Since the event $\{T^{x,\varepsilon} \leqslant T\}$ contains $(\sup_{s \leqslant t} |X^{x,\varepsilon}(s) - y^{x,u(\cdot)}(s)| < \delta\}$, therefore

$$\mathbb{P}(\tau^{x,\varepsilon} \leqslant T) \geqslant e^{-(1/\varepsilon^2)(\eta + \beta)}, \qquad |x| \leqslant r_1, \quad \varepsilon < \varepsilon_0.$$

It follows from Proposition 6 that for arbitrary $p \in (0, 1)$ one finds $T_1 > 0$ such that for all $x \in \bar{D}$ and sufficiently small ε, say $\varepsilon < \varepsilon_1$

$$\mathbb{P}(|X^{x,\varepsilon}(T_1)| \leqslant r_1) \geqslant p.$$

Markov property implies that for all $x \in \bar{D}$ and $\varepsilon < \varepsilon_1 \wedge \varepsilon_2$

$$\mathbb{P}(\tau^{x,\varepsilon} \leqslant T + T_1) \geqslant pe^{-(1/\varepsilon^2)(\eta + \varepsilon)}$$

or equivalently that

$$\mathbb{P}(\tau^{x,\varepsilon} > T + T_1) \leqslant 1 - pe^{-(1/\varepsilon^2)(\eta + \beta)}$$

By a simple induction argument and Markov property, for $k = 1, 2, \ldots$

$$\mathbb{P}(\tau^{x,\varepsilon} > k(\tau + \tau_1)) \leqslant (1 - pe^{-(1/\varepsilon^2)(\eta + \beta)})^k$$

But $1/(T+T_1)\mathbb{E}(\tau^{x,\varepsilon})\leqslant\sum_{k=0}^{+\infty}\mathbb{P}(\tau^{x,\varepsilon}>k(T+T_1))\leqslant\sum_{k=0}^{+\infty}(1-pe^{-(1/\varepsilon^2)(\eta+\beta)})^k$ $\leqslant$ $(1/p)e^{-(1/\varepsilon^2)(\eta+\beta)}$ and $\varepsilon^2\ln\mathbb{E}(\tau^{x,\varepsilon})\leqslant\varepsilon^2\ln(T+T_1)/p+(\eta+\beta)$ consequently $\overline{\lim}_{\varepsilon\downarrow 0}\varepsilon^2\ln\mathbb{E}(\tau^{x,\varepsilon})\leqslant\eta+\beta$. Passing with M and β to $\tilde{\eta}$ and 0 respectively one gets (i).

(ii) To simplify notation we make an assumption slightly stronger than (H1) namely that for arbitrarily $\eta>\tilde{\eta}>0$ and $\delta>0$ there exists $r>0$ such that

$$\gamma^x(\tilde{\eta})\subset B(\gamma^0(\eta),\delta)\qquad\text{for all}\quad x,\ |x|\leqslant r. \tag{8}$$

In the proof only those points from $\gamma_x(\tilde{\eta})$ which are close to ∂D matter.

Let $\eta>0$ be a number such that $\gamma^0(\eta)\subset D$. Then for some $\delta>0$, $\rho(\gamma^0(\eta),D^c)\geqslant\delta$. Moreover for $\tilde{\eta}<\eta$ one can find $r>0$ such that for all x, $|x|\leqslant r$

$$\gamma^x(\tilde{\eta})\subset B\left(\gamma^0(\eta),\frac{\delta}{2}\right).$$

Let $r_1>r_0>0$ be arbitrary numbers smaller than r. For arbitrary x, $|x|=r_1$ one defines stopping times $\tau_1^x\leqslant\kappa_1^x\leqslant\tau_2^x\leqslant\kappa_2^x\leqslant\cdots$ as follows:

$$\tau_1^{x,\varepsilon}=\inf\{t\geqslant 0;\ X^{x,\varepsilon}(t)\in\partial D\text{ or }|X^{x,\varepsilon}(t)|=r_0\}$$
$$\kappa_1^{x,\varepsilon}=\inf\{t\geqslant\tau_1^x;\ |X^{x,\varepsilon}(t)|=r_1\}.$$

And for arbitrary $k=1,2,\ldots$

$$\tau_{k+1}^{x,\varepsilon}=\inf\{t\geqslant\kappa_k^{x,\varepsilon},\ X^{x,\varepsilon}(t)\in\partial D\text{ or }|X^{x,\varepsilon}(t)|=r_0\}$$
$$\kappa_{k+1}^{x,\varepsilon}=\inf\{t\geqslant\tau_{k+1}^{x,\varepsilon};\ X^{x,\varepsilon}(t)\in\partial D\text{ or }|X^{x,\varepsilon}(t)|=r_1\}.$$

We estimate first

$$p_1=\mathbb{P}(X^{x,\varepsilon}(\tau_1^{x,\varepsilon})\in\partial D).$$

Note that for arbitrary $T>0$

$$\begin{aligned}p_1&\leqslant\mathbb{P}(|X^{x,\varepsilon}(s)|>r_0\text{ for all }s\leqslant T\text{ and }\tau_1^{x,\varepsilon}>T)\\&\quad+\mathbb{P}(\tau_1^{x,\varepsilon}\leqslant T\text{ and }\tau_1^{x,\varepsilon}=\tau^{x,\varepsilon})\\&\leqslant\mathbb{P}(X^{x,\varepsilon}(s)\in F\text{ for all }s\leqslant T)+\mathbb{P}(\tau^{x,\varepsilon}\leqslant T),\end{aligned}$$

where

$$F=\bar{D}\backslash B(0,r_0).$$

By Proposition 7 there exists $T>0$ and $\varepsilon_0>0$ such that for $\varepsilon<\varepsilon_0$ and $x\in F$

$$\mathbb{P}(X^{x,\varepsilon}(s)\in F \text{ for all } s\leqslant T)\leqslant e^{-(1/\varepsilon^2)\tilde\eta}$$

However $\{\tau^{x,\varepsilon}\leqslant T\}\subset\{\rho_T(X^{x,\varepsilon},\Gamma_T^x(\tilde\eta))\geqslant(\delta/2)\}$ and therefore

$$\mathbb{P}(\tau^{x,\varepsilon}\leqslant T)\leqslant\mathbb{P}\left(\rho_T(X^{x,\varepsilon},\Gamma_T^x(\eta))>\frac{\delta}{2}\right)$$

From Proposition 5 there exists $\varepsilon_1>0$ such that for $\varepsilon<\varepsilon_1$ and $|x|=r$,

$$\mathbb{P}(\tau^{x,\varepsilon}\leqslant T)\leqslant e^{-(1/\varepsilon^2)(\tilde\eta-\beta)}$$

Consequently one can assume that for $\varepsilon<\varepsilon_0\wedge\varepsilon_1$, $|x|=r_1$

$$\mathbb{P}(|X^{x,\varepsilon}(\tau_1^{x,\varepsilon})|=r_0)\geqslant 1-e^{-(1/\varepsilon^2)(\tilde\eta-\beta)}$$

If $A_k^{x,\varepsilon}=\{|X^{x,\varepsilon}(\tau_k^{x,\varepsilon})|=r_0\}$, then the strong Markov property implies

$$\mathbb{P}(\mathbb{A}^{x,\varepsilon}\cap\cdots\cap A_k^{x,\varepsilon})\geqslant(1-e^{-(1/\varepsilon^2)(\eta-\beta)})^k$$

for $|x|=r_1$ and $\varepsilon<\varepsilon_0\wedge\varepsilon_1$. Consider now x, $|x|=r_0$ and define

$$\begin{aligned}\hat\tau_1^{x,\varepsilon}=\inf\{t\geqslant 0;\ &|X^{x,\varepsilon}(s)|=r_1 \text{ for some } s\in[0,t] \text{ and}\\ &|X^{x,\varepsilon}(t)|=r_0 \text{ or } X^{x,\varepsilon}(t)\in\partial D\}\\ \hat\tau_{k+1}^{x,\varepsilon}=\inf\{t\geqslant\hat\tau^{x,\varepsilon},\ &|X^{x,\varepsilon}(s)|=r_1 \text{ for some } s\in[\hat\tau_k^{x,\varepsilon},t] \text{ and}\\ &|X^{x,\varepsilon}(t)|=r_0 \text{ or } X^{x,\varepsilon}(t)\in\partial D\},\\ \hat A_k^{x,\varepsilon}=\{|X^{x,\varepsilon}(\tau_k^{x,\varepsilon})|=r_0\},&\qquad k=1,2,\ldots.\end{aligned}$$

One can assume, see the proof of (i), that

$$\mathbb{P}(|X^{x,\varepsilon}(t)|=r_1 \text{ for some } t)=1$$

for $|x|=r_0$ and therefore for arbitrary $k=1,\ldots$ one gets also that

$$\mathbb{P}(\hat A_1^{x,\varepsilon}\cap\hat A_2^{x,\varepsilon}\cap\cdots\cap\hat A_k^{x,\varepsilon})\geqslant(1-e^{-(1/\varepsilon^2)(\tilde\eta-\beta)})^k,\ \varepsilon<\varepsilon_0\wedge\varepsilon_1,\ |x|=r_0$$

Define $\xi_1^{x,\varepsilon} = \hat{\tau}_1^{x,\varepsilon}$ and $\xi_{k+1}^{x,\varepsilon} = \hat{\tau}_{k+1}^{x,\varepsilon} - \hat{\tau}_k^{x,\varepsilon}$, $k = 1,\ldots$. Then

$$\begin{aligned}
\mathbb{E}(\tau^{x,\varepsilon}) &= \mathbb{E}(\xi_1^{x,\varepsilon}; (\hat{A}_1^{x,\varepsilon})^c) \\
&\quad + \sum_{k=2}^{+\infty} \mathbb{E}(\xi_1^{x,\varepsilon} + \cdots + \xi_k^{x,\varepsilon}; \hat{A}_1^{x,\varepsilon} \cap \cdots \cap \hat{A}_{k-1}^{x,\varepsilon} \cap (\hat{A}_k^{x,\varepsilon})^c) \\
&= \mathbb{E}(\xi_1; \hat{A}_1^{x,\varepsilon}) + \sum_{k=2}^{+\infty} \mathbb{E}(\xi_1^{x,\varepsilon} \\
&\quad + \cdots + \xi_k^{x,\varepsilon}; \hat{A}_1^{x,\varepsilon} \cap \cdots \cap \hat{A}_k^{x,\varepsilon} \backslash \hat{A}_1^{x,\varepsilon} \cap \cdots \cap \hat{A}_{k-1}^{x,\varepsilon}) \\
&= \mathbb{E}(\xi_1^{x,\varepsilon}) + \sum_{k=2}^{+\infty} \mathbb{E}(\xi_k^{x,\varepsilon}; \hat{A}_1^{x,\varepsilon} \cap \cdots \cap \hat{A}_{k-1}^{x,\varepsilon})
\end{aligned}$$

Without any loss of generality one can assume that there exists a constant $M > 0$ such that for $\varepsilon < \varepsilon_0 \wedge \varepsilon_1$ and $|x| = r_0$, $\mathbb{E}(\xi_1^{x,\varepsilon}) \geqslant M$. Consequently

$$\mathbb{E}(\tau^{x,\varepsilon}) \geqslant M + M \sum_{k=2}^{+\infty} \mathbb{P}(A_1^{x,\varepsilon} \cap \cdots \cap A_{k-1}^{x,\varepsilon}) \geqslant M e^{-(1/\varepsilon^2)(\tilde{\eta} - \beta)}$$

for all x, $|x| = r_0$ and $\varepsilon < \varepsilon_0 \wedge \varepsilon_1$. Thus if $|x| = r_0$ then

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^2 \ln \mathbb{E}(\tau^{x,\varepsilon}) \geqslant \tilde{\eta} - \beta$$

Passing with $\tilde{\eta}$ and β to η and 0 respectively one gets that for $|x| \leqslant r_0$

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^2 \ln \mathbb{E}(\tau^{x,\varepsilon}) \geqslant \eta$$

where η was an arbitrary number smaller than $\underline{\eta}$, and therefore

$$\liminf_{\varepsilon \downarrow 0} \varepsilon^2 \ln \mathbb{E}(\tau^{x,\varepsilon}) \geqslant \underline{\eta}, \qquad |x| \leqslant r_0.$$

If now $x \in D_0$ then $z^x(t) \in D$ for all $t \geqslant 0$ and for some $T > 0$, $|z^x(T)| \leqslant r_0$. Taking into account sufficiently small tube around $z^x(t)$, $t \leqslant T$ and Proposition 6 one gets that (ii) holds true for all $x \in D_0$.

5. Proof of Theorem 2

The proof is an adaptation of that given in [11, Section 7].

Let $\delta > 0$ be a fixed number and let

$$E_\delta = \{x \in \partial D; \rho(x, E) < \delta\}, \qquad E_\delta^c = x \in \partial D; \rho(x, E) \geqslant \delta\}.$$

Let moreover $A^{x,\varepsilon}$ and $B^{x,\varepsilon}$ be the following events

$$A^{x,\varepsilon} = \{X^{x,\varepsilon}(\tau^{x,\varepsilon}) \in E_\delta\}, \qquad B^{x,\varepsilon} = \{X^{x,\varepsilon}(\tau^{x,\varepsilon}) \in E_\delta^c\}.$$

Assumption (H3) implies that for $\varepsilon > 0$ small enough and $x \in D$

$$\mathbb{P}(A^{x,\varepsilon} \cup B^{x,\varepsilon}) = \mathbb{P}(A^{x,\varepsilon}) + \mathbb{P}(B^{x,\varepsilon}) = 1,$$

see the proof of Theorem 1, (i). Our aim is to show that $\mathbb{P}(A^{x,\varepsilon}) \to 0$ as $\varepsilon \downarrow 0$ for all $x \in D_0$. Without any loss of generality one can assume that the set E_δ^c is nonempty.

Let us fix $r_1 > r_0 > 0$ and $T > 0$ and define Markov times $\tau_T^{x,\varepsilon}$

$$\tau_T^{x,\varepsilon} = \inf\{t \geqslant T;\ |X^{x,\varepsilon}(t)| = r_0 \text{ or } X^{x,\varepsilon}(t) \in \partial D\}$$

Moreover let

$$A_T^{x,\varepsilon} = \{\tau^{x,\varepsilon} \leqslant \tau_T^{x,\varepsilon} \text{ and } X^{x,\varepsilon}(\tau^{x,\varepsilon}) \in E_\delta\}$$

$$B_T^{x,\varepsilon} = \{\tau^{x,\varepsilon} \leqslant \tau_T^{x,\varepsilon} \text{ and } X^{x,\varepsilon}(\tau^{x,\varepsilon}) \in E_\delta^c\}$$

We will show first that one can choose r_0, r_1, and T such that

$$\frac{\sup_{|x| = r_1} \mathbb{P}(B_T^{x,\varepsilon})}{\operatorname{int}_{|x| = r_1} \mathbb{P}(A_T^{x,\varepsilon})} \to 0 \qquad \text{as} \quad \varepsilon \downarrow 0.$$

Remark that

$$\mathbb{P}(A_T^{x,\varepsilon}) \geqslant \mathbb{P}(\tau^{x,\varepsilon} \leqslant T \qquad \text{and} \qquad X^{x,\varepsilon}(\tau^{x,\varepsilon}) \in E_\delta),$$

and for arbitrary $T_1 > 0$,

$$\begin{aligned} \mathbb{P}(B_T^{x,\varepsilon}) &\leqslant \mathbb{P}(\{{}_T^{x,\varepsilon} \leqslant T + T_1\} \cap B_T^{x,\varepsilon}) + \mathbb{P}(\tau_T^{x,\varepsilon} > T + T_1) \\ &\leqslant \mathbb{P}(X^{x,\varepsilon}(t) \in E_\delta^c \text{ for some } t \leqslant T + T_1) \\ &\quad + \mathbb{P}(X^{x,\varepsilon}(t) \in F \text{ for all } t \in [T, T + T_1]) \end{aligned}$$

where $F = \bar{D} \setminus B(0, r_0)$. We will need the following lemma.

LEMMA 2. *There exist* $r > 0$, $\tilde{\eta} > \bar{\eta}$, $\delta_1 > 0$, $\delta_2 > 0$ *such that if* $|x| \leqslant r$ *then*

$$\rho(B(\gamma^x(\tilde{\eta}), \delta_1), E_\delta^c) \geqslant \delta_2 \tag{9}$$

Proof. From the very definition of the set E and Corollary 1, Section 3 it follows that the lemma is true for $x = 0$. Because of (H1) it is true for all x close enough to 0.

To estimate further the three probabilities of interest let us fix numbers $\eta_1 > 0$, $\beta_1 > 0$, $\eta_2 > 0$, $\beta_2 > 0$ as follows

$$\bar{\eta} < \eta_1 < \eta_1 + \beta_1 + \beta_2 < \eta_2 < \tilde{\eta}$$

From the very definition of $\bar{\eta}$ and Lemma 2 there exist $T > 0$, $\delta_3 > 0$, $r_1 > 0$, $r_1 > r$ and control $u \in L^2[0, T; R^m]$ such that for all x, $|x| \leqslant r_1$:

$$\frac{1}{2}\int_0^T |u(s)|^2\, ds < \eta_1 \qquad \text{and} \qquad \rho(y^{x,u}(T), \bar{D}) \geqslant \delta_3$$

Let r_0 be an arbitrary positive number such that $r_0 < r_1$. It follows from Proposition 7 that there exists $T_1 > 0$ and $\varepsilon_1 > 0$ such that for all $x \in F = \bar{D} \backslash \{x\colon |x| < r_0\}$ and $\varepsilon < \varepsilon_1$

$$\mathbb{P}(X^{x,\varepsilon}(s) \in F \text{ for all } s \leqslant T_1) \leqslant e^{-(1/\varepsilon^2)(\eta_2 - \beta_2)}$$

The Markov property implies in turn that for all $x \in F$ and $\varepsilon < \varepsilon_1$

$$\mathbb{P}(X^{x,\varepsilon}(s) \in F \text{ for all } s \in [T, T + T_1]) \leqslant e^{-(1/\varepsilon^2)(\eta_2 - \beta_2)}$$

Moreover, if $|x| \leqslant r_1 < r$, then

$$\begin{aligned} &\mathbb{P}(\tau^{x,\varepsilon} \leqslant T \text{ and } X^{x,\varepsilon}(\tau^{x,\varepsilon}) \in E_\delta) \\ &\qquad \geqslant \mathbb{P}(\sup_{s \leqslant T} |X^{x,\varepsilon}(s) - y^{x,u}(s)| \leqslant \delta_3 \wedge \delta_1). \end{aligned}$$

This is because $y^{x,u}(s) \in \gamma^x(\eta_1) \subset \gamma^x(\eta)$ and lemma. Consequently for all x, $|x| \leqslant r_1$ and $\varepsilon < \varepsilon_2$, $\varepsilon_2 > 0$ suitably chosen

$$\begin{aligned} &\mathbb{P}(\tau^{x,\varepsilon} \leqslant T \text{ and } X^{x,\varepsilon}(\tau^{x,\varepsilon}) \in E_\delta) \\ &\qquad \geqslant e^{-(1/\varepsilon^2)((1/2)\int_0^T |u(s)|^2 ds + \beta_1)} \geqslant e^{-(1/\varepsilon^2)(\eta_1 + \beta_1)} \end{aligned}$$

In a similar way for $\varepsilon < \varepsilon_3$,

$$\begin{aligned} &\mathbb{P}(X^{x,\varepsilon}(t) \in E_\delta^c \text{ for some } t \leqslant T + T_1) \\ &\qquad \leqslant \mathbb{P}(\rho_{T+T_1}(X^{x,\varepsilon}, \Gamma^x_{T+T_1}(\eta_2)) \geqslant \delta_2) \leqslant e^{-(1/\varepsilon^2)(\eta_2 - \beta_2)} \end{aligned}$$

The first estimate follows from the fact that the set covered by all trajectories from $\Gamma^x_{T_1+T_2}(\eta_2)$ is contained in $\gamma^x(\eta_2) \subset \gamma^x(\tilde{\eta})$. Summing up all the obtained estimates one has for $\varepsilon < \varepsilon_1 \wedge \varepsilon_2 \wedge \varepsilon_2$.

$$\frac{\sup_{|x| = r_1} \mathbb{P}(B_T^\varepsilon)}{\text{int}_{|x| = r_1} \mathbb{P}(A_T^\varepsilon)} \leqslant \frac{2e^{-(1/\varepsilon^2)(\eta_2 - \beta_2)}}{e^{-(1/\varepsilon^2)(\eta_1 + \beta_1)}} \leqslant 2e^{-(1/\varepsilon^2)(\eta_2 - (\eta_1 + \beta_1 + \beta_2))} \to 0 \text{ as } \varepsilon \downarrow 0.$$

However, again by an easy application of the strong Markov property for all $x \in D$ and $\varepsilon > 0$

$$\mathbb{P}(B^{x,\varepsilon}) \leqslant \frac{\sup_{|x|=r_1} \mathbb{P}(B_T^{x,\varepsilon})}{\inf_{|x|=r_1} \mathbb{P}(A_T^{x,\varepsilon})} \mathbb{P}(A^{x,\varepsilon})$$

Consequently $\mathbb{P}(A^{x,\varepsilon}) \to 1$ as $\varepsilon \downarrow 0$ for all x, $|x| = r_1$. The case of all $x \in D_0$ one gets in the same way as at the end of the proof of Theorem 1, (ii).

6. Comments and Open Questions

In this final section we make some comments on the imposed conditions and obtained results and we pose some questions.

6.1. *Hypotheses* (H1)–(H3)

Hypothesis (H2) is natural as our main concern was to study behaviour of perturbed systems in a neighbourhood of a stable equilibrium. Condition (H3) is also natural. If all controlled trajectories $y^{0,u}(\cdot)$ are contained in D then, it follows from support theorems of Stroock–Varadhan, that stochastic trajectories $X^{0,\varepsilon}(\cdot)$, $x \in D$, are also contained in $\bar{D}$ and the exit problem looses its meaning. Condition (H1)-being a kind of smoothness property of the metric Δ—is of a more special character. First it is not at all obvious that (H3) does not follow from the fact that f and σ are of class C^1. Proposition 8 below shows that it is not the case. Moreover the same proposition implies that Theorem 1 and Theorem 2 hold true, in some special cases, without (H1). The following question is therefore of some interest.

Question 1. Are Theorem 1 and Theorem 2 true if f and σ are of class C^1 and only the conditions (H2) and (H3) are satisfied?

Proposition 8 concerns a two dimensional system

$$\begin{aligned} dX_1 &= -X_1\, dt + \varepsilon\, dW_t^1 \\ dX_2 &= -\operatorname{sgn}(X_2)\, X_2^2\, dt + \varepsilon X_2\, dW_t^2 \end{aligned} \tag{10}$$

and the reference set $D = \{\binom{x_1}{x_2} \in R^2;\ -a < x_1 < a,\ -b < x_2 < b\}$ where a and b are arbitrary positive numbers.

Proposition 8. *For system* (10) *conclusions of Theorem* 1 *and Theorem* 2 *hold true, hypotheses* (H2) *and* (H3) *are satisfied but hypothesis* (H1) *is violated.*

Proof. Let

$$\tau_1^{x_1,\varepsilon} = \inf\{t \geqslant 0; |X_1^{x,\varepsilon}(t)| \geqslant a\},$$
$$\tau_2^{x_2,\varepsilon} = \inf\{t \geqslant 0; |X_2^{x,\varepsilon}(t)| \geqslant b\}$$

where $x = \binom{x_1}{x_2} \in D$. Let $y_1^{x_1,u_1}$ and $y_2^{x_2,u_2}$ be solutions of the associated controlled systems:

$$dy_1 = -y_1\, dt + u_1\, dt, \qquad y_1(0) = x_1 \tag{11}$$

$$dy_2 = -\operatorname{sgn}(y_2)\, y_2^2\, dt + u_2\, y_2\, dt, \qquad y_2(0) = x_2, \tag{12}$$

and let $\gamma_1^{x_1}(\eta)$, $\gamma_2^{x_2}(\eta)$ be the quasi balls corresponding to the one dimensional systems (11) and (12) respectively. It is easy to calculate, see, e.g., [12], that

$$\gamma_1^0(\eta) = [-\sqrt{\eta}, \sqrt{\eta}] \qquad \text{and} \qquad \gamma_2^0(\eta) = \{0\}.$$

Thus

$$\underline{\eta} = \bar{\eta} = a^2$$

and

$$E = \left\{\begin{pmatrix} -a \\ 0 \end{pmatrix}, \begin{pmatrix} a \\ 0 \end{pmatrix}\right\}.$$

Note that conclusions of Theorem 1 and Theorem 2 are certainly true for initial conditions $\binom{x_1}{0} \in D$ because system (11) satisfies all the requirements. Let now $x = \binom{x_1}{x_2} \in D$ and $x_2 \neq 0$, say $x_2 > 0$ and let $\delta > 0$ be a fixed number. Without any loss of generality we can assume that $x_2 < \delta$. From standard formulae the probability $q(x_2, \varepsilon)$ of the event $A(x_2, \varepsilon)$ that trajectory $X_1^{x,\varepsilon}$ will never hit $\{\delta\}$ is

$$(e^{(2/\varepsilon^2)\delta} - e^{(2/\varepsilon^2)x_2})/(e^{(2/\varepsilon^2)\delta} - 1)$$

and does converge to 1 as $\varepsilon \downarrow 0$. But

$$\mathbb{E}(\tau^{x,\varepsilon}) = \mathbb{E}(\tau_1^{x_1,\varepsilon} \wedge \tau_2^{x_2,\varepsilon}) \leqslant \mathbb{E}(\tau_1^{x_1,\varepsilon}),$$
$$\mathbb{E}(\tau^{x,\varepsilon}) \geqslant \mathbb{E}(\tau_1^{x_1,\varepsilon}; A(x_2, \varepsilon)) \geqslant \mathbb{E}(\tau_1^{x_1,\varepsilon})\, q(x_2, \varepsilon).$$

Thus

$$\lim_{\varepsilon \downarrow 0} \varepsilon^2 \ln \mathbb{E}(\tau^{x,\varepsilon}) = \lim_{\varepsilon \downarrow 0} \dot{\varepsilon}^2 \ln \mathbb{E}(\tau_1^{x_1,\varepsilon}) = a^2,$$

and since $\mathbb{P}(\rho(X^{x,\varepsilon}, E) < \delta) \geqslant q(x_2, \varepsilon)$, therefore

$$\lim_{\varepsilon \downarrow 0} \mathbb{P}(\rho(X^{x,\varepsilon}, E) < \delta) = 1.$$

To see that (H1) does not hold let us assume that $x = \binom{0}{x_2}$, $x_2 > 0$ and consider feedback law $u_2 = 2y_2$ for the system (12). Direct computations lead to the conclusion that

$$\gamma_2^{x_2}(\eta) \supset [0, \tfrac{1}{2}x_2 + \eta]$$

Therefore $\gamma^0(\eta) = \{\binom{z_1}{z_2};\ -\sqrt{\eta} \leqslant z_1 \leqslant \sqrt{\eta},\ z_2 = 0\}$,

$$\gamma^x(\eta) = \left\{\begin{pmatrix} z_1 \\ z_2 \end{pmatrix};\ -\sqrt{\eta} \leqslant z_1 \leqslant \sqrt{\eta}, 0 \leqslant z_2 \leqslant \frac{1}{2}(\eta - z_1^2) + x_2\right\}$$

and we see that condition (H1) can not be satisfied for system (11)–(12).

6.2

In general $\underline{\eta} \neq \bar{\eta}$. To see this it is enough to consider linear system $\dot{y} = -y + u$ in R^n, $n \geqslant 2$ and $D = B(0, 1) \setminus \{xR^n;\ \frac{1}{2} \leqslant x_1 \leqslant 1\}$. Since $\gamma^0(\eta) = B(0, \sqrt{\eta})$, therefore $\underline{\eta} = \frac{1}{4}$ and $\bar{\eta} = 1$.

6.3

Finally we want to pose some questions related to the size of E.

It was shown in the paper [6] that for nth order systems $n = 2, 3$

$$z^{(n)} + a_1 z^{(n-1)} + \cdots + a_n z = \varepsilon \dot{W} \tag{13}$$

the exist set relative to the unit ball $B(0, 1)$ was, with only one exception, a two point set. The situation in the case of $n \geqslant 4$ is not clear.

Question 2. Is it so that generically all n-dimensional systems (13) have two point exit sets?

Question 3. Do generically all systems (1) have exit sets composed of finite number of points?

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