The Crash-NIG copula model

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Abstract

We calibrated the three-state version of the model to the history of iTraxx tranches and showed that the fitting ability of the model is much better than that of a corresponding one-factor LHC NIG model. We also introduced liquidity premiums into the Crash-NIG copula model and demonstrated that the actual credit crisis is substantially driven by liquidity effects.

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1. Motivation

The aim of this article is the development of a credit portfolio model that allows realistic risk analysis of Collateralized Debt Obligations. CDOs arose from the securitization tool that was widely used by banks already years before the financial crisis of 2007. The originators of the mortgages/credits could not keep all of them themselves and worked with the originate-to-distribute model selling the credits as structured credit products to investors. ABS and CDOs became very popular alternative investment products because these products promised considerably higher returns than corporate bonds with the same credit rating. However, the recent credit crisis showed that the risks of CDOs were not correctly taken into account by the ratings. Many investors were not able to analyze and understand these complex products by themselves and relied on the rating agencies.

The popular model used by practitioners and rating agencies was introduced already in 1987 by Vasicek. This model (see Vasicek [1], [2], [3]) is the asymptotic version of the one factor Gaussian model of correlated defaults.

Definition 1.1 (One factor Gaussian copula). Consider a portfolio of m credit instruments. The standardized asset return up to time $t$ of the $i$-th issuer in the portfolio, $A_i(t)$, is assumed to be of the form:

$$A_i(t) = a_i M(t) + \sqrt{1 - a_i^2} X_i(t),$$

(1.1)

where $M(t)$ and $X_i(t), i = 1,\ldots,m$ are independent standard normally distributed random variables. Under this copula model the variable $A_i(t)$ is mapped to default time $\tau_i$ of the $i$-th issuer using a percentile-to-percentile transformation, i.e. the issuer $i$ defaults before time $t$ when

$$\Phi(A_i(t)) \leq Q_i(t),$$

(1.2)

or equivalently

$$A_i(t) \leq \Phi^{-1}(Q_i(t)) =: C_i(t),$$

(1.3)

where $Q_i(t)$ denotes the probability of the issuer $i$ to default before time $t$

$$Q_i(t) = Q[\tau_i \leq t].$$

The risk-neutral probabilities are implied from the observable market prices of credit default instruments (e.g. bonds or CDS).

Vasicek derived the credit portfolio loss distribution applying the large homogeneous portfolio assumption to the one factor Gaussian copula model:

$$F_\infty(t,x) = \Phi \left( \frac{\sqrt{1 - a^2} \Phi^{-1}(x) - C(t)}{a} \right),$$

(1.4)

with $x \in [0,1]$ the percentage portfolio loss.

Due to its simplicity, the Gaussian copula model and Vasicek’s analytical approximation of it was immediately accepted and employed by practitioners. More sophisticated research on credit portfolio modeling started 2004 when tranched credit indices iTraxx and CDX started trading. Before this, research was not really possible due to a lack of appropriate data for testing the models. Then it was very fast clear
that the Vasicek model is too simple to describe the reality and it is impossible to fit the quotes of different tranches with the same correlation parameter. Practitioners started using the model in the way the Black-Scholes model is used for equity options and so the notion of a correlation smile or skew arose.

The correlation smile gave an indication of not correct modeling of the systematic risks and further research (Andersen et al. [4], Laurent and Gregory [5], Andersen and Sidenius [6], O’Kane and Schloegl [7], Hull and White [8], Hull et al. [9], Trinh et al. [10]) concentrated on the effort of improving the Gaussian copula model by using different distributions or introducing more stochastic factors. However, the aim of the research was still to keep the model simple and fast in computation.

Motivation of our research is a development of a credit portfolio model that is powerful enough to describe the reality and which is able to generate reasonable scenarios but also numerically fast in pricing CDO tranches.

In our opinion, the Normal Inverse Gaussian (NIG) distribution is an appropriate distribution for our objective. The family of NIG distributions is a special case of the generalized hyperbolic distributions (see Barndorff-Nielson [11]). In the basic version of the NIG factor-copula model, the Gaussian distribution is exchanged with the heavy-tailed NIG distribution allowing more joint defaults.

Definition 1.2 (One factor NIG copula model; see Kalemanova et al. [12]) Consider a homogeneous portfolio of \( m \) credit instruments. The standardized asset return up to time \( t \) of the \( i \)-th issuer in the portfolio, \( A_i(t) \), is assumed to be of the form:

\[
A_i(t) = aM(t) + \sqrt{1-a^2} X_i(t),
\]

with independent random variables

\[
M(t) \sim \text{NIG}\left(\alpha, \beta, -\frac{\beta y^2}{\alpha^2}, \frac{\gamma^3}{\alpha^2}\right)
\]

\[
X_i(t) \sim \text{NIG}\left(\frac{\sqrt{1-a^2}}{a}, \frac{\sqrt{1-a^2}}{a}, \frac{\sqrt{1-a^2}}{a}, \frac{\sqrt{1-a^2}}{a}, \frac{\gamma^3}{\alpha^2}\right),
\]

where \( \gamma = \sqrt{\alpha^2 - \beta^2} \). Then, the distribution of the asset return is:

\[
A_i \sim \text{NIG}\left(\frac{\alpha}{a}, \frac{\beta}{a}, -\frac{1}{a}, \frac{1}{a}, \frac{\gamma^3}{\alpha^2}\right).
\]

Further extensions, presented in this article, make the model even more flexible to describe the time dimension, heterogeneity of the credit portfolio and changing market regimes while the model is still consistent, arbitrage-free and analytically tractable.

2. Term structure extension

The term-structure dimension is as important as the dimension of the attachment points of the different CDO tranches. The Vasicek model does not incorporate the term-structure dimension. It only averages the correlations and other model parameters over the complete lifetime of the tranche. Thus, applying the model to the long-dated tranches is not consistent with the short-dated ones. The practitioners tried to fix
the problem of the Vasicek model with the term-structure dimension by extending the method of base correlations. Opposite to the Vasicek model, the NIG factor copula model gives a possibility for an extension into the term-structure dimension. This extension is not only helpful for the pricing of CDO tranches with different maturities, but also important for defining a consistent simulation framework. So, the model factors can be defined as stochastic processes and discretized in an arbitrary frequency for a simulation.

3. Large Homogeneous Cell extension

The models which we considered before, attempt to describe all tranches and maturities of a CDO with only one correlation parameter assuming that the portfolio is homogeneous. Already for one point in time, this assumption is quite strong. For the iTraxx tranches, there are at least 15 market quotes on one trading day, and it is very ambitious to argue that they all can be explained by only one parameter in the case of the Vasicek model or by two parameters in the case of the NIG copula model. We apply the extension of Large Homogeneous Cells, introduced by Desclee et al. [13] for the Vasicek model, to the NIG copula model. This extension allows more heterogeneity for the reference portfolio by considering, e.g., several different rating cells. However, it is still faster than a model allowing for a completely heterogeneous portfolio. This extension is going to be especially useful when modeling the dynamics of the credit spread of the underlying portfolio or, equivalently, of the default probability over time. The reason for that is that the quality of a portfolio, for example, of the iTraxx portfolio, depends not only on the usual credit spread fluctuations, but also on the changes in the rating composition in the real iTraxx portfolio. It would be difficult to model this with only one stochastic process representing the “average” portfolio spread. The LHC extension allows to take the rating migrations into account in a simulation framework while having a consistent and more flexible pricing model.

4. Crash-NIG model: regime-switching extension

Finally, we present a regime-switching model extension of the NIG model, called Crash-NIG copula model. This model allows for several correlation regimes and is especially highly topical in view of the recent sub-prime crisis. Although the extension is less important for a stand-alone pricing application, since for pricing CDO tranches on a particular day the parameters can be simply updated, this extension represents a very important feature for simulation purposes and risk management. So, a possibility of an economic crash can be taken into account in a simulation framework.

The main result of this article is the Crash-NIG model. The big advantage of this model is that it can be used for both, simulation and pricing: the Crash-NIG copula model is powerful enough to generate realistic simulation paths and at the same time admits a semi-analytical solution for the fair value computation.

For a scenario simulation framework, it is very important to have a possibility to generate different market regimes that are characterized by different levels of correlation. Such scenarios take into account the possibility of market turbulences with the correlation shifts as could be observed during the financial crisis.

5. Crash-NIG copula model for credit portfolios

For simplicity, we consider two regimes: the first regime represents a normal correlation and the second a high (crash) correlation state. The model is derived to satisfy some requirements that are important for the simulation framework:
A1 The distributions of the factors in both states have zero mean.
A2 The distributions of both factors in different states are stable under convolution.
A3 The distribution of the market factor does not depend on the correlation.
A4 The (standardized) return has the same distribution in both states to ensure an easy derivation of the default thresholds.

Then it can be shown that the model is defined in the following way:

**Definition 5.1 (Crash-NIG copula model).** The asset return of the \( i \)-th issuer in rating cell \( j \) for \( j = 1, \ldots, J \), \( A_{ij}(t) \), is assumed to be of the form:

\[
dA_{ij}(t) = a_j dM(t) + \sqrt{1-a_j^2} dX_{ij}(t),
\]

where \( M(t), X_{ij}(t), i = 1, \ldots, m \) are independent processes with the following distributions:

\[
dM(t) \sim NIG \left( \alpha, \beta, -\Lambda_i, \frac{\beta y^2}{\alpha^2} dt, \Lambda_i, \frac{\gamma^3}{\alpha^2} dt \right),
\]

\[
dX_{ij}(t) \sim NIG \left( \sqrt{1-a_j^2} \frac{\alpha}{a_j}, \sqrt{1-a_j^2} \frac{\beta}{a_j}, \frac{1-\Lambda_i^2}{1-a_j^2} \frac{\alpha^2}{a_j^2} dt, \frac{1-\Lambda_i^2}{1-a_j^2} \frac{\gamma^3}{a_j^2} dt \right).
\]

\( \Lambda_i \) is a Markov process with state space \( \{1, \lambda\} \), an initial distribution \( \pi = \{\pi_1, \pi_2\} \) and a \((2 \times 2)\) transition function \( \{P(h)\}_{h \geq 0} \). The distribution of the asset return is

\[
dA_{ij}(t) \sim NIG \left( \frac{1}{a_j}, \frac{1}{a_j}, -\frac{1}{a_j}, \frac{\beta y^2}{a_j^2} dt, \frac{1}{a_j}, \frac{\gamma^3}{a_j^2} dt \right).
\]

The Crash-NIG copula model has the following properties:

- In the first correlation regime, the variance of all factor changes is \( dt \). The variance of the factors in the second regime is given by

\[
V_{d\hat{M}} = \dot{\lambda}^2 dt, \quad V_{d\hat{x}_j} = \frac{1-\dot{\lambda}^2 a_j^2}{1-a_j^2} dt.
\]

Furthermore, the correlation of asset returns of an issuer \( i_1 \) from rating cell \( j_1 \) and an issuer \( i_2 \) from the rating cell \( j_2 \) is

\[
Corr(dA_{i_1j_1}(t), dA_{i_2j_2}(t)) = \frac{a_{i_1} a_{i_2} V_{d\hat{M}}}{\sqrt{V_{dA_{i_1j_1}} V_{dA_{i_2j_2}}}} = a_{i_1} a_{i_2} \dot{\lambda}^2.
\]
A higher correlation in the second regime comes along with a higher variance of the market factor compared to the normal correlation regime, i.e. by choosing $\lambda > 1$. The variance of the idiosyncratic factor is then lower than in the normal regime. This is exactly the behavior observed in crisis regimes: the complete market becomes more volatile and correlated at the same time and the idiosyncratic developments play a comparably small role.

- The Crash-NIG copula model is defined in such way that it can be discretized in an arbitrary way and the distributions of the increments of all factors are stable under convolution. The parameters of the model are the same for any time horizon since the time component is taken into account by $dt$.

- The Crash-NIG copula model can be easily extended to a higher number of regimes. Then, the Markov process $\Lambda_t$ has the state space $\{1, \Lambda_1, \ldots, \Lambda_{n-1}\}$, an initial distribution $\pi = \{\pi_1, \pi_2, \ldots, \pi_n\}$ and a $(n \times n)$ transition function $\{P(h)\}_{h \geq 0}$.

6. Application to pricing of synthetic CDOs

Lemma 6.1 The loss distribution of an infinitely large homogenous cell portfolio with the asset returns following a Crash-NIG copula model with two states is given by

$$F_x^{LHC}(t, x) = 1 - F_{\text{NIG}} \left( l_{i-1}^{-1}(x); \alpha, \beta, (h_1^\prime(t) + \lambda^2 h_2^\prime(t)) \frac{\beta \gamma^2}{\alpha^2}, (h_1^\prime(t) + \lambda^2 h_2^\prime(t)) \frac{\gamma^3}{\alpha^2} \right),$$

(6.1)

with $x \in [0,1]$ denoting the percentage portfolio loss. Here,

$$h_i^\prime(t) = \int_0^t (\exp(sO))_{i,j} ds$$

(6.2)

for $i = 1,2$, where $O := \lim_{h \to 0} \frac{P(h) - I}{h}$ is the intensity matrix of the transition function $P(h)$. The function $l_i(M(t))$ is the portfolio loss conditional on the realization of the systematic factor $M(t)$ and is given by:

$$l_i(M(t)) = \sum_{j=1}^d (1-R) \omega_j F_{\text{NIG}} \left( \frac{C_j(t) - a_j M(t)}{\sqrt{1-a_j^2}}, \frac{1-a_j^2}{a_j}, \frac{1-a_j^2}{\alpha^2}, \frac{1-a_j^2}{\alpha^2}, \frac{1-a_j^2}{\alpha^2}, \frac{1-a_j^2}{\alpha^2} \right),$$

(6.3)

where $R$ is a constant recovery and $\omega_j$, such that $\sum_{j=1}^d \omega_j = 1$, are the weights of the rating cells. The default thresholds are computed as

$$C_j(t) = F_{\text{NIG}}^{-1} \left( Q_j(t); \frac{1}{a_j} \alpha, \frac{1}{a_j} \beta, \frac{1}{a_j} \frac{\beta \gamma^2}{\alpha^2}, \frac{1}{a_j} \frac{\gamma^3}{\alpha^2} \right),$$
where $Q_j(t)$ is the default probability in the cell $j$ up to time $t$.

Again, it is straightforward to change the formulas for the model with more than two states by changing the expression $h_1'(t) + \lambda^2 h_2'(t)$ accordingly.

There exist no analytical expressions for the expected tranche losses. They have to be computed numerically by approximating the corresponding integrals over the portfolio loss distribution function. However, the computations are very fast. Our implementation takes us only 3 seconds to price the iTraxx tranches on one day, and 25 seconds for a simultaneous pricing on 200 days (on a computer with Intel Core Duo with 2.2GHz Processor).

We calibrated the three-state version of the model to the history of iTraxx tranches and showed that the fitting ability of the model is much better than that of a corresponding one-factor LHC NIG model. We also introduced liquidity premiums into the Crash-NIG copula model and demonstrated that the actual credit crisis is substantially driven by liquidity effects.

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