Abstract

If $X$ is a completely regular Hausdorff topological space we show that: (i) if $H$ is a compact subset of $C_p(X)$ and $X$ contains a quasi-Souslin subspace which separates the functions of $H$, then $H$ is Talagrand compact, and (ii) $C_p(X)$ is quasi-Souslin if and only if it is $K$-analytic.

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1. Notation and terminology

All topological spaces considered are supposed to be Hausdorff. We represent by $C(X)$ (by $C_p(X)$) the linear space of all real-valued continuous functions on a topological space $X$ (equipped with the pointwise convergence topology). We denote by $\mathbb{N}$ the set of positive integers endowed with the discrete topology and equip $\mathbb{N}^\mathbb{N}$ with the product topology. If $\alpha, \beta \in \mathbb{N}^\mathbb{N}$ we write $\alpha \leq \beta$ if $\alpha(i) \leq \beta(i)$ for each $i \in \mathbb{N}$. If $X$ is a topological space for which there is a family of compact sets $\{A_\alpha: \alpha \in \mathbb{N}^\mathbb{N}\}$ covering $X$ such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$, we say that $X$ is dominated by the irrationals [6]. It is well known that if $X$ is compact, then $C_p(X)$ is $K$-analytic iff it is dominated by the irrationals [7, Proposition 6.13]. This result has recently been extended in [6, Theorem 2.8] by showing that the previous statement holds for every completely regular space $X$.

A completely regular space $X$ is called $K$-analytic (respectively quasi-Souslin) [8] if there is a map $T$ from $\mathbb{N}^\mathbb{N}$ into the family of all compact sets in $X$ (respectively of all subsets of $X$) such that (i) $\{T(\alpha): \alpha \in \mathbb{N}^\mathbb{N}\}$ covers $X$ and (ii) if $\alpha_n \to \alpha$ in $\mathbb{N}^\mathbb{N}$ and $x_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$, then $\{x_n\}$ has a cluster point $x \in T(\alpha)$. If $\alpha_n \to \alpha$ and $x_n \in T(\alpha_n)$ imply that $\{x_n\}$ has a subsequence converging to a point of $T(\alpha)$, we shall say that $X$ is a quasi-Souslin space of sequential type. A compact set $K$ is called Talagrand compact if $C_p(K)$ is $K$-analytic (alternatively, iff it is homeomorphic to a compact subset of a space $C_p(X)$ with $X$ being $K$-analytic). If $X$ is a topological space and $\mathcal{E}$ and $\mathcal{A}$ are two families of subsets of $X$, $\mathcal{E}$ is called a network modulo $\mathcal{A}$ if for each set $A \in \mathcal{A}$ and each open neighborhood $U$ of $A$ in $X$ there is some $E \in \mathcal{E}$ with $A \subseteq E \subseteq U$. The class of Lindelöf $\Sigma$-spaces is the smallest class of topological

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spaces containing all compact spaces, all second countable spaces and being closed under finite (in fact countable) products, closed subspaces and continuous images. Each $K$-analytic space is a Lindelöf $\Sigma$-space and each Lindelöf $\Sigma$-space is Lindelöf.

2. Introduction

In [4] is shown that the weak* dual $(E', \sigma(E', E))$ of a locally convex space $E$ of the class $\mathfrak{G}$ of Cascales–Orihuela [3] is always quasi-Souslin and highlighted the fact that some properties enjoyed by the spaces of the class $\mathfrak{G}$ are also shared by the locally convex spaces with quasi-Souslin weak* dual. Examples of locally convex spaces in class $\mathfrak{G}$ with non-$K$-analytic weak* dual, as well as of locally convex spaces $E$ not in class $\mathfrak{G}$ whose weak* dual is quasi-Souslin but not $K$-analytic, are exhibited in [4]. Our next example, now within $C_p$-theory, is of this last kind with $E = C_p(X)$ and is not included in [4]. We shall represent by $L_p(X)$ the weak* dual of $C_p(X)$.

Example 1. If $X$ is a quasi-Souslin space of sequential type which is not Lindelöf, then $L_p(X)$ is quasi-Souslin but not $K$-analytic.

Proof. Since the class of quasi-Souslin spaces of sequential type is stable under transition to continuous images, countable unions and finite (even countable) products [8, I.4.2]; and, in addition, $\mathbb{R}$ is a quasi-Souslin space of sequential type, the results of [1, Proposition 0.5.13] show that $L_p(X)$ is a quasi-Souslin space. Given that $X$ is not Lindelöf, the space $L_p(X)$ is not Lindelöf either because $X$ embeds in $L_p(X)$ as a closed subspace; in particular, $L_p(X)$ is not $K$-analytic. \hfill \Box

Corollary 2. If $X$ is a non-compact sequentially compact space, then $L_p(X)$ is quasi-Souslin but not $K$-analytic.

In this paper we get some new results on $C_p$-theory related to quasi-Souslin spaces. In particular we show (i) if $H$ is a compact subset of $C_p(X)$ and $X$ contains a quasi-Souslin subspace which separates the functions of $H$, then $H$ is Talagrand compact, and (ii) the fact that $C_p(X)$ is quasi-Souslin is equivalent to the $K$-analyticity of $C_p(X)$.

3. On Talagrand compact sets

We shall say that a topological space $X$ is dominated by a family of relatively countably compact subsets if there is a family of relatively countably compact subsets $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of $X$, covering $X$, such that $A_\alpha \subseteq A_\beta$ whenever $\alpha \leq \beta$.

Theorem 3. Let $X$ be a completely regular space and let $H$ be a compact subset of $C_p(X)$. If $X$ contains a subspace $Y$ dominated by a family of relatively countably compact sets which separates the functions of $H$, then $H$ is Talagrand compact.

Proof. The mapping $\delta : X \to (C_p(X)', \sigma(C_p(X)' , C_p(X)))$ defined by $\delta(x) = \delta_x$, where $\delta_x$ denotes the evaluation map at $x \in X$, is a homeomorphism from $X$ onto its range in the weak* dual of $C_p(X)$. Let $\{A_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ be a family of relatively countably compact subsets of $X$ with $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$ whose union $Y$ separates the points of $H$. If we set $C_\alpha := \{\delta_x |_H : x \in A_\alpha\}$ for every $\alpha \in \mathbb{N}^\mathbb{N}$, then each $C_\alpha$ is a relatively countably compact subset of $C_p(H)$. Indeed, since $\delta_x$ is a continuous map on $C_p(X)$, its restriction $\delta_x |_H$ to $H$ belongs to $C(H)$, so that $C_\alpha \subseteq C(H)$. If $\{\delta_{x_n} |_H : n \in \mathbb{N}\}$ is a sequence in $C_\alpha$, due to the fact that $\{x_n : n \in \mathbb{N}\} \subseteq A_\alpha$ there is a subnet $\{z_d : d \in D\}$ of $\{x_n : n \in \mathbb{N}\}$ which converges to some $z \in X$. Thus $f(z_d) \to f(z)$ for each $f \in C(X)$ and, particularly, for each $f \in H$, which means that $\delta_{z_d} |_H \to \delta_{z} |_H$ under the pointwise convergence topology of $C(H)$. But since $H$ is compact, $C_p(H)$ is angelic. So $C_\alpha$ is a relatively compact subset of $C_p(H)$. Setting $K_\alpha := \{C_{\alpha} \cap K_p(H)\}$ for each $\alpha \in \mathbb{N}^\mathbb{N}$, then $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ is a family of compact sets in $C_p(H)$ such that $K_\alpha \subseteq K_\beta$ if $\alpha \leq \beta$. Since $Y$ separates the functions of $H$ and $\{\delta_x |_Y : x \in Y\} \subseteq \bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} C_\alpha \subseteq \bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} K_\alpha$, then the subset $\bigcup_{\alpha \in \mathbb{N}^\mathbb{N}} K_\alpha$ of $C(H)$ separates the points of $H$ showing that $H$ is Talagrand compact [7, Proposition 6.13]. \hfill \Box

Corollary 4. Let $X$ be a completely regular space and let $H$ be a compact subset of $C_p(X)$. If $X$ contains a quasi-Souslin subspace which separates the functions of $H$, then $H$ is Talagrand compact.
Proof. Let $Y$ be a quasi-Souslin space which separates the points of $H$. It suffices to show that $Y$ is dominated by a family of relatively countably compact sets.

Let $T$ be a map from the Polish space $\mathbb{N}^\mathbb{N}$ into the family of all subsets of $Y$ such that $\{T(\alpha): \alpha \in \mathbb{N}^\mathbb{N}\}$ covers $Y$ and if $\alpha_n \to \alpha$ in $\mathbb{N}^\mathbb{N}$ and $x_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$, then the sequence $\{x_n\}$ has a cluster point $x \in T(\alpha)$. For each $\alpha \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$ put

$$T(\alpha | n) = \bigcup \{T(\beta): \beta \in \mathbb{N}^\mathbb{N}, \beta(i) \leq \alpha(i), 1 \leq i \leq n\}$$

and $A_\alpha = \bigcap_{n=1}^{\infty} T(\alpha | n)$. Then the family $\mathcal{A} = \{A_\alpha: \alpha \in \mathbb{N}^\mathbb{N}\}$ covers $Y$ and $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$. If $\{x_n\} \subseteq A_\alpha$ then $x_n \in T(\alpha | n)$ for each $n \in \mathbb{N}$, so that there is $\beta_n \in \mathbb{N}^\mathbb{N}$ with $\beta_n(i) \leq \alpha(i)$ for $1 \leq i \leq n$ such that $x_n \in T(\beta_n)$. Let $\delta(i) = \max\{\beta_n(i): n \in \mathbb{N}\}$ for each $i \in \mathbb{N}$; then $\delta \in \mathbb{N}^\mathbb{N}$ and $\beta_n \leq \delta$ for every $n \in \mathbb{N}$, so that $\{\beta_n: n \in \mathbb{N}\}$ is a bounded subset of $\mathbb{N}^\mathbb{N}$. Thus there is a subsequence $\{\beta_{n_k}\}$ of $\{\beta_n\}$ which converges to some $\beta \in \mathbb{N}^\mathbb{N}$. Since $Y$ is quasi-Souslin, the sequence $\{x_{n_k}\}$, and hence the sequence $\{x_n\}$, has a cluster point $x \in T(\beta)$. Due to the fact that $\beta \leq \alpha$, we have $x \in T(\beta) \subseteq A_\beta \subseteq A_\alpha$, which shows that $A_\alpha$ is countably compact. 

Corollary 5. If $E$ is a Banach space, then $E$ is weakly quasi-Souslin if and only if it is weakly $K$-analytic.

Proof. If the Banach space $E$ with the weak topology $\sigma(E, E')$ is quasi-Souslin then, by the previous corollary, every compact set in $C_p(E, \sigma(E, E'))$ is Talagrand compact. In particular, the closed dual unit ball $B_{E'}(\text{weak}^*)$ equipped with the weak* topology is Talagrand compact. Since, as is well known, a Banach space $E$ is weakly $K$-analytic if and only if $B_{E'}(\text{weak}^*)$ is Talagrand compact, the corollary follows.

Corollary 6. Let $A$ be a compact subset of $X$. If $C_p(X)$ contains a subspace dominated by a family of relatively countably compact sets which separates the points of $A$, then $A$ is Talagrand compact.

Proof. If $C_p(X)$ contains a subspace dominated by a family of relatively countably compact sets which separates the points of $A$, by the previous theorem $\delta(A)$ is a Talagrand compact subset of $C_p(C_p(X))$. Since $\delta$ embeds $X$ into $C_p(C_p(X))$, the conclusion follows.

Example 7. The space $C_p([0, \omega_1])$ is not dominated by a family of relatively countably compact sets. Observe that $C_p([0, \omega_1])$ has uncountable extent while every space dominated by a family of relatively compact sets has countable extent.

4. On quasi-Souslin spaces $C_p(X)$

We have seen in Example 1 that there are quasi-Souslin spaces which are not $K$-analytic. On the other hand, if $X$ is compact then $C_p(X)$ is angelic and hence $C_p(X)$ is quasi-Souslin if it is $K$-analytic. Now we show that this statement is true for every completely regular space $X$. Although the proof of the next theorem is essentially contained in [6], some explanation is required in order to see how it adapts to the present setting. For the sake of completeness we summarize the main ideas, explain some specific points and refer the reader to [6] for other details. So, the following result extends [6, Theorem 2.8].

Theorem 8. If $X$ is a completely regular space, then the following are equivalent

1. $C_p(X)$ is a quasi-Souslin space.
2. $C_p(X)$ is $K$-analytic.

Proof. It suffices to prove that (1) $\Rightarrow$ (2). So let us suppose that $C_p(X)$ is a quasi-Souslin space and let $T$ be a map from the Polish space $\mathbb{N}^\mathbb{N}$ into the family of all subsets of $C_p(X)$ such that (i) $\{T(\alpha): \alpha \in \mathbb{N}^\mathbb{N}\}$ covers $C(X)$ and (ii) if a sequence $\{\alpha_n\}$ in $\mathbb{N}^\mathbb{N}$ converges to $\alpha$ and $f_n \in T(\alpha_n)$ for each $n \in \mathbb{N}$, the sequence $\{f_n\}$ has a cluster point $f$ in $C_p(X)$ contained in $T(\alpha)$. For each $\alpha \in \mathbb{N}^\mathbb{N}$ and $n \in \mathbb{N}$ put again

$$T(\alpha | n) = \bigcup \{T(\beta): \beta \in \mathbb{N}^\mathbb{N}, \beta(i) \leq \alpha(i), 1 \leq i \leq n\}$$
and $A_\alpha = \bigcap_{n=1}^{\infty} T(\alpha \mid n)$. Then we have seen before that (i) each $A_\alpha$ is countably compact, (ii) the family $\mathcal{A} = \{A_\alpha; \ \alpha \in \mathbb{N}^N\}$ covers $C(X)$ and (iii) $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$.

Let us see that the countable family $\mathcal{E} = \{T(\alpha \mid n); \ \alpha \in \mathbb{N}^N, \ n \in \mathbb{N}\}$ is a network in $C_p(X)$ modulo $\mathcal{A}$, i.e. that for each $A_\alpha \in \mathcal{A}$ and each open neighborhood $U_\alpha$ of $A_\alpha$ in $C_p(X)$ there is $n \in \mathbb{N}$ such that $A_\alpha \subseteq T(\alpha \mid n) \subseteq U_\alpha$. Indeed, if $T(\alpha \mid n) \nsubseteq U_\alpha$ for each $n \in \mathbb{N}$ there is a sequence $\{\beta_n\}$ in $\mathbb{N}^N$ with $\beta_n(i) \leq \alpha(i)$ for $1 \leq i \leq n$ and $n \in \mathbb{N}$, and a sequence $\{f_n\} \subseteq C_p(X) \setminus U_\alpha$, such that $f_n \in T(\beta_n)$ for every $n \in \mathbb{N}$. Setting $\delta(i) = \max\{\beta_n(i); \ n \in \mathbb{N}\}$ then $\beta_n \leq \delta$ for every $n \in \mathbb{N}$, so that $\{\beta_n\}$ is bounded in $\mathbb{N}^N$. Hence there is a subsequence $\{\beta_n\}$ of $\{\beta_n\}$ which converges to some $\beta \in \mathbb{N}^N$. Since $C_p(X)$ is quasi-Souslin the sequence $\{f_n\}$ and, consequently, the sequence $\{f_n\}$, has a cluster point $f \in T(\beta)$. Due to the fact that $\beta \leq \alpha$, then $f \in T(\beta) \subseteq A_\beta \subseteq A_\alpha \subseteq U_\alpha$, but since $\{f_n\}$ was contained in the closed set $C_p(X) \setminus U_\alpha$, necessarily $f \notin U_\alpha$, a contradiction.

It can be easily seen (this is mentioned for instance in [6, Proposition 2.7]) that Uspenskiï’s result [1, Proposition IV.9.3] is also valid for our family $\mathcal{A}$ of countably compact sets. So the fact that $\mathcal{E}$ is a countable network in $C_p(X)$ modulo $\mathcal{A}$ implies that there exists a Lindelöf $\Sigma$-space $S$ such that $C_p(C_p(X)) \subseteq S \subseteq \mathbb{R}^{C(X)}$. Since the realcompactification $\nu X$ of $X$ coincides with the closure in $\mathbb{R}^{C(X)}$ of the canonical copy of $X$ contained in $C_p(X)$, the fact that $S$ is realcompact guarantees, up to homeomorphisms, that $\nu X$ is a closed subset of the Lindelöf $\Sigma$-space $S$, so that $\nu X$ is a Lindelöf $\Sigma$-space as well.

Consider the restriction map $\Phi: C_p(\nu X) \to C_p(X)$ defined by $\Phi(f) = f \mid X$. Setting $B_\alpha = \Phi^{-1}(A_\alpha)$ and using a result of Okunev which asserts that $\Phi$ becomes a homeomorphism when restricted onto the countable subsets of $C_p(\nu X)$, we can see that $B_\alpha$ is a countably compact set in $C_p(\nu X)$. Indeed, if $\{\varphi_n\}$ is a sequence in $B_\alpha$, $\psi \in A_\alpha$ is a cluster point in $C_p(X)$ of the sequence $\{\varphi_n\}$ and $\psi^\nu$ denotes the Stone extension of $\psi$ restricted to $\nu X$, the fact that $\Phi$ is a homeomorphism when restricted onto the set $\{\psi^\nu, \varphi_n; \ n \in \mathbb{N}\}$ assures that $\psi^\nu$ is a cluster point in $C_p(\nu X)$ of the sequence $\{\varphi_n\}$. It follows from $\Phi(\psi^\nu) = \psi$ that $\psi^\nu \in B_\alpha$ and we are done.

Since $\nu X$ is a Lindelöf $\Sigma$-space, [1, Corollary III.6.2] shows that each set $B_\alpha$ is compact, so that $C_p(\nu X)$ is a quasi-Souslin space [2, Proposition 1]. Alternatively, since each Lindelöf $\Sigma$-space is $K$-countably determined, it is web-compact in the sense of Orihuela; hence $C_p(\nu X)$ is angelic [5] and consequently each $B_\alpha$ is compact, so that $C_p(\nu X)$ is quasi-Souslin. Applying again [1, Corollary III.6.2] to the definition of quasi-Souslin space, we conclude that $C_p(\nu X)$ is $K$-analytic. As $\Phi$ maps continuously $C_p(\nu X)$ onto $C_p(X)$, we can see that $C_p(X)$ is $K$-analytic.

5. Quasi-Souslin spaces of continuous and bounded functions

As we are going to see, Theorem 8 does not hold if we replace $C_p(X)$ by the subspace $C_p^b(X)$ of all continuous and bounded real-valued functions on $X$.

Let us consider the semiclosed ordinal interval $[0, \omega_1]$ provided with the discrete topology and consider as neighborhoods of $\omega_1$ all those subsets $U$ of $[0, \omega_1]$ which contain $\omega_1$ and $[0, \omega_1] \setminus U$ is countable. Let us denote by $Z$ the closed interval $[0, \omega_1]$ equipped with such topology (the so-called one-point Lindelöfication of a discrete space of cardinality $\aleph_1$). Then $Z$ is a Lindelöf $P$-space and [1, Example IV.2.15], the linear subspace $L = \{f \in C(Z); \ f(\omega_1) = 0\}$ of $C_p(Z)$ coincides with the subspace $E_0$ of $\mathbb{R}^{[0,\omega_1]}$ formed by all those elements of countable support, so that $C_p(Z)$, being homeomorphic to $L \times \mathbb{R}$, is homeomorphic to $E_0$.

Example 9. The space $C_p^b(Z)$ is quasi-Souslin but not $K$-analytic.

**Proof.** If $B$ denotes the closed unit ball of the Banach space $(C^b(Z), \|\|_\infty)$ endowed with the supremum norm, i.e. $B = \{f \in C^b(Z); \ |f|_\infty \leq 1\}$, then $B$ is a non-compact, closed and countably compact subset of $C_p^b(Z)$. This is because $B \cap E_0$ is a non-compact sequentially compact subset of $\mathbb{R}^{[0,\omega_1]}$. Due to the fact that $C^b(Z) = \bigcup_{n=1}^{\infty} nB$, if we set $A_\alpha = \alpha(1)B$ for each $\alpha \in \mathbb{N}^N$ then $\{A_\alpha; \ \alpha \in \mathbb{N}^N\}$ is a family of closed countably compact subsets in $C_p^b(Z)$ covering $C_p^b(Z)$ such that $A_\alpha \subseteq A_\beta$ if $\alpha \leq \beta$. So, according to [2, Proposition 1], $C_p^b(Z)$ is a quasi-Souslin space. However $C_p^b(Z)$ is not $K$-analytic, otherwise $B$ would be Lindelöf and hence compact, a contradiction.

**Problem 10.** We do not know if there exists a completely regular space $X$ such that $C_p(X)$ is a non-quasi-Souslin space dominated by a family of relatively countably compact sets.
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