Global Minimization of Constrained Problems with Discontinuous Penalty Functions

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Abstract—With the integral approach to global optimization, a class of discontinuous penalty functions is proposed to solve constrained minimization problems. Optimality conditions of a penalized minimization problem are generalized to a discontinuous case; necessary and sufficient conditions for an exact penalty function are examined; a nonsequential algorithm is proposed. Numerical examples are given to illustrate the effectiveness of the algorithm. © 1999 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Let \( X \) be a topological space, \( S \) a nonempty subset of \( X \), and \( f : X \rightarrow \mathbb{R} \) a real-valued function. Consider the following constrained optimization problem:

\[ c^* = \inf_{x \in S} f(x). \tag{1} \]

In general, minimizers of (1) may not exist. We will not examine particularly the existence problem of global minimizers here. Assume that (A): \( f \) is lower semicontinuous, \( S \) is inf-compact. Under (A) minimizers of (1) exist. Here, inf-compactness means that there is a real number \( b > c^* \) such that the level set

\[ H_b = \{ x : f(x) \leq b \} \]

is a nonempty compact set.

The problem of minimizing a function over a constrained set has been investigated since the 17th century with the concepts of derivative and Lagrangian multiplier. The gradient-based approach to optimization is the mainstream of that research. However, the requirement of differentiability restricts its application to many practical problems. Moreover, it can only be utilized to characterize and find a local solution of a general optimization problem. In this work, we will investigate a constrained minimization problem with discontinuous objective function by using the integral approach.
The penalty function method, representing a constrained minimization problem in terms of unconstrained ones, is one of the popular numerical methods of nonlinear programming because the idea is simple and quite universal. The penalty approach to constrained optimization is attributed to Courant [1], and was developed and popularized by Fiacco and McCormick [2] and others. In recent years, a considerable amount of investigation has been devoted to methods that attempt to solve a constrained problem by means of a single unconstrained minimization. It is termed exact penalty method [3–10].

A major disadvantage of the penalty approach is the choice of penalty parameters. The use of large values of the penalty coefficient leads to a minimization problem where the Hessian is ill-conditioned, if one uses a gradient-based method. Moreover, for an exact penalty function, a constraint qualification is required.

Taking advantage of the integral approach of global optimization, a class of discontinuous penalty functions is proposed in this work. Using the theory and algorithms of the integral global minimization, one can solve a constrained problem by unconstrained minimization technique without requirement of a constraint qualification.

In this paper, we first recall basic concepts of robust sets, functions, and the integral approach to global minimization (Section 2).

In Section 3, we consider general penalty functions which may be discontinuous. We derive conditions for a penalty function to be exact and propose several discontinuous exact penalty functions in Section 4. We study optimality conditions for the penalized problem with the integral approach in Section 5 and propose an algorithm for approximating solutions of constrained optimization problems in Section 6; these problems may have discontinuous objective function with disconnected constraint set. Numerical examples are given in Section 7 to illustrate the effectiveness of the algorithm.

2. ROBUST SETS AND FUNCTIONS.
INTEGRAL GLOBAL MINIMIZATION

In this section, we will summarize several concepts and properties of the integral global minimization of robust discontinuous functions, which will be utilized in the following sections. For more details, see [11,12].

2.1. Robust Sets and Functions

Let $X$ be a topological space, a subset $D$ of $X$ is said to be robust if

$$\text{cl} D = \text{cl int} D,$$

where $\text{cl} D$ denotes the closure of the set $D$ and $\text{int} D$ denotes the interior of $D$.

A robust set consists of robust points of the set. A point $x \in D$ is said to be a robust point of $D$, if for each neighborhood $N(x)$ of $x$, $N(x) \cap \text{int} D \neq \emptyset$. A set $D$ is robust if and only if each point of $D$ is a robust one. A point $x \in D$ is a robust point of $D$ if and only if there exists a net $\{x_{\lambda}\} \subseteq \text{int} D$ such that $x_{\lambda} \to x$.

The interior of a nonempty robust set is nonempty. A union of robust sets is robust. An intersection of two robust sets may be nonrobust; but the intersection of an open set and a robust set is robust. A set $D$ is robust if and only if $\partial D = \partial \text{int} D$, where $\partial D = \text{cl} D \setminus \text{int} D$ denotes the boundary of the set $D$.

A function $f : X \to R$ is said to be upper robust if the set

$$F_c = \{x : f(x) < c\}$$

is robust for each real number $c$. A sum or a product of two upper robust functions may be nonupper robust; but the sum of an upper robust function and an upper semicontinuous (u.s.c.,
for the product case (nonnegativity is required) function is upper robust. A function \( f \) is upper robust if and only if it is upper robust at each point; \( f \) is upper robust at a point \( x \) if \( x \in F_c \) implies \( x \) is a robust point of \( F_c \). An example of a nonupper robust function on \( R^1 \) is

\[
f(x) = \begin{cases} 
  x^2, & x \neq 0, \\
  -1, & x = 0.
\end{cases}
\]

\( f \) is not upper robust at \( x = 0 \).

Let \( S \) be a robust set in a topological space \((X, \tau)\), where \( \tau \) is the topology of \( X \). We can introduce a relative topology \( \tau_S \) and obtain a new topological space \((S, \tau_S)\). In this new topological space, we also have concepts of robust set and upper robust function with this relative topology. Then, we have concepts of relative robust set and relative upper robust function.

### 2.2. \( Q \)-Measure Spaces and Integration

In order to investigate a minimization problem with an integral approach, a special class of measure spaces, which are called \( Q \)-measure spaces, should be examined.

Let \( X \) be a topological space, \( \Omega \) a \( \sigma \)-field of subsets of \( X \), and \( \mu \) a measure on \( \Omega \). A triplet \((X, \Omega, \mu)\) is called a \( Q \)-measure space iff

1. each open set in \( X \) is measurable;
2. the measure \( \mu(G) \) of a nonempty open set \( G \) in \( X \) is positive: \( \mu(G) > 0 \);
3. the measure \( \mu(K) \) of a compact set \( K \) in \( X \) is finite.

The \( n \)-dimensional Lebesgue measure space \((R^n, \Omega, \mu)\) is a \( Q \)-measure space; a nondegenerate Gaussian measure \( \mu \) on a separable Hilbert space \( H \) with Borel sets as measurable sets constitutes an infinite dimensional \( Q \)-measure space. A specific optimization problem is related to a specific \( Q \)-measure space which is suitable for consideration in this approach. Once a measure space is given, we can define integration in a conventional way.

Since the interior of a nonempty open set is nonempty, the \( Q \)-measure of a measurable set containing a nonempty robust set is always positive. This is an essential property we need in the integral approach of minimization. Hence, the following assumptions are usually required.

1. \( f \) is lower semicontinuous (l.s.c.) and \( S \) is inf-compact.
2. \( f \) is upper robust on \( S \).
3. \((X, \Omega, \mu)\) is a \( Q \)-measure space.

### 2.3. Integral Optimality Conditions for Global Minimization

We now proceed to define the concepts of mean value and modified variance of \( f \) over its level set. These concepts are closely related to optimality conditions and algorithms for global minimization.

Suppose that Assumptions (A), (M), and (R) hold, and \( c > c^* = \min_{x \in S} f(x) \). We define the mean value and modified variance, respectively, as follows:

\[
M(f, c; S) = \frac{1}{\mu(H_c \cap S)} \int_{H_c \cap S} f(x) \, d\mu,
\]

\[
V_1(f, c; S) = \frac{1}{\mu(H_c \cap S)} \int_{H_c \cap S} (f(x) - c)^2 \, d\mu.
\]

They are well defined. These definitions can be extended to the case \( c \geq c^* \) by a limit process. For instance,

\[
V_1(f, c; S) = \lim_{c_k \to c^*} \frac{1}{\mu(H_{c_k} \cap S)} \int_{H_{c_k} \cap S} (f(x) - c_k)^2 \, d\mu.
\]

The limits exist and are independent of the choice of \( \{c_k\} \). The extended concepts are well defined and consistent with the above definitions.

With these concepts, we characterize the global optimality as follows.
THEOREM 2.1. Under Assumptions (A), (M), and (R), the following statements are equivalent:

1. \( x^* \in S \) is a global minimizer of \( f \) over \( S \) and \( c^* = f(x^*) \) is the global minimum value,
2. \( M(f, c^*; S) = c^* \) (the mean value condition),
3. \( V_1(f, c^*; S) = 0 \) (the modified variance condition).

2.4. An Integral Algorithm

An integral global minimization algorithm for finding the global minimum value and the set of global minimizers of an upper robust function over a robust constraint set is given as follows [13].

STEP 1. Take \( c_0 > c^* \) and \( \epsilon > 0 \), \( k := 0 \).

STEP 2. \( c_{k+1} := M(f, c_k; S) \), \( v_{k+1} = V_1(f, c_k; S) \), \( H_{c_{k+1}} \cap S := \{ x \in S : f(x) \leq c_{k+1} \} \).

STEP 3. If \( v_{k+1} \geq \epsilon \), then \( k := k + 1 \), go to Step 2.

STEP 4. \( c^* \leftarrow c_{k+1} \), \( H^* \leftarrow H_{c_{k+1}} \); Stop.

If we take \( \epsilon = 0 \), the algorithm may stop in a finite number of iterations, and we obtain the global minimum value with the set of global minimizers. Or, we obtain two monotone sequences

\[
\begin{align*}
&c_0 \geq c_1 \geq \cdots \geq c_k \geq c_{k+1} \geq \cdots \\
&H_{c_0} \supseteq H_{c_1} \supseteq \cdots \supseteq H_{c_k} \supseteq H_{c_{k+1}} \supseteq \cdots.
\end{align*}
\]

Let

\[
\bar{c} = \lim_{k \to \infty} c_k \quad \text{and} \quad H^* = \bigcap_{k=1}^{\infty} H_{c_k} \cap S.
\]

THEOREM 2.2. Under Assumptions (A), (M), and (R), \( \bar{c} \) is the global minimum value of \( f \) over \( S \), and \( H^* \) is the set of global minimizers.

Note, that errors at each iteration in the algorithm are not accumulated. The algorithm has been implemented by a properly designed Monte-Carlo method. The numerical tests show that the algorithm is competitive with other algorithms.

3. DISCONTINUOUS PENALTY FUNCTIONS

Let \( X \) be a metric space, \( S \) a subset of \( X \), and \( f \) a real-valued function. Consider the constrained problem

\[
c^* = \inf_{x \in S} f(x),
\]

with the penalty approach. Recall, that a continuous and nonnegative function \( p : X \to \mathbb{R}^1 \) is said to be a penalty function associated with the constraint set \( S \) if

\[
p(x) = 0, \quad \text{if and only if } x \in S.
\]

With such a penalty function, we can find the set of global minimizers of a constrained problem by an integral algorithm [14]. In this section, we will generalize this definition to the discontinuous case.

Now, suppose \( S \) is a closed robust subset of a metric space \( X \) and \( f \) a real-valued function on \( X \). Under Assumption (A), the set of global minimizers of the constrained problem (4) is nonempty. Moreover, (A) also implies that \( f \) is bounded below on \( X \), i.e., there is a constant \( L \) such that

\[
f(x) \geq L, \quad \text{for all } x \in X.
\]

The minimizers of the constrained problem (4) can be approximated by a sequence of solutions of associated penalized unconstrained problems.
The discontinuous penalty function associated with the constraint set \( S \) is defined as follows.

**Definition 3.1.** A function \( p(x) \) on a metric space \( (X, d) \) is a penalty function associated with a constraint set \( S \) if:

1. \( p \) is lower semicontinuous;
2. \( p(x) = 0 \) if \( x \in S \);
3. \( \inf_{x \notin S} p(x) > 0 \), where \( S_\beta = \{ u : d(u, v) \leq \beta, \forall v \in S \} \) and \( \beta > 0 \).

**Remark 3.1.** In the above definition, we relax the requirement of continuity from the traditional definition [2,5] as we wish to utilize discontinuous penalty functions.

**Remark 3.2.** It is expected that the penalty increases when the distance from a point \( x \notin S \) to the constraint set \( S \) increases. We replace the traditional property

\[
p(x) > 0, \quad \text{if} \ x \notin S,
\]

by assumption (3).

With a penalty function \( p \), we examine a penalized unconstrained minimization problem associated with (4),

\[
\min_{x \in X} \{ f(x) + \alpha p(x) \}, \tag{5}
\]

where \( \alpha > 0 \) is a penalty parameter. Under Assumption (A), the penalized level set

\[
H_\alpha^\beta = \{ x : f(x) + \alpha p(x) \leq \beta \}
\]

is a nonempty closed subset of \( H_\beta \). Thus, \( H_\alpha^\beta \) is compact in \( X \). It follows that the minimizers of (5) also exist. Furthermore,

\[
\min_{x \in X} \{ f(x) + \alpha p(x) \} \leq \min_{x \in S} \{ f(x) + \alpha p(x) \} = \min_{x \in S} f(x) = c^*.
\]

We will construct two sequences \( \{ \alpha_n \} \) and \( \{ c_n \} \) so that \( \alpha_n \uparrow \infty \) and \( c_n \downarrow c \geq c^* \), assuming \( b > c \), as \( n \to \infty \) with the property that

\[
\min_{x \in H_n} \{ f(x) + \alpha_n p(x) \} \to c^*, \quad \text{as} \ n \to \infty, \tag{6}
\]

where, in order to simplify the notation, we denote

\[
H_n = \{ x : f(x) + \alpha_n p(x) \leq c_n \}. \tag{7}
\]

**Proposition 3.1.** If \( c_n \downarrow c \geq c^* \), then

\[
\lim_{n \to \infty} H_n = \bigcap_{n=1}^\infty H_n = H_c \cap S. \tag{8}
\]

**Proof.** We first show that \( \{ H_n \} \) is a monotone sequence. It follows that the limit in (8) exists and equals the intersection. Suppose \( x \in H_{n+1} \). Since \( \alpha_{n+1} \geq \alpha_n \) and \( c_{n+1} \leq c_n \),

\[
f(x) + \alpha_n p(x) \leq f(x) + \alpha_{n+1} p(x) \leq c_{n+1} \leq c_n.
\]

Therefore, \( x \in H_n \). This proves \( H_{n+1} \subset H_n \). Now, we show that

\[
\bigcap_{n=1}^{\infty} H_n = H_c \cap S.
\]
If \( x \in H_c \cap S \), then \( p(x) = 0 \) and \( f(x) + \alpha_n p(x) = f(x) \leq c \leq c_n, \forall n = 1, 2, \ldots \). Hence, \( x \in H_n \), for \( n = 1, 2, \ldots \). This proves

\[
H_c \cap S \subset \bigcap_{n=1}^{\infty} H_n.
\]

On the other hand, suppose \( x \in \bigcap_{n=1}^{\infty} H_n \). Then, \( f(x) \leq f(x) + \alpha_n p(x) \leq c_n, \) for \( n = 1, 2, \ldots \). Letting \( n \to \infty \), we have \( f(x) \leq c, \) i.e., \( x \in H_c \). If \( x \notin S \), then \( p(x) > 0 \) and \( f(x) + \alpha_n p(x) \to \infty \) as \( n \to \infty \). This contradicts that \( f(x) + \alpha_n p(x) \leq c \leq c_n, \) for \( n = 1, 2, \ldots \). Hence, \( x \in H_c \cap S \). This proves

\[
\bigcap_{n=1}^{\infty} H_n \subset H_c \cap S.
\]

The proof of Proposition 3.1 is completed.

**Remark 3.3.** We will use the concepts of mean value and modified variance to study a global minimization problem. If \( c < c^* = \min_{x \in S} f(x) \), then \( H_c \cap S = \emptyset \). From the above proposition, there is an integer \( N \) such that \( H_n = \emptyset \) for \( n \geq N \). In this case, we cannot even define mean values and variances on \( X \). Thus, this situation should not be allowed to happen in the integral algorithm.

The following proposition shows that in the above framework, the global minimum value of a constrained problem is the limit of the global minimum values of penalized problems.

**Proposition 3.2.** Suppose that \( \{\alpha_n\} \) is a positive increasing sequence which tends to infinity as \( n \to \infty \) and \( \{c_n\} \) is a decreasing sequence which tends to \( c \geq c^* \) as \( n \to \infty \). Under Assumption (A), we have

\[
\min_{x \in X} \{f(x) + \alpha_n p(x)\} = \min_{x \in H_n} \{f(x) + \alpha_n p(x)\} = \alpha_n \to c^* = \min_{x \in S} f(x).
\]

**Proof.** Since \( f \) and \( p \) are l.s.c., \( H_n \) is closed, and, thus, compact. Therefore,

\[
\min_{x \in H_n} \{f(x) + \alpha_n p(x)\}
\]

exists for each \( n \). Since \( H_c \cap S \subset H_n \), we have

\[
\min_{x \in H_n} \{f(x) + \alpha_n p(x)\} \leq \min_{x \in H_n \cap S} \{f(x) + \alpha_n p(x)\} = \min_{x \in H_n \cap S} f(x) = \min_{x \in S} f(x) = c^*.
\]

Hence,

\[
\liminf_{n \to \infty} \min_{x \in H_n} \{f(x) + \alpha_n p(x)\} \leq c^*.
\]

We now prove

\[
\limsup_{n \to \infty} \min_{x \in H_n} \{f(x) + \alpha_n p(x)\} \leq c^*.
\]

Suppose, on the contrary, \( \hat{c} < c^* \). Let \( c^* - \hat{c} = 2\eta > 0 \); then, there is a subsequence of \( \{\alpha_n\} \) (we denote it with the same notation) and an integer \( N \) such that \( \alpha_n \to \hat{c} \) and \( \alpha_n < c - \eta, \forall n \geq N \). Let \( \hat{x}_n \in H_n \) be a global minimizer of \( \min_{x \in H_n} \{f(x) + \alpha_n p(x)\} \), then

\[
0 \leq f(\hat{x}_n) - f(x_n) 
\]

\( \leq \alpha_n p(\hat{x}_n) \leq c^* - 2\eta, \) \( n = 1, 2, \ldots \). We now have \( \hat{x}_n \in H_{c^* - \eta} \cap H_n, n = N + 1, N + 2, \ldots \). Because of the monotonicity of \( \{H_n\} \), \( H_{c^* - \eta} \cap H_n \neq \emptyset \) implies that \( H_{c^* - \eta} \cap H_k \neq \emptyset, k = 1, \ldots, n - 1, n \). Hence, the intersection of these nested closed (compact) sets is also nonempty:

\[
\bigcap_{n=1}^{\infty} (H_{c^* - \eta} \cap H_n) = \bigcap_{n=1}^{\infty} H_n = \bigcap_{n=1}^{\infty} H_{c^* - \eta} \cap H_c \cap S \neq \emptyset.
\]

Therefore, we have a point \( \hat{x} \) which is in both \( S \) and \( H_{c^* - \eta} \). This contradicts the fact that \( c^* \) is the global minimum value of \( f \) over \( S \). Combining (10) and (11), we obtain (9).
4. DISCONTINUOUS EXACT PENALTY FUNCTIONS

In this section, we will derive conditions for a penalty function to be exact. With these conditions, several discontinuous exact penalty functions are proposed.

**Definition 4.1.** A penalty function \( p \) for the constraint set \( S \) is exact for (4), if there is a real number \( \alpha_0 > 0 \) such that for each \( \alpha \geq \alpha_0 \), we have

\[
\min_{x \in X} \{ f(x) + \alpha p(x) \} = \min_{x \in \bar{S}} f(x) = c^* \tag{13}
\]

and

\[
\{ x : f(x) + \alpha p(x) = c^* \} = \{ x \in S : f(x) = c^* \} = H^*. \tag{14}
\]

**Lemma 4.1.** A necessary condition for a penalty function \( p(x) \) to be exact is as follows.

**Condition (E1).** There are \( \alpha_0 > 0 \) and \( \beta = \beta(\alpha_0) > 0 \) such that

\[
p(x) \geq \frac{c^* - f(x)}{\alpha_0}, \quad \text{for all } x \in S_\beta. \tag{15}
\]

**Proof.** Suppose that \( p(x) \) is an exact penalty function, but (E1) does not hold. Then, there are sequences \( \alpha_k \uparrow \infty, \beta_k > 0 \) and \( x_k \in S_{\beta_k} \) such that

\[
c^* - f(x_k) \beta_k < 0, \tag{16}
\]

or

\[
f(x_k) + \alpha_k \beta_k < c^*. \tag{17}
\]

Let \( \tilde{x}_k \) be a solution to the penalized minimization problem

\[
\min_{x \in X} \{ f(x) + \alpha_k \beta_k \}, \tag{18}
\]

then

\[
f(\tilde{x}_k) + \alpha_k \beta_k \leq f(x_k) + \alpha_k \beta_k < c^*, \quad \text{for } k = 1, 2, \ldots. \tag{19}
\]

It implies

\[
\min_{x \in X} \{ f(x) + \alpha_k \beta_k \} < c^*, \quad \text{for } k = 1, 2, \ldots. \tag{20}
\]

This contradicts the definition of the exact penalty function. \( \blacksquare \)

**Remark 4.1.** Condition (E1) states the following properties.

(i) If \( x \in S \), then \( p(x) = 0 \) and (15) becomes \( f(x) \geq c^* \); this is just the definition of \( c^* \).

(ii) There is a nonnegative function \( b(x) \) such that if \( x \notin S \), then we have

\[
p(x) \geq \alpha_0 b(x) \quad \text{and} \quad f(x) \geq c^* - b(x). \tag{21}
\]

These mean that for points outside of the constraint set \( S \), the objective function \( f(x) \) cannot decrease too quickly and the penalty function \( p(x) \) cannot increase too slowly.

**Example 4.1.** Consider the problem \( \min_{x \geq 0} x \). The penalty function

\[
p(x) = \begin{cases} x^2, & x < 0, \\ 0, & x \geq 0 \end{cases}
\]

is not exact because Condition (E1) or (21) does not hold.

Condition (E1) cannot ensure the feasibility of solution of the associated penalized problem. Thus, one more trivial necessary condition is stated.

**Condition (E2).** There is \( \alpha_0 > 0 \) such that if \( \alpha > \alpha_0 \) and \( x_\alpha \) is a solution of

\[
\min_{x \in X} \{ f(x) + \alpha p(x) \} = c^*, \tag{22}
\]

then \( x_\alpha \) is feasible.
In their paper, Di Pillo and Grippo [6] state a feasibility assumption \((a_4)\) (see [6, Theorem 1, p. 1339]). It is easy to verify that \((a_4)\) implies \((E2)\). Condition \((E2)\) is easy to verify when we study discontinuous penalty functions.

We are now ready to prove that \((E1)\) and \((E2)\) are necessary and sufficient for an exact penalty function.

**Theorem 4.1.** A penalty function \(p(x)\) is exact for the minimization problem \((4)\) if and only if \((E1)\) and \((E2)\) hold.

**Proof.** We have shown that conditions \((E1)\) and \((E2)\) are necessary. For the sufficiency, we first prove that there is \(\alpha_p > 0\) such that

\[
\min_{x \in X} [f(x) + \alpha p(x)] = c^* \quad \text{for all } \alpha > \alpha_p. \tag{23}
\]

Condition \((E1)\) implies that there are \(\alpha_0 > 0\) and \(\beta > 0\) such that

\[
f(x) + \alpha p(x) \geq f(x) + \alpha_0 p(x) \geq c^*, \quad \text{for all } x \in S_\beta, \quad \alpha > \alpha_0. \tag{24}
\]

Thus,

\[
c^* < \min_{x \in S_\beta} [f(x) + \alpha p(x)] \leq \min_{x \in S} [f(x) + \alpha p(x)] = \min_{x \in S} f(x) = c^*. \tag{25}
\]

Since \(\min_{x \in S} p(x) = \eta > 0\) and \(f\) is bounded below, \(f(x) > L\), there is \(\alpha_L\) such that \(\alpha_L \eta > |L| + c^*\).

It follows,

\[
f(x) + \alpha_L p(x) \geq c^*, \quad \text{for all } x \notin S_\beta. \tag{26}
\]

Thus, if we take \(\alpha \geq \alpha_p = \max(\alpha_0, \alpha_L)\), then

\[
c^* \leq \min_{x \in X} [f(x) + \alpha p(x)] \leq \min_{x \in S_\beta} [f(x) + \alpha p(x)] \leq c^*. \tag{27}
\]

This implies \((23)\).

If \(\hat{x} \in \{x \in S : f(x) = c^*\}\), then \(p(\hat{x}) = 0\) and \(f(\hat{x}) + \alpha p(\hat{x}) = f(\hat{x}) = c^*\), i.e., \(\hat{x} \in \{x : f(x) + \alpha p(x) = c^*\}\), for all \(\alpha\). If \(\hat{x} \in \{x : f(x) + \alpha p(x) = c^*\}\) for \(\alpha \geq \alpha_p\), then from \((E2)\), \(\hat{x}\) should be feasible, i.e., \(\hat{x} \in S\). Thus, \(\hat{x} \in \{x \in S : f(x) = c^*\}\).

We now construct a class of discontinuous penalty functions for the constrained problem

\[
\min_{x \in S} f(x), \tag{28}
\]

where \(S\) is a robust set and \(f\) is upper robust on \(S\). Let

\[
p(x) = \begin{cases} 
0, & x \in S, \\
\delta + d(x), & x \notin S,
\end{cases} \tag{29}
\]

where \(\delta\) is a positive number and \(d(x)\) is a penalty-like function.

**Theorem 4.2.** The discontinuous penalty function \((29)\) is exact.

**Proof.** Take \(\alpha_0 \geq (c^* - m_\eta) / \delta\), where \(m_\eta = \min_{x \in S_\eta} f(x)\). Then, if \(x \in S_\eta\), we have

\[
p(x) \geq \delta \geq \frac{c^* - m_\eta}{\alpha_0}, \quad \geq \frac{c^* - f(x)}{\alpha_0}. \tag{30}
\]

This is \((E1)\). Suppose, for \(\alpha \geq \alpha_0\), we have

\[
\min_{x \in X} [f(x) + \alpha p(x)] = c^*. \tag{31}
\]
If a solution $\tilde{x}$ of (31) is not feasible, then $p(\tilde{x}) \geq \delta$ and

$$\alpha p(\tilde{x}) > \alpha_0 p(\tilde{x}) \geq \alpha_0 \delta \geq (c^* - m_\eta) \geq c^* - f(\tilde{x}).$$

This implies a contradiction

$$f(\tilde{x}) + \alpha p(\tilde{x}) > c^*, \quad \text{for } \alpha > \alpha_0. \tag{32}$$

**Remark 4.2.** No constraint qualification is required for this kind of penalty functions. For example, for the inequality-constraint set

$$S = \{x : g_i(x) \leq 0, i = 1, \ldots, r\},$$

we can take

$$d(x) = \sum_{i=1}^{r} \| \max(g_i(x), 0) \| \rho \quad \text{or} \quad d(x) = \max_i \| \max(g_i(x), 0) \| \rho,$$

where $\rho > 0$. If $g_i, i = 1, \ldots, r$, are upper semicontinuous, so is $d$.

In order to apply an integral global algorithm, we still need robustness of $f + \alpha p$.

**Proposition 4.1.** If $d$ is upper robust on $S$, then $p(x)$ is also upper robust on $S$.

**Proof.** For each $c$, we have

$$\{x \in S : p(x) < c\} = \begin{cases} \emptyset, & \text{if } c < 0, \\ S, & \text{if } 0 \leq c \leq \delta, \\ \{x \in S : d(x) < c\}, & \text{if } c > \delta. \end{cases} \tag{33}$$

We know that $\emptyset$ and $S$ are robust. The set $\{x \in S : \delta + d(x) < c\}$ is also robust because $d(x)$ is assumed to be upper robust on $S$. It follows that $\{x \in S : p(x) \leq c\}$ is robust for every real number $c$. Hence, $p(x)$ is upper robust on $S$.

**Proposition 4.2.** If $f$ is upper robust on $S$, or $f$ is upper robust and $d$ is u.s.c. on $S$, then $f + \alpha p$ is upper robust on $S$ for every $\alpha > 0$.

**Proof.** If $d$ is upper robust on $S$, then $\alpha p, \alpha > 0$, is also upper robust on $S$. If $f$ is u.s.c., then as the sum of an u.s.c. function and a upper robust function, $f + \alpha p$ is upper robust.

If $f$ is upper robust on $S$, we cannot directly apply this result to prove $f + \alpha p$ is upper robust on $S$. We enumerate all rational numbers $\tau_1, \tau_2, \ldots$. For each real number $c$, we have

$$\{x \in S : f(x) + \alpha p(x) < c\} = \bigcup_{k=1}^{\infty} \{x \in S : f(x) \leq \tau_k\} \cap \{x \in S : p(x) < c - \tau_k\}. \tag{34}$$

We know that

$$\{x \in S : \alpha p(x) < c - \tau_k\} = \{x \in S : p(x) < g_k\} = \begin{cases} \emptyset, & \text{if } g_k < 0, \\ S, & \text{if } 0 \leq g_k \leq \delta, \\ G \cap S, & \text{if } g_k > \delta, \end{cases} \tag{35}$$

where

$$g_k = \frac{c - \tau_k}{\alpha} \quad \text{and} \quad G = \{x : \delta + d(x) < g_k\} = \{x : d(x) < g_k - \delta\}.$$
5. PENALTY OPTIMALITY CONDITIONS

We now generalize the penalty optimality conditions [15, 16] for continuous functions to those for upper robust functions. In this section, we will examine the concepts of penalized mean value, modified variance, and higher moments conditions.

Let $S$ be a subset of a metric space $X$, $f$ a real-valued function on $X$, and $p$ a penalty function for the constraint set $S$.

**Definition 5.1.** Let $c_n < c^* = \inf_{x \in S} f(x)$. We define the penalty mean value, modified variance and $m$th moment (centered at $a$), respectively, of $f + \alpha_n p$ over the penalized level set

$$H_n = \{x : f(x) + \alpha_n p(x) \leq c_n\},$$

with a $Q$-measure $\mu$ on $X$ as follows:

$$M(f, c_n; p) = \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x)] \, d\mu,$$

$$V_1(f, c_n; p) = \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x) - c_n]^2 \, d\mu,$$

$$M_m(f, c; a; p) = \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x) - a]^m \, d\mu, \quad m = 1, 2, \ldots.$$

Under Assumptions (A), (R), and (M), they are well defined.

Now, we consider the convergence properties of the penalized mean value, modified variance, and higher moments as $n \to \infty$. As usual, we assume that

$$c_n \downarrow c \geq c^* = \inf_{x \in S} f(x). \quad (36)$$

**Theorem 5.1.** Suppose $S$ is robust and $f + \alpha p$ ($\alpha > 0$) is robust on $S$. Under Assumptions (A) and (M), we have, for $c \geq c^*$,

$$\lim_{n \to \infty} M(f, c_n; p) = M(f, c; S). \quad (37)$$

**Proof.** We first prove that when $c > c^*$, (37) holds. Since $\mu(H_c \cap S) > 0$, we have $\mu(H_n) > 0$ because $S \cap H_c \subset H_n$, $n = 1, 2, \ldots$. Thus,

$$|M(f, c_n; p) - M(f, c; S)| \leq I_1 + I_2,$$

where

$$I_1 = \left| \frac{1}{\mu(H_n)} - \frac{1}{\mu(H_c \cap S)} \right| \cdot \int_{H_n} [f(x) + \alpha_n p(x)] \, d\mu,$$

and

$$I_2 = \left| \frac{1}{\mu(H_c \cap S)} \int_{H_n} [f(x) + \alpha_n p(x)] \, d\mu - \int_{H_c \cap S} [f(x) + \alpha_n p(x)] \, d\mu \right|.$$

We have, $L \leq f(x) \leq f(x) + \alpha_n p(x) \leq c_n \leq c_1$, for all $n = 1, 2, \ldots$. Thus,

$$|I_1| \leq \left| \frac{1}{\mu(H_n)} - \frac{1}{\mu(H_c \cap S)} \right| \cdot A \cdot \mu(H_1),$$

where $A = \max(c_1, |L|)$. It follows, by Proposition 3.1, $I_1 \to 0$ as $n \to \infty$. Next, we have

$$|I_2| \leq \frac{2A}{\mu(H_c \cap S)} \cdot |\mu(H_n) - \mu(H_c \cap S)|,$$

which tends to zero as $n \to \infty$. 
When \( c = c^* \), since \( f(x) + \alpha_n p(x) \leq c_n \) on \( H_n, \forall n \), we have

\[
M(f, c_n; p) = \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x)] d\mu \leq c_n, \quad n = 1, 2, \ldots.
\]

It follows that

\[
\limsup_{n \to \infty} M(f, c; p) \leq \lim_{n \to \infty} c_n = c = c^*. \tag{38}
\]

We now prove

\[
\liminf_{n \to \infty} \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x)] d\mu \geq c^*. \tag{39}
\]

Suppose, on the contrary, that (39) does not hold. Then, there is a subsequence of

\[
\frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x)] d\mu,
\]

which we denote with the same notation, such that

\[
\lim_{n \to \infty} \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x)] d\mu = \hat{c} < c^*.
\]

Let \( 2\eta = c^* - \hat{c} > 0 \). Thus, there is a positive integer \( N \) such that for \( n \geq N \),

\[
\frac{1}{\mu(H_n)} \int_{H_n} f(x) d\mu \leq \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x)] d\mu \leq c^* - \eta.
\]

This implies that

\[
H_{c^* - \eta} \cap H_n \neq \emptyset, \quad \text{for } n \geq N,
\]

and hence,

\[
H_{c^* - \eta} \cap H_n \cap S \neq \emptyset.
\]

That is to say, we have points both in \( H_{c^* - \eta} \) and \( S \). This contradicts the assumption that \( c^* \) is the global minimum value of \( f \) over \( S \).

**Theorem 5.2.** Suppose \( S \) is a robust set and \( f + \alpha p (\alpha > 0) \) is robust on \( S \). Under Assumptions (A) and (M), we have, for \( c \geq c^* \),

\[
\lim_{n \to \infty} V_1(f, c_n; p) = V_1(f, c; S). \tag{40}
\]

**Proof.** When \( c > c^* \), the proof is similar to that of the mean value case. Suppose \( c = c^* \). Since

\[
V_1(f, c_n; p) \geq 0, \quad n = 1, 2, \ldots,
\]

it follows that

\[
\liminf_{n \to \infty} V_1(f, c_n; p) \geq 0. \tag{41}
\]

We prove that \( \limsup_{n \to \infty} V_1(f, c_n; p) = 0 \). Suppose, on the contrary, it does not hold. Then, there is a subsequence (for which we keep the same notation) such that

\[
V_1(f, c_n; p) \to 2\eta > 0.
\]

Thus, there is an integer \( N \) such that

\[
V_1(f, c_n; p) > \eta, \quad \text{when } n \geq N.
\]
Since $f$ is bounded below on $H_1 \cap S$, there exists a real number $g \geq 0$ such that
\[ f(x) + \alpha_n p(x) + g \geq 0, \quad \forall x \in H_1. \tag{42} \]

Therefore,
\[
V_1(f, c_n; p) = \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x) - c_n]^2 \, d\mu
\]
\[
= \frac{1}{\mu(H_n)} \left\{ \int_{H_n} [f(x) + \alpha_n p(x) + g]^2 \, d\mu + \int_{H_n} (g + c_n)^2 \, d\mu \right. \\
- 2(g + c_n) \int_{H_n} [f(x) + \alpha_n p(x) + g] \, d\mu \right\} > \eta.
\]

It follows that
\[(c_n + g)^2 + (g + c_n)^2 > 2(g + c_n) \cdot (g + c^*) + \eta.\]

Letting $n \to \infty$ in the above inequality, we obtain
\[(c^* + g)^2 + (g + c^*)^2 \geq 2(g + c^*) \cdot (g + c^*) + \eta,\]

and have a contradiction: $0 \geq \eta > 0$. Therefore, when $c = c^*$, the limit of (40) exists and is equal to 0. But, according to the modified variance condition, $c^*$ is the global minimum value if and only if $V_1(f, c^*; S) = 0$. Hence, when $c = c^*$, we have
\[
\lim_{n \to \infty} V_1(f, c_n; p) = 0 = V_1(f, c^*; S). \]

**Theorem 5.3.** Under the assumption of Theorem 5.1, we have, for $c \geq c^*$,
\[
\lim_{n \to \infty} M_m(f, c_n; c_n; p) = M_m(f, c; c; S). \tag{43}
\]

**Proof.** When $c > c^*$ or when $c = c^*$ and $m$ is odd, the proof is similar to that of the mean value case. Suppose $m = 2r$ and $r > 1$, then
\[
M_m(f, c_n; c_n; p) \geq 0.
\]

Thus,
\[
\liminf_{n \to \infty} M_m(f, c_n; c_n; p) \geq 0. \tag{44}
\]

On the other hand,
\[
M_m(f, c_n; c_n; p) = \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x) - c_n]^m \, d\mu
\]
\[
\leq A^{2(r-1)} \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x) - c_n]^2 \, d\mu
\]
\[= A^{2(r-1)} V_1(f, c_n; p) \to 0, \quad \text{as } n \to \infty,
\]

where $|f(x) + \alpha_n p(x) - c_n| \leq A, \forall x \in H_1 \cap S$.

Therefore, we have proven that
\[
\lim_{n \to \infty} M_m(f, c_n; c_n; p) = 0 = M_m(f, c^*; c^*; S). \tag{46}
\]

The last equality holds because of the higher moment conditions for global minimization.

**Theorem 5.4.** Under the assumptions of Theorem 5.1, $c^*(c_n \downarrow c = c^*)$ is the global minimum value of $f$ over $S$ if and only if one of the following conditions holds:

1. $\lim_{n \to \infty} M(f, c_n; p) = c^*$,
2. $\lim_{n \to \infty} V_1(f, c_n; p) = 0$,
3. $\lim_{n \to \infty} M_m(f, c_n; c_n; p) = 0$, for some positive integer $m = 1, 2, \ldots$. 

The above theorem, in fact, also gives us necessary and sufficient conditions for global minimization with a penalty function.
6. A PENALTY ALGORITHM

In this section, we propose a penalty algorithm in terms of a penalty mean value and modified variance. We then prove that the algorithm produces a sequence which converges to the global minimum.

Take a real number
\[ c_1 > \min_{x \in S} f(x) \]
an exact penalty function \( p(x) \) and a penalty parameter \( \alpha_1 \). Let
\[ c_2 = M(f, c_1; p) = \frac{1}{\mu(H_1)} \int_{H_1} [f(x) + \alpha_1 p(x)] d\mu. \]
Replacing \( c_1 \) by \( c_2 \) and \( \alpha_1 \) by \( \alpha_2 = \alpha_1 \cdot \beta \) (where \( \beta \geq 1.0 \) is a prespecified constant) and go to the next iteration.

**Lemma 6.1.** If \( \mu(H_1) > 0 \), then \( \mu(H_2) > 0 \).

**Proof.** By the definition of \( H_1 \), we see that \( c_2 \leq c_1 \). If \( c_2 = c_1 \), then \( \mu(H_2) > 0 \). Indeed, suppose, on the contrary, that \( \mu(H_2) = 0 \); then \( c_2 \) is the global minimum value of \( f + \alpha_2 p \). But,
\[ c_1 > \min_{x \in S} f(x) \geq \min_{x \in S} f(x) = \min_{x \in X} [f(x) + \alpha_2 p(x)] = c_2. \]
The last equality holds because we have an exact penalty function. This contradicts that \( c_1 = c_2 \).

Now, suppose \( c_2 < c_1 \) and suppose, on the contrary, that \( \mu(H_2) = 0 \); then \( c_2 \) is the global minimum of \( f + \alpha_2 p \) in \( X \), i.e.,
\[ f(x) + \alpha_2 p(x) \geq c_2, \quad \forall x \in X. \]
Since \( \mu(H_1) > 0 \), there exists \( \epsilon > 0 \) such that \( 0 < \epsilon < c_1 - c_2 \) and
\[ \mu(G_{\epsilon} \cap S) > 0, \]
where
\[ G_{\epsilon} = \{ x : c_2 + \epsilon < f(x) \leq c_1 \}, \]
otherwise, \( c_1 \) would be the global minimum of \( f \) over \( S \). Hence,
\[ c_2 = M(f, c_1; p) = \frac{1}{\mu(H_1)} \int_{H_1} [f(x) + \alpha_1 p(x)] d\mu \]
\[ = \frac{1}{\mu(H_1)} \left( \int_{H_1 \setminus G_{\epsilon} \cap S} [f(x) + \alpha_1 p(x)] d\mu + \int_{G_{\epsilon} \cap S} f(x) d\mu \right) \]
\[ \geq \frac{c_2}{\mu(H_1)} (\mu_1(H) - \mu(G_{\epsilon} \cap S)) + (c_2 + \epsilon) \frac{\mu(G_{\epsilon} \cap S)}{\mu(H_1)} \]
\[ = c_2 + \epsilon \cdot \frac{\mu(G_{\epsilon} \cap S)}{\mu(H_1)} > c_2. \]
This is a contradiction. The proof is now complete.

Continuing the process described above, we obtain a sequence of real numbers \( c_n \) which converges to the global minimum of \( f(x) \) on \( S \cap X \).

A penalty algorithm is proposed as follows.

**Step 1.** Take \( c_0 > \min_{x \in S} f(x), \epsilon > 0, n := 0, \beta > 1.0, H_0 = \{ x : f(x) + \alpha_0 p(x) \leq c_0 \} \).

**Step 2.** Calculate the mean value
\[ c_{n+1} = \frac{1}{\mu(H_n)} \int_{H_n} [f(x) + \alpha_n p(x)] d\mu. \]
STEP 3. Calculate the modified variance

\[ v_{n+1} = \frac{1}{\mu(H_n)} \int_{H_n} (f(x) + \alpha_n p(x) - c_n)^2 \, d\mu. \]

If \( v_{n+1} \geq \epsilon \), then \( n := n + 1 \) and \( \alpha_{n+1} = \alpha_n \cdot \beta \), and go to Step 2; otherwise, go to Step 4.

STEP 4. \( c^* \leftarrow c_{n+1} \), \( H^* \leftarrow H_{c_{n+1}} \). Stop.

Applying this algorithm with \( \epsilon = 0 \), we obtain a decreasing sequence

\[ c_1 \geq c_2 \geq \cdots \geq c_n \geq c_{n+1} \geq \cdots, \]

and a sequence of sets

\[ H_1 \supseteq H_2 \supseteq \cdots \supseteq H_n \supseteq H_{n+1} \supseteq \cdots. \]

THEOREM 6.1. Suppose that \( S \) is robust and \( f + \alpha p \) (\( \alpha > 0 \)) is upper robust on \( S \), and Assumptions (A) and (M) hold. With this algorithm, we have

\[ \lim_{n \to \infty} c_n = c^* = \min_{x \in S} f(x) \]

and

\[ \lim_{n \to \infty} H_n = \bigcap_{k=1}^{\infty} H_k = H^*. \]

PROOF. According to the algorithm, we know that \( c_n \geq c^* \) for \( n = 1, 2, \ldots \), and the sequence \( \{c_n\} \) is decreasing. Thus, the limit

\[ \lim_{n \to \infty} c_n = \hat{c} \geq c^* \]

exists. Letting \( n \to \infty \) in (47), we obtain

\[ \hat{c} = \lim_{n \to \infty} c_{n+1} = \lim_{n \to \infty} M(f, c_n; p) = M(f, \hat{c}; S). \]

It follows from Theorem 5.4 that \( \hat{c} \) is the global minimum value of \( f \) over \( S \), i.e., \( \hat{c} = c^* \).

The equality (51) is valid by Proposition 3.1.

7. NUMERICAL TESTS

An important way to ascertain the performance of a global minimization algorithm is to see if it can pass numerical tests successfully.

There are a lot of test problems for constrained minimization available in the literature. We select four problems, testing as follows.

EXAMPLE 1. (From [17].) The objective function is

\[ f(x) = f_1(x_1) + f_2(x_2), \]

where

\[
 f_1(x_1) = \begin{cases} 
 30x_1, & 0 \leq x_1 < 300, \\
 31x_1, & 300 \leq x_1 < 400, 
\end{cases} \quad f_2(x_2) = \begin{cases} 
 28x_2, & 0 \leq x_2 < 100, \\
 29x_2, & 100 \leq x_2 < 200, \\
 30x_2, & 200 \leq x_2 < 1000,
\end{cases}
\]
with constraints

\[
\begin{align*}
    x_1 &= 300 - \frac{x_3x_4}{131.078} \cos(1.48577 - x_6) + \frac{0.90798x_2^2}{131.078} \cos(1.47588), \\
    x_2 &= -\frac{x_3x_4}{131.078} \cos(1.48577 + x_6) + \frac{0.90798x_2^2}{131.078} \cos(1.47588), \\
    x_5 &= -\frac{x_3x_4}{131.078} \sin(1.48577 + x_6) + \frac{0.90798x_2^2}{131.078} \sin(1.47588), \\
    200 - \frac{x_3x_4}{131.078} \sin(1.48577 - x_6) + \frac{0.90798x_2^2}{131.078} \sin(1.47588) &= 0,
\end{align*}
\]

\[
0 \leq x_1 \leq 400, \\
0 \leq x_2 \leq 1000, \\
340 \leq x_3 \leq 420, \\
340 \leq x_4 \leq 420, \\
-1000 \leq x_5 \leq 1000, \\
0 \leq x_6 \leq 0.5236.
\]

The objective of this problem is a discontinuous robust function with four nonlinear equality constraints. We take \( x_3 \) and \( x_6 \) as independent variables. Then, \( x_1, x_2, x_4, \) and \( x_5 \) are functions of \( x_3 \) and \( x_6 \). Thus, except box constraints on these independent variables, we have eight more nonlinear inequality constraints. The discontinuous penalty function is applied to these inequality constraints.

With the penalty algorithm, we obtain the solution

\[
x^* = (202.9967, 99.99992, 383.071, 420.0000, -10.90771, 0.007314806),
\]

\[
f^* = 8889.899.
\]

**Example 2.** (From [13,18].) Let

\[
f(x) = 0.7854x_1x_2^2 (3.3333x_3^2 + 14.9334x_3 - 43.0934) - 1.5080x_1 (x_2^2 + x_2^3) + 7.4770 (x_3^2 + x_3^3) + 0.7854 (x_4x_6 + x_5x_7),
\]

with constraints

\[
\begin{align*}
    x_1x_2^2x_3 &\geq 27, \\
    x_1x_2^2x_3 &\geq 397.5, \\
    x_2x_3x_6^4 &\geq 1.93, \\
    x_2x_3x_6^4 &\geq 1.93, \\
    \frac{1}{0.1x_3^2} \sqrt{\left[\frac{745x_4}{x_2x_3}\right]^2} + 16.91 \times 10^6 &\leq 1100, \\
    \frac{1}{0.1x_3^2} \sqrt{\left[\frac{745x_5}{x_2x_3}\right]^2} + 157.5 \times 10^6 &\leq 850,
\end{align*}
\]

\[
x_2x_3 \leq 40, \\
5 \leq \frac{x_1}{x_2} \leq 12, \\
1.5x_6 + 1.9 \leq x_4,
\]
We recalculate this problem with the discontinuous penalty method, which is more efficient, and obtain the following solution:

\[ x^* = (3.5, 0.7, 17.0, 7.30, 7.71531991, 3.35054095, 5.28665446) \]

and

\[ f^* = 2994.425. \]

EXAMPLE 3. Consider a nonlinear integer programming problem from [19,20]. The objective function is

\[ f(x) = x_1x_2x_3 + x_1x_4x_5 + x_2x_4x_6 + x_6x_7x_8 + x_2x_5x_7, \]

with constraints,

\[
\begin{align*}
2x_1 + 2x_4 + 8x_8 & \geq 12, \\
11x_1 + 7x_4 + 13x_6 & \geq 41, \\
6x_2 + 9x_4x_6 + 5x_7 & \geq 60, \\
3x_2 + 5x_5 + 7x_8 & \geq 42, \\
6x_2x_7 + 9x_3 + 5x_5 & \geq 53, \\
4x_3x_7 + x_5 & \geq 13, \\
2x_1 + 4x_2 + 7x_4 + 3x_5 + x_7 & \leq 69, \\
x_1 & \leq 7, \quad i = 1, 3, 4, 6, 8, \\
x_i & \leq 15, \quad i = 2, 5, 7, \\
x & \text{ integer, } \quad i = 1, \ldots, 8.
\end{align*}
\]

Solution:

\[ x^* = (5, 4, 1, 1, 6, 3, 2, 0), \quad f^* = 110. \]

REMARK 7.1. The discontinuous penalty function is applied to handle the constraints. After 919 function evaluations, the global minimizer is found. The modified variance does not equal zero until 1370 function evaluations. The acceptance-rejection technique could not be applied here because the acceptance-rate is extremely low.

EXAMPLE 4. Consider a mixed programming problem from [17,19]. The objective function is

\[ f(x) = 5.3578547x_3^2 + 0.835689x_1x_5 + 37.293239x_1 - 40792.141, \]
with constraints

\[\begin{align*}
0 & \leq 85.334407 + 0.0056858x_2x_5 + 0.0062622x_1x_4 - 0.0022053x_3x_5 \leq 92, \\
90 & \leq 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3x_5^2 \leq 110, \\
20 & \leq 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 \leq 25, \\
78 & \leq x_1 \leq 102, \quad 23 \leq x_2 \leq 45, \quad x_1, x_2 \text{ are integers,} \\
27 & \leq x_i \leq 45, \quad i = 3, 4, 5.
\end{align*}\]

Solution:

\[x^* = (78, 33, 29.995256, 45.0, 36.77581), \quad f^* = -30665.54.\]

8. CONCLUSIONS

In this paper, the methodology of integral global optimization is applied to constrained minimization problems by discontinuous penalty technique. Under very weak assumptions, the discontinuous function is exact without any constraint qualification requirement.

The examples presented in this paper are illustrative of several noteworthy ideas. Example 1 has discontinuous objective function. We recalculate Example 2 in [13,21] with a discontinuous penalty function. Example 3 is a nonlinear integer programming problem for which one cannot use the acceptance-rejection technique because the rate of acceptance is very low, as mentioned in [19]. Example 4 is a mixed programming problem. For these examples, the new solution methodology works remarkably well, making computation seem like an almost routine task. It is our claim that there is no existing methodology which can match that performance.

REFERENCES