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ScienceDirect
Applied Mathematics Letters 21 (2008) 786–790
Applied Mathematics Letters
www.elsevier.com/locate/aml

Permanental Mates: Perturbations and Hwang's conjecture

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Received 28 November 2005; received in revised form 15 July 2007; accepted 14 August 2007

Abstract

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices. Two unequal matrices A and B in Ω_n are called permanental mates if the permanent function is constant on the line segment $tA + (1-t)B,\ 0 \le t \le 1$, connecting A and B. We study the perturbation matrix $A + E$ of a symmetric matrix A in Ω_n as a permanental mate of A. Also we show an example to disprove Hwang's conjecture, which states that, for $n \ge 4$, any matrix in the interior of Ω_n has no permanental mate.

Keywords: Perturbation matrix; Doubly stochastic matrix; One matching matrices; Permanent; Permanental mates

1. Introduction

Let Ω_n be the set of all $n \times n$ doubly stochastic matrices. Given a function f whose domain is the set of doubly stochastic matrices, two doubly stochastic matrices A and B, $A \ne B$, are called f-mates if $f(tA+(1-t)B)$ is constant for all t in [0, 1]. That is, f is constant on the line segment connecting A and B. Wang [6] took f as the permanent of a doubly stochastic matrix and defined permanental mates as follows. $A, B \in \Omega_n$, $A \ne B$, are said to be permanental mates if per$[tA + (1-t)B]$ is constant for all t in [0, 1]. If $A = B$ then A and B are trivial mates, otherwise nontrivial mates. Here mates refer to nontrivial permanental mates. From the definition of mates, the matrices in Ω_2 have no permanental mates. It is proved in [2] that $\begin{pmatrix} x & 1-x \\ 1-x & x \end{pmatrix} \oplus 1,\ 0 \le x \le 1$, has no permanental mate in Ω_3, where $\oplus$ denotes the direct sum. Karuppanchetty and Maria Arulraj [4] characterized the set of all matrices Ω_3 and Ω_4 having their transposes as permanental mates.

Let A^{T} denote the transpose of A and let trace of A be denoted by tra(A). For integers r, n $(1 \le r \le n)$, let $Q_{r,n}$ denote the set of all sequences $(i_1, i_2, \ldots, i_r)$ such that $1 \le i_1 < \cdots < i_r \le n$. For fixed $\alpha, \beta \in Q_{r,n}$, let $A(\alpha/\beta)$ be the submatrix of A obtained by deleting the rows α and columns β of A, let $A[\alpha/\beta]$ denote the submatrix of A formed by rows α and columns β of A and A_i denote the first $n-3$ rows of the ith column of A. We denote $A + E$ as $\tilde{A}$, a perturbation of a matrix $A \in \Omega_n$.

For brevity, let us use the notation $M(a, b; c, d)$ to denote the 3×3 doubly stochastic matrix $\begin{pmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & a+b+c+d-1 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & \varepsilon & -\varepsilon \\ -\varepsilon & 0 & \varepsilon \\ \varepsilon & -\varepsilon & 0 \end{pmatrix}$, $\varepsilon > 0$. An $n \times n$ doubly stochastic matrix $A = (a_{ij})$ is

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.08.005

called a k-matching matrix, if there exists precisely one k $(1 \leq k \leq n)$ such that $a_{kj} = a_{jk}$ for $j = 1, 2, \ldots, n$. Let $M_{4,k}$ denote the set of all k-matching matrices in Ω_4.

In this paper, we frequently use the following results (Minc [5]).

If $A = (a_{ij})$ and $B = (b_{ij})$ are $n \times n$ matrices, then

$$\text{per}A = \sum_{\beta \in Q_{r,n}} \text{per}A[\alpha/\beta]\text{per}A(\alpha/\beta), \quad \text{for } \alpha \in Q_{r,n}, \tag{1}$$

$$\sum_{\alpha,\beta \in Q_{r,n}} \text{per}A[\alpha/\beta]\text{per}A(\alpha/\beta) = \binom{n}{r} \text{per}A \quad \text{and} \tag{2}$$

$$\text{per}(A+B) = \sum_{r=o}^{n} S_r(A,B), \quad \text{where } S_r(A,B) = \sum_{\alpha,\beta \in Q_{r,n}} \text{per}A[\alpha/\beta]\text{per}B(\alpha/\beta) \tag{3}$$

$\text{per}A[\alpha/\beta] = 1$ when $r = 0$ and $\text{per}B(\alpha/\beta) = 1$ when $r = n$.

The Eq. (1) is called the Laplace expansion for permanents.

In the next section, we prove a necessary and sufficient condition for a symmetric matrix in Ω_n having a permanental mate with its perturbation matrix and a counter example is also given to disprove the Hwang's conjecture [3].

2. Permanental mates

Brenner and Wang [1] have obtained a necessary and sufficient condition for A and B in Ω_n to be permanental mates, which is given below.

$$\text{per}A = \text{per}B \quad \text{and} \quad S_k(A,B) = \binom{n}{k} \text{per}A = S_{n-k}(A,B). \tag{4}$$

Theorem 2.1 (*Hwang [3]*). *For $A, B \in \Omega_n$, the following are equivalent:*

1. *A and B form a permanental pair.*
2. $\sum_{\alpha,\beta \in Q_{r,n}} \text{per}A[\alpha/\beta]\text{per}B(\alpha/\beta) = \binom{n}{r} \text{per}A$, *for all* $r = 0, 1, \ldots, n$.
3. $\sum_{\alpha,\beta \in Q_{r,n}} \text{per}A[\alpha/\beta]\text{per}[B(\alpha/\beta) - A(\alpha/\beta)] = 0$, *for all* $r = 0, 1, \ldots, n-1$.

Theorem 2.2. *Let $A = (a_{ij}) \in \Omega_n$ be a symmetric matrix, let $E_1 = \mathbf{O}_{n-3\times n-3} \oplus E$, $\mathbf{O}_{n-3\times n-3}$ be the zero matrix of order $n-3$, such that the perturbation $\tilde{A} = A + E_1 \in \Omega_n$. Then A and $\tilde{A}$ are permanental mates if and only if*

$$\left(\sum_{i,j=n-2}^{n} a_{ij} - 2 \sum_{i=n-2}^{n} a_{ii} \right) \text{per}X + \sum_{i,j=1}^{n-3} \left(\sum_{k,r=n-2}^{n} a_{ir}a_{jk} - 2 \sum_{k=n-2}^{n} a_{ik}a_{jk} \right) \text{per}\, X(i/j) = 0, \tag{5}$$

where $X = (a_{ij})_{n-3\times n-3}$ is a submatrix of A formed by taking first $n-3$ rows and $n-3$ columns of A.

Proof. From the Theorem 2.1, a necessary and sufficient condition for A and $\tilde{A}$ to be permanental mates is

$$S_r(A, E_1) = \sum_{\alpha,\beta \in Q_{r,n}} \text{per}A[\alpha/\beta]\text{per}E_1(\alpha/\beta) = 0, \quad r = 0, 1, \ldots, n-1.$$

It is easy to see that $S_r(A, E_1) = 0$ for $r = 0, 1, \ldots, n-3$ and $S_{n-1}(A, E_1) = 0$, since A is symmetric.

Now,

$$S_{n-2}(A, E_1) = \varepsilon^2 \sum_{\substack{i,j=n-2 \\ i \neq j}}^{n} \text{per} \begin{pmatrix} X & A_j \\ A_i^{\mathrm{T}} & a_{ij} \end{pmatrix} - \varepsilon^2 \sum_{i=n-2}^{n} \text{per} \begin{pmatrix} X & A_i \\ A_i^{\mathrm{T}} & a_{ii} \end{pmatrix}.$$

Taking the permanent through the last row, we get

$$S_{n-2}(A, E_1) = \varepsilon^2 \left\{ \left(\sum_{\substack{i,j=n-2 \\ i\neq j}}^{n} a_{ij} - \sum_{i=n-2}^{n} a_{ii} \right) \mathrm{per} X + \sum_{i=1}^{n-3} \sum_{j=n-2}^{n} \left(\sum_{\substack{r=n-2 \\ j\neq r}}^{n} a_{ri} \mathrm{per}(X(i)A_j) - a_{ji} \mathrm{per}(X(i)A_j) \right) \right\},$$

where $(X(i)A_j)$ is a submatrix of order $n-3$ formed by deleting the ith column of X and includes the jth column of A.

Now, taking permanent of $(X(i)A_j)$ by expanding along the column A_j we get,

$$S_{n-2}(A, E_1) = \varepsilon^2 \left\{ \left(\sum_{i,j=n-2}^{n} a_{ij} - 2 \sum_{i=n-2}^{n} a_{ii} \right) \mathrm{per} X + \sum_{i,j=1}^{n-3} \left(\sum_{k,r=n-2}^{n} a_{ir} a_{jk} - 2 \sum_{k=n-2}^{n} a_{ik} a_{jk} \right) \mathrm{per} X(i/j) \right\}.$$

Hence the necessary and sufficient condition for A and $\tilde{A}$ to be permanental mates is $(\sum_{i,j=n-2}^{n} a_{ij} - 2\sum_{i=n-2}^{n} a_{ii})$ $\mathrm{per} X + \sum_{i,j=1}^{n-3} (\sum_{k,r=n-2}^{n} a_{ir} a_{jk} - 2\sum_{k=n-2}^{n} a_{ik} a_{jk}) \mathrm{per} X(i/j) = 0$. □

Let A be a symmetric matrix in Ω_4, we define,

$$T(A) = 4a_{kk}^2 - 2a_{kk}\mathrm{tra}(A) + 1 - 2\sum_{\substack{i=1 \\ i\neq k}}^{4} a_{ki}^2, \quad k = 1, 2, 3, 4.$$

Corollary 2.3. *Let A be a symmetric matrix in Ω_4 and choose $\varepsilon > 0$ such that the perturbation $\tilde{A} = A + E_1 \in \Omega_4$, where $E_1 = 0 \oplus E$. A and $\tilde{A}$ are permanental mates if and only if $T(A) = 0$.*

Proof. Let $X = (a_{11})$. From (3), $\mathrm{per} X(1/1) = 1$. Now, the condition (5) becomes,

$$\begin{aligned}
&\left(\sum_{i,j=n-2}^{n} a_{ij} - 2 \sum_{i=n-2}^{n} a_{ii} \right) \mathrm{per} X + \sum_{i,j=1}^{n-3} \left(\sum_{k,r=n-2}^{n} a_{ir} a_{jk} - 2 \sum_{k=n-2}^{n} a_{ik} a_{jk} \right) \mathrm{per} X(i/j) \\
&\quad = a_{11}(3 - (a_{21} + a_{31} + a_{41}) - 2(a_{22} + a_{33} + a_{44})) + a_{12}(1 - a_{11}) \\
&\qquad + a_{13}(1 - a_{11}) + a_{14}(1 - a_{11}) - 2(a_{12}^2 + a_{13}^2 + a_{14}^2) \\
&\quad = a_{11}(2 + 3a_{11} - 2\mathrm{tra}(A)) + (1 - a_{11})(1 - a_{11}) - 2(a_{12}^2 + a_{13}^2 + a_{14}^2) \\
&\quad = (4a_{11}^2 - 2a_{11}\mathrm{tra}(A) + 1 - 2(a_{12}^2 + a_{13}^2 + a_{14}^2)) \\
&\quad = T(A). \quad \square
\end{aligned}$$

Construct a symmetric matrix $A = (a_{ij}) \in \Omega_4$ as follows: Let $a_{11} = x$, $a_{1j} = y$, for $j = 2, 3, 4$ and $\mathrm{tra}(A) = \frac{10x^2+4x+1}{6x}$. Choose arbitrary small x, $x > 0$, such that $y = \frac{1-x}{3} > 0$. It is easy to see that $T(A) = 0$, hence by the Corollary 2.3, A and $\tilde{A} = A + E_1$ are permanental mates in Ω_4.

In particular, if we take $x = y = \frac{1}{4}$, then

$$A = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{8} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{8} & \frac{1}{2} \end{pmatrix}, \quad 0 < \varepsilon < \frac{1}{8} \quad \text{and} \quad T(A) = 0. \tag{6}$$

Therefore A and $\tilde{A}$ form permanental mates in Ω_4. Hence for some symmetric matrices $A \in \Omega_4$, we can find an infinite number of perturbation matrices, which are permanental mates of A in Ω_4.

Let A be a symmetric matrix in Ω_4. Then the perturbation matrix $\tilde{A} \in M_{4,1}$, where $M_{4,1}$ denote the set of all one matching matrices in Ω_4. As $\tilde{A} \neq \tilde{A}^{\mathrm{T}}$ and $T(\tilde{A}) = T(A) = 0$, by the Theorem 2.2 in [4], $\tilde{A}$ and $\tilde{A}^{\mathrm{T}}$ are permanental mates in Ω_4. Hence the pairs $(A, \tilde{A})$ and $(\tilde{A}, \tilde{A}^{\mathrm{T}})$ are permanental mates in Ω_4. The Theorem 2.2 is only for a symmetric matrix in Ω_n. We can extend to an asymmetric matrix in Ω_3.

Theorem 2.4. *Let $A = M(a, b; c, d)$, $c \neq b$, be in Ω_3 and choose $\varepsilon > 0$ such that $\tilde{A} = A + E \in \Omega_3$. A and the perturbation $\tilde{A}$ are permanental mates if and only if* $\mathrm{tra}(A) = \frac{3}{2}$.

Proof. Necessary and sufficient condition for A and $\tilde{A}$ to be permanental mates is $S_2(A, E) = 0 = S_1(A, E)$, since per $E = 0$.

$$S_2(A, E) = 0 \quad \text{and} \quad S_1(A, E) = 0 \quad \text{if and only if } \mathrm{tra}(A) = \frac{3}{2}. \quad \square$$

Suppose $c = b$ in $A = M(a, b; c, d)$ then $S_2(A, E) = 0$. In this case also A and $\tilde{A}$ are permanental mates if and only if $\mathrm{tra}(A) = \frac{3}{2}$, which is easily verified from the Theorem 2.2.

Consider a matrix

$$A = M\left(\frac{1}{2}, \frac{1}{8}; \frac{3}{8}, \frac{1}{2}\right) \in \Omega_3 \tag{7}$$

and choose $0 < \varepsilon < \frac{3}{8}$ such that $\tilde{A} = A + E \in \Omega_3$, A and $\tilde{A}$ are permanental mates, since $\mathrm{tra}(A) = \frac{3}{2}$.

In [3], Hwang discussed some nontrivial permanental mates of doubly stochastic matrices. Mte(A) denotes the set of all mates of A for $A \in \Omega_n$. Let Int Ω_n denote the interior of Ω_n. In concluding remarks, Hwang [3] asked the question whether there exists an $A \in \mathrm{Int}\,\Omega_n$ with a nontrivial permanental mate. We established such a matrix (7) in Ω_3. The matrices $\tilde{A}$ and A are permanental mates, since $\mathrm{tra}(A) = \frac{3}{2}$. $A \in \mathrm{Int}\Omega_3$ and $\tilde{A} = A + E \in \mathrm{Mte}(A)$ for all ε, $0 < \varepsilon < \frac{3}{8}$. In Ω_4, consider the matrix (6),

$$A \in \mathrm{Int}\Omega_4 \quad \text{and} \quad \tilde{A} = A + E_1 \in \mathrm{Mte}(A) \quad \text{for all } \varepsilon,\ 0 < \varepsilon < \frac{1}{8}.$$

For larger n's, Hwang thought that an affirmative answer to the question regarding the existence of matrices in Int Ω_n having nontrivial mates seems to be not possible. Therefore, Hwang proposed a conjecture, which states that, "for $n \geq 4$, if $A \in \mathrm{Int}\,\Omega_n$, then Mte($A$) is a trivial set".

Karuppanchetty and Maria Arulraj [4] disproved this conjecture by proving necessary and sufficient conditions for $A = \begin{pmatrix} X & U \\ U^{\mathrm{T}} & Y \end{pmatrix} \in \Omega_n$ and that A^{T} are permanental mates, where X is a symmetric matrix and of order $(n-3) \times (n-3)$, U is $(n-3) \times 3$, and Y is $3 \times 3(\mathrm{Y} \neq \mathrm{Y}^{\mathrm{T}})$. Here we have shown the perturbation matrix $\tilde{A}$ as a permanental mate of a symmetric matrix A in Ω_n, which will also disprove the Hwang's conjecture.

Construct a symmetric matrix $A = (a_{ij}) \in \Omega_n$ as follows,

$$a_{ij} = x, \quad \text{for } i, j = 1, 2, \ldots, n-3 \text{ and}$$
$$a_{ij} = y, \quad \text{for } i = 1, 2, \ldots, n-3 \text{ and } j = n-2, n-1, n.$$

Choose arbitrary small x, $x > 0$, such that $y = \frac{1-(n-3)x}{3} > 0$.

Choose the 3×3 bottom right sub matrix (a_{ij}), $i, j = n-2, n-1, n$, such that

$$\left(\sum_{i,j=n-2}^{n} a_{ij} - 2\sum_{i=n-2}^{n} a_{ii}\right) = \frac{-3(n-3)y^2}{x}.$$

Therefore, per $X = (n-3)!x^{n-3}$, per$(X(i/j)) = (n-4)!x^{n-4}$ and

$$\sum_{i,j=1}^{n-3}\left(\sum_{k,r=n-2}^{n} a_{ir}a_{jk} - 2\sum_{k=n-2}^{n} a_{ik}a_{jk}\right) = 3(n-3)^2y^2.$$

It is easy to verify the condition (5) of the Theorem 2.2,

$$\left(\sum_{i,j=n-2}^{n} a_{ij} - 2\sum_{i=n-2}^{n} a_{ii}\right) \operatorname{per} X + \sum_{i,j=1}^{n-3} \left(\sum_{k,r=n-2}^{n} a_{ir}a_{jk} - 2\sum_{k=n-2}^{n} a_{ik}a_{jk}\right) \operatorname{per} X(i/j)$$
$$= \frac{-3(n-3)y^2}{x}(n-3)!x^{n-3} + 3(n-3)^2y^2(n-4)!x^{n-4} = 0.$$

By the Theorem 2.2, A and the perturbation $\tilde{A} = A + E_1 \in \Omega_n$ are permanental mates. Hence for $A \in \operatorname{Int} \Omega_n$, there exist infinitely many perturbation matrices, which are permanental mates of A in Ω_n.

If A and B have permanental mates with C and D respectively then $A \oplus B$ and $C \oplus D$ are permanental mates. In particular, if A and A^{T} are permanental mates in Ω_{n_1}, B and B^{T} are permanental mates in Ω_{n_2} then $A \oplus B$ and $A^{\mathrm{T}} \oplus B^{\mathrm{T}}$ are permanental mates in Ω_n, where $n = n_1 + n_2$.

From the definition of permanental mates, we observed the fact that, if a permanent function is constant on the line segment connecting A and B, then it is constant on its line subsegment connecting a given convex combination of two with any other convex combination. Hence, the convex combination of two permanental mates has a permanental mate.

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