Generalization of the Power Means and Their Inequalities

JOSIP E. PEČARIĆ

Faculty of Technology, University of Zagreb,
Ive Lole Ribara 126, 41000 Zagreb, Yugoslavia

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In this paper we gave a generalization of power means which include positive nonlinear functionals. For these means we obtained generalizations of fundamental mean inequality, Hölder’s and Minkowski’s, and their converse inequalities.


1. INTRODUCTION

Let \( E \) be a nonempty set and let \( L \) be a class of nonnegative functions \( f: E \to \mathbb{R} \). We shall consider functionals \( A: L \to \mathbb{R} \) which satisfy the following conditions:

1. \( f \in L \Rightarrow A(f) \geq 0 \),
2. \( f \in L, \lambda > 0 \Rightarrow \lambda f \in L \) and \( A(\lambda f) = \lambda A(f) \),
3. \( 1 \in L \), that is, if \( f(t) \equiv 1 \ (t \in E) \), then \( f \in L \) and \( A(1) = 1 \),
4. \( f, g \in L \) with \( f(t) \leq g(t) \ (\forall t \in E) \Rightarrow A(f) \leq A(g) \).

Let \( p \in \mathbb{R} \setminus \{0\} \). We shall consider the following generalization of the power means:

\[
M_p(f) = A(f^p)^{1/p},
\]
where \( f \) and \( A \) satisfy the above conditions. It is indeed mean since a simple consequence of (1°)–(4°) is the following implication:

5. \( a \leq f(t) \leq b \ (\forall t \in E, a, b \geq 0) \Rightarrow a \leq M_p(f) \leq b \).

In Section 2 we shall give some inequalities for this generalization of power means. Section 3 contains related results for the above means but in the case when \( A \) satisfies additional supposition

6. \( A(f + g) \leq A(f) + A(g) \ (f, g \in L \Rightarrow f + g \in L) \)
or

\[(7^o) \quad A(f + g) \geq A(f) + A(g).\]

We suppose that all expressions are well-defined.

### 2. Inequalities for a General Class of Means

**Theorem 1.** Let \( r \) and \( s \) be two real numbers such that \( r \leq s, r \neq 0, s \neq 0 \).

If \( 0 < a \leq f(t) \leq b \ (\forall t \in E), f \in L, \) then the following inequalities are valid:

(i) If \( r > 0, \) then

\[
(b/a)^{1-s/r} M_s(f) \leq M_r(f) \leq (a/b)^{1-s/r} M_s(f). \tag{1}
\]

If \( r < 0 \) the reverse inequalities in (1) are valid.

(ii) If \( s > 0, \) then

\[
(a/b)^{1-r/s} M_s(f) \leq M_r(f) \leq (b/a)^{1-r/s} M_s(f). \tag{2}
\]

If \( s < 0 \) the reverse inequalities in (2) are valid.

**Proof.** From \( f \geq a, \) and \( M_s \leq b \) it follows that \( bf \geq aM_s(f), \) i.e.,

\[
bf/(aM_s(f)) \geq 1. \tag{3}
\]

Similarly we get

\[
bf/(aM_s(f)) \geq 1. \tag{4}
\]

From \( M_r \geq a \) and \( f \leq b \) it follows that \( af \leq bM_r(f), \) i.e.,

\[
0 < af/(bM_r(f)) \leq 1. \tag{5}
\]

Similarly we get

\[
0 < af/(bM_s(f)) \leq 1. \tag{6}
\]

Using the monotonic character of exponential functions we obtain

\[
(bf/(aM_r))^r \leq (bf/(aM_r))^r, \tag{7}
\]

\[
(bf/(aM_s))^r \leq (bf/(aM_r))^r, \tag{8}
\]

\[
(af/(bM_r))^r \leq (af/(bM_r))^r, \tag{9}
\]

\[
(af/(bM_s))^r \leq (af/(bM_s))^r. \tag{10}
\]
In the following we use (7):

\[ A(f^r) = A\left(\left(\frac{bf}{aM_r}\right)^r \left(\frac{a/b}{M_r}\right)^r\right) = \left(\frac{a/b}{M_r}\right)^s A\left(\frac{bf}{aM_r}\right)^r \geq (a/b)^s M_r(f)^r A\left(\frac{bf}{aM_r}\right)^r = (a/b)^{r-s} M_r(f)^r. \]

Analogous consequences of (8)–(10) are

\[ A(f^r) \leq (a/b)^{r-s} M_s(f)^r, \quad A(f^s) \leq (b/a)^{s-r} M_s(f)^s \]

and

\[ A(f^r) \geq (b/a)^{r-s} M_s(f)^r. \]

Now, (1) and (2) are simple consequences of these results.

**Theorem 2.** If \( g_1, g_2 \in L, 0 < m_i \leq g_i(t) \leq M_i (\forall t \in E), i = 1, 2, A: L \to R \) satisfy (1°)–(4°), \( 1/k + 1/k' = 1/r \), we write

\[ T(g_1, g_2) = M_r(g_1 g_2)/(M_k(g_1) M_k(g_2)), \]

then we have

\[ (m_1/M_1)^{r/k'} (m_2/M_2)^{r/k} \leq T(g_1, g_2) \leq (M_1/m_1)^{r/k'} (M_2/m_2)^{r/k} \] (11)

when \( k, k'>0 \);

\[ (M_1/m_1)^{r/k'} (m_2/M_2)^{r/k} \leq T(g_1, g_2) \leq (m_1/M_1)^{r/k'} (M_2/m_2)^{r/k} \] (12)

when \( r, k>0, k'<0 \);

\[ (m_1/M_1)^{r/k'} (M_2/m_2)^{r/k} \leq T(g_1, g_2) \leq (M_1/m_1)^{r/k'} (m_2/M_2)^{r/k} \] (13)

when \( r, k'<0, k>0 \);

\[ (m_1/M_1)^{r/k'} (M_2/m_2)^{r/k} \leq T(g_1, g_2) \leq (M_1/m_1)^{r/k'} (M_2/m_2)^{r/k} \] (14)

when \( k, k'<0 \).

**Proof.** The following result is a simple consequence of Theorem 1.

If \( 0 < a \leq x \leq b, y > 0, x, y \in L, A: L \to R \) is defined as in the above with \( A(y) = 1 \), then for \( r \geq 1 \),

\[ (a/b)^{1-1/r} A(xy) \leq A(xy)^{1/r} \leq (b/a)^{1-1/r} A(xy). \] (15)

If \( r \in (0, 1] \), the reverse inequalities in (15) are valid.
Now we obtain (11) in the following way. Let

\[ r - \frac{k}{r}, \quad x = g_1'^r g_2'^{-k} (A(g_2^r))^r/k, \quad y = g_2^k/(A(g_2^r)), \]

\[ a = m_1'^r M_2'^{-k} (A(g_2^r))^r/k, \quad b = M_1'^r m_2'^{-k} (A(g_2^r))^r/k, \]

then (15) becomes

\[ (m_1'/M_1')^{r/k'} (m_2'/M_2')^{r/k} A(g_1^r g_2'^r)/(A(g_j^r))^r/k', \]

\[ \leq (A(g_1^r))^r/k \leq (M_1'/m_1')^{r/k'} (M_2'/m_2')^{r/k} A(g_1^r g_2'^r)/(A(g_j^r))^r/k', \]

i.e., (11).

Inequalities (12) can be proved in a similar way. Let

\[ x = m_1^{r^j} m_2'^{-k} (A(g_2^r))^r/k, \quad b = M_1'^r M_2'^{-k} (A(g_2^r))^r/k, \]

and \( r, x, y \) be the same as above when \( r, k > 0 \) and \( k' < 0 \); then we use the reverse inequalities in (15).

Similarly we can prove (13) and (14).

**Theorem 3.** If \( g_1, g_2, \ldots, g_n \in L, 0 < m_i \leq g_i(t) \leq M_i (\forall t \in E), i = 1, 2, \ldots, n, \)

\( A: L \to R \) satisfies \((1')-(4''), k_1, k_2, \ldots, k_n \) are all positive numbers such that

\[ \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_n} = 1/r, \]

let \( \frac{1}{k_i'} = 1/r - 1/k_i, i = 1, 2, \ldots, n; \)

let

\[ \prod_{i=1}^{n} \left( \frac{m_i}{M_i} \right)^{r/k_i} \leq M_s(g_1 g_2 \cdots g_n)/(\prod_{i=1}^{n} M_{k_i}(g_i)) \]

\[ \leq \prod_{i=1}^{n} \left( \frac{M_i/m_i}{M_j/m_j} \right)^{r/k_i}; \]

(16)

**Proof.** Let us use induction on \( n \). By inequality (11), we obtain inequality (16) in the case \( n = 2 \).

Suppose now that the results hold for \( n = m \); we shall prove that this implies that it holds for \( n = m + 1 \). Since \( 1/k_{m+1} + 1/k_{m+1} = 1/r \) and

\[ m_1 m_2 \cdots m_m \leq g_1 g_2 \cdots g_m \leq M_1 M_2 \cdots M_m, \]

using inequality (11) (the result in the case \( n = 2 \)) for \( g_1, g_2, \ldots, g_m, g_{m+1} \) and exponents \( k_{m+1}, k_{m+1} \), respectively, we have

\[ ((m_1 m_2 \cdots m_m)/(M_1 M_2 \cdots M_m))^{r/k_m+1} (m_{m+1}/M_{m+1})^{r/k_{m+1}} \]

\[ \leq M_s((g_1 g_2 \cdots g_m) \cdot g_{m+1})/(M_{k_{m+1}}(g_1 g_2 \cdots g_m) M_{k_{m+1}}(g_{m+1})) \]

\[ \leq ((M_1 M_2 \cdots M_m)/(m_1 m_2 \cdots m_m))^{r/k_m+1} (M_{m+1}/m_{m+1})^{r/k_{m+1}}. \]

(17)
But since \( \frac{1}{k_1} + \frac{1}{k_2} + \cdots + \frac{1}{k_m} = \frac{1}{k_{m+1}} - \frac{1}{k_j} \), \( j = 1, 2, \ldots, m \); by hypothesis, we have

\[
\prod_{j=1}^{m} \left( \frac{m_j}{M_j} \right)^{r/k_j} \leq M_{k_{m+1}} \left( g_1 g_2 \cdots g_m \right)^{r/k_j} \prod_{j=1}^{m} M_{k_j} (g_j)
\]

\[
\leq \prod_{j=1}^{m} \left( \frac{M_j/m_j}{r/k_j} \right)^{r/k_j}.
\]

(18)

Since \( r/k_{m+1} + r/k_j = r(1/k_j - 1/k_j) = r/k'_j \), \( j = 1, 2, \ldots, m \), multiplying (17) by (18), we get

\[
\left( \frac{m_j}{M_j} \right)^{r/k_j} \leq M_r \left( g_1 \cdots g_m g_{m+1} \right)^{r/k_j} \prod_{j=1}^{m+1} M_{k_j} (g_j)
\]

\[
\leq \prod_{j=1}^{m+1} \left( \frac{M_j/m_j}{r/k_j} \right)^{r/k_j},
\]

which means that the results holds for \( n = m + 1 \).

Hence, by induction, the theorem is true for all \( n \geq 2 \).

3. Fundamental Mean Inequality, Hölder's, Minkowski's, and Related Inequalities

Theorem 4. If \( g_1, g_2 \in L \), \( A: L \to R \) satisfy \((1^o)\)-(4") and \((6")\), \( 1/k + 1/k' = 1/r \), and \( T(g_1, g_2) \) is defined as in Theorem 2, then we have

\[
T(g_1, g_2) \leq 1 \quad \text{when either} \quad k, k' > 0 \text{ or } r, k' < 0, k > 0, \quad (19)
\]

and

\[
T(g_1, g_2) \geq 1 \quad \text{when either} \quad r, k > 0, k' < 0 \text{ or } k, k' < 0. \quad (20)
\]

Proof. If \( u > 0, v > 0, 1/u + 1/v = 1, a > 0, b > 0 \), then

\[
ab \leq (1/u)a^u + (1/v)b^v. \quad (21)
\]

By substitutions, \( u = r/k \), \( v = r/k' \), \( a = g_1^k/M_k(g_1)' \), \( b = g_2^k/M_k(g_2)' \), we get from (21)

\[
g_1^k g_2^k/(M_k(g_1) M_k(g_2))'
\]

\[
\leq (r/k)(g_1^k/A(g_1^k)) + (r/k')(g_2^k/A(g_2^k)).
\]
Now, using properties \((4^o)\) and \((6^o)\) we get

\[
A(g_1 g_2/(M_k(g_1) M_k(g_2)))^r \\
\leq (r/k) A(g_1^r)/A(g_1^r) + (r/k') A(g_2^r)/A(g_2^r) = 1,
\]
i.e., \((19)\), if \(k, k' > 0\), and \((20)\) if \(k, k' < 0\).

Since \(1/k = 1/r + 1/(-k')\) and \(r > 0, -k' > 0\), using \((19)\) (for \(k, k' > 0\)) we get \((20)\) for \(r, k > 0, k' < 0\). Similarly, using \((20)\) for \(k, k' < 0\) we get \((19)\) for

\[r, k' < 0, k > 0.\]

By mathematical induction (as in the proof of Theorem 3) we can prove:

**Theorem 5.** If \(g_1, g_2, \ldots, g_n \in L, A: L \rightarrow R\) satisfies \((1^o)-(4^o)\) and \((6^o)\), \(k_1, k_2, \ldots, k_n\) are all positive numbers such that \(1/k_1 + 1/k_2 + \cdots + 1/k_n = 1/r\), then

\[
M_r(g_1 g_2 \cdots g_n) \left(\prod_{i=1}^n M_{k_i}(g_i)\right) \leq 1. \tag{22}
\]

**Theorem 6.** If \(g_1, g_2, \ldots, g_n \in L, A: L \rightarrow R\) satisfies \((1^o)-(4^o)\) and \((6^o)\), \(r > 0, k_1 > 0, k_2 < 0, \ldots, k_n < 0\) are real numbers such that \(1/k_1 + 1/k_2 + \cdots + 1/k_n = 1/r\), then the reverse inequality in \((22)\) holds.

*Proof.\) By substitutions, \(r \rightarrow k_1, k_1 \rightarrow r, k_i \rightarrow -k_i\) \((i = 2, \ldots, n)\)

\[g_1 \rightarrow g_1 g_2 \cdots g_n, \quad g_i \rightarrow g_i^{-1}\]

\((i = 2, \ldots, n)\), we obtain Theorem 6 from Theorem 5.

**Theorem 7.** Let \(r\) and \(s\) be two real numbers such that \(r \leq s, r \neq 0, s \neq 0\). Then, for \(f \in L, \) and \(A: L \rightarrow R\) which satisfies \((1^o)-(4^o)\) and \((6^o)\),

\[
M_r(f) \leq M_s(f). \tag{23}
\]

*Proof.\) This is a simple consequence of Theorem 4 for \(g_1 = f, g_2 = 1\).

**Theorem 8.** Let \(f, g \in L, \) and \(A: L \rightarrow R\) satisfies \((1^o)-(4^o)\) and \((6^o)\). If \(p > 1\), then

\[
M_p(f + g) \leq M_p(f) + M_p(g). \tag{24}
\]

*Proof.\) As in the proof of the ordinary Minkowski inequality, we write

\[(f + g)^p = f(f + g)^{p-1} + g(f + g)^{p-1}.\]

Applying \(A\) to this, \((19)\) then yields

\[A((f + g)^p) \leq \{A(f^p)^{1/p} + A(g^p)^{1/p}\} A((f + g)^p).\]

Hence, \((24)\) follows.
Of course, results for Hölder's and Minkowski's inequalities can be formulated without conditions $A(1) = 1$, i.e., for nonnormalized functional $A: L \to R$. For example, the following result is a special case of (19).

If $0 < \lambda < 1$, then

$$A(f^\lambda g^{1-\lambda}) \leq A(f)^\lambda A(g)^{1-\lambda}. \quad (25)$$

For $f \to f^r$, $g \to f^s$, we get

$$A(f^{r\lambda} g^{1-r\lambda}) \leq A(f)^r A(g)^{1-r}, \quad (0 \leq \lambda \leq 1).$$

Hence the function $G(r) = A(f^r)$ is logarithmically convex (and so is convex) on $R$ if $A: L \to R$ satisfies the conditions (1'), (2'), (4'), and (6').

This result is a generalization of a result from [2]. In fact all previous results are generalizations of some results from [1, 4]. Similarly we can prove the following generalizations of some results from these papers:

1. Let $p$ and $x$ be two real $n$-tuples such that $\sum_{i=1}^n p_i = 0$, and for $k = 1, ..., n$, $\sum_{i=1}^n p_i |x_i - x_k| \geq 0$. Then

$$1 \leq A(f^{x_1})^{p_1} \cdots A(f^{x_n})^{p_n}. \quad (26)$$

2. Let

$$g(x) = \prod_{j=1}^n A \left( f_{q_j} \left( \prod_{k=1}^n f_{q_k}^{x} \right)^{-q_j} \right)^{1/q_j},$$

where $q_i > 0$ ($i = 1, ..., n$) with $1/q_1 + \cdots + 1/q_n = 1$, $r \in R$. If $|x| \leq |y|$ $(xy > 0)$, then

$$g(x) \leq g(y). \quad (27)$$

3. Denote by $S_r(g) := A(g^r)^{1/r}$ ($r \neq 0$). If $A(1) = 1$ we have $S_r(g) = M_r(g)$. Liapunov's inequality now becomes

$$S_t(g) \leq S_s(g)^{(r/s)/(r-s)/(r-t)} S_r(g)^{(t/s)/(s-r)/(r-t)} \quad (0 < t < s < r). \quad (28)$$

Hence, by the arithmetic-geometric mean inequality

$$S_t(g) \leq \frac{r-s}{s-r} S_r(g) + \frac{t-s}{s-r} S_s(g)$$

or for $S_r(g) < S_s(g)$

$$\frac{S_r(g) - S_s(g)}{S_r(g) - S_s(g)} \leq \frac{s(r-t)}{t(r-s)} \quad (0 < t < s < r), \quad (29)$$

and reverse inequality for $S_r(g) < S_s(g)$.
Also we have that the function \( f(s) = S_{1/s}(g) \) is logarithmically convex (hence convex) for \( s > 0 \).

**Remark.** In fact (2) is a generalization of a result from [5].

As in [3] we can prove the following results.

**Theorem 9.** Let \( f, g \in L \) and \( A: L \to R \) satisfies conditions \((1^o), (2^o), (4^o), (6^o)\). If
\[
g_0^2 > A(g^2) > 0 \quad \text{(or } f_0^2 > A(f^2) > 0)\text{,}
\]
where \( g_0, f_0 \) are real numbers, then
\[
(f_0 g_0 - A(fg))^2 \geq (f_0^2 - A(f^2))(g_0^2 - A(g^2)).
\]

**Theorem 10.** Let \( f, g \in L \) and \( A: L \to R \) satisfies conditions \((1^o), (2^o), (4^o), (6^o)\). If \( p > 1 \), \( p^{-1} + q^{-1} = 1 \), and \( f_0, g_0 \) are positive numbers such that
\[
g_0^p > A(g^p) > 0 \quad \text{and} \quad f_0^p > A(f^p) > 0,
\]
then
\[
(f_0^p - A(f^p))^{1/p} (g_0^q - A(g^q))^{1/q} \leq f_0 g_0 - A(fg).
\]
In case \( p < 1 \) (\( p \neq 0 \), \( p^{-1} + q^{-1} = 1 \)) the inequality (31) is reversed.

**Theorem 11.** Let \( f, g \in L \) and \( A: L \to R \) satisfies conditions \((1^o), (2^o), (4^o), (6^o)\). If \( p > 1 \) and if \( f_0, g_0 \) are positive numbers such that
\[
g_0^p > A(g^p) > 0 \quad \text{and} \quad f_0^p > A(f^p) > 0
\]
then
\[
(f_0^p - A(f^p))^{1/p} + (g_0^p - A(g^p))^{1/p} \\
\leq ((f_0 + g_0)^p - A((f + g)^p))^{1/p}.
\]

These results are a generalization of the well-known Aczel, Popoviciu, and Bellman inequalities.

Now, as in [3], we can consider the function \( P(r) = f_0^r - A(f^r) \), positive on an interval \( J \). This function is logarithmically concave on \( J \), and we can give some applications of this result similar to results given in (1)–(3).

A very important functional which satisfies \((1^o), (2^o), (4^o), (6^o)\) is an increasing gauge in the vector space of numerical functions (see [4]).

In the case when \((7^o)\) (instead \((6^o)\)) is valid we have the following results.
THEOREM 12. Let \( f, g \in L \), and \( A: L \to R \) satisfies (1'), (2'), (4'), (7'). If either (i) \( p, q, r > 0 \), or (ii) \( p, -q, -r > 0 \), or (iii) \( -p, q, -r > 0 \) is valid and if \( 0 < m \leq f(x)^p g(x)^q \leq M \) for \( x \in E \), then

\[
A(f^p g^q)^{1/r} \leq |p|^{1/p} |q|^{1/q} |r|^{-1/r} \frac{(M - m)^{1/p} |mM^{p/r} - Mm^{p/r}|^{1/q}}{|M^{p/r} - m^{p/r}|^{1/r}} \times A(f^p g^q)^{1/q}, \tag{33}
\]

and if either (iv) \( p, q, r < 0 \), or (v) \( -p, q, r > 0 \), or (vi) \( p, -q, r > 0 \), is valid then the reverse inequality in (33) is valid.

In the case when either (i), or (ii), or (iv), or (v) is valid then

\[
(M - m) A(f^p) + (mM^{p/r} - Mm^{p/r}) A(g^q) \leq (M^{p/r} - m^{p/r}) A(f^p g^q), \tag{34}
\]

and the reverse inequality is valid if either (iii) or (vi) holds.

Proof. If \( 0 < m \leq x^p y^{-q/r} \leq M \), then, for example, in the case \( p, q, r > 0 \), the following inequality is valid (see [3]):

\[
(M - m)x^p + (mM^{p/r} - Mm^{p/r}) y^q \leq (M^{p/r} - m^{p/r})x^p y^q.
\]

Put in this inequality \( x \to f(t) \) and \( y \to g(t) \), and apply \( A \), we get (34). (Other cases can be proved similarly.) Now, rewrite (34) in the form

\[
\frac{(r/p)((p/r)(M - m) A(f^p)) + (r/q)((q/r)(mM^{p/r} - Mm^{p/r}) A(g^q))}{\leq (M^{p/r} - m^{p/r}) A(f^p g^q)},
\]

and apply the arithmetic-geometric means inequality on the left-hand side and we get (33). The proof is similar in the other cases.

For \( g \equiv 1 \) (\( \forall t \in E \)), we get the well-known conversion of fundamental mean inequality.

THEOREM 13. Let \( f \in L \), and \( A: L \to R \) satisfies conditions (1')-(4') and (7'). If \( r \leq s, r \neq 0, s \neq 0, 0 < m \leq f(t) \leq M \) (\( \forall t \in E \)), \( \lambda = M/m \), then

\[
M_s(f)/M_r(f) \leq \left( \frac{s - r}{\lambda^s - \lambda^r} \right)^{1/r - 1/s} \left( \frac{\lambda^{s-1}}{s} \right)^{1/r} \left( \frac{r}{\lambda^{r-1}} \right)^{1/s}. \tag{35}
\]

Also, we have

\[
(M' - m')(M_s(f))^r - (M_s - m') (M_r(f))^s \leq M'm^s - M^s m'
\]

if \( 0 < r < s \), or \( r < 0 < s \), and the opposite inequality if \( r < s < 0 \).
REFERENCES


