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To see symmetry in a forest of trees

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Abstract

The exact symmetry identities among four-point tree-level amplitudes of bosonic open string theory as derived by G.W. Moore are re-examined. The main focuses of this work are: (1) Explicit construction of kinematic configurations and a new polarization basis for the scattering processes. These setups simplify greatly the functional forms of the exact symmetry identities, and help us to extract easily high-energy limits of stringy amplitudes appearing in the exact identities. (2) Connection and comparison between D.J. Gross's high-energy stringy symmetry and the exact symmetry identities as derived by G.W. Moore. (3) Observation of symmetry patterns of stringy amplitudes with respect to the order of energy dependence in scattering amplitudes.

1. Introduction

Given many tremendous progresses and miraculous achievements, string theory as we know it today is still a beautiful work under construction [1,2]. While the lack of a full non-perturbative background-independent definition [3,4] may await for an unexpected breakthrough, the current formulation does not follow the wisdom of previous paradigms such as Einstein's general theory of relativity or the standard model of the particle physics. Nevertheless, one can hardly imagine that a symmetry principle would be irrelevant under a proper formulation of string theory. To this

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end, many people have addressed this issue over past decades. See, for example, [5]. Notable examples are the high-energy symmetry as proposed by D.J. Gross [6,7] and the exact symmetry identities derived by G.W. Moore [8]. In the former context, one observes linear relations among high-energy scattering amplitudes among stringy excitations at the same level, and inter-level symmetry patterns for leading four-point tree-level amplitudes were proposed in [9]. In the latter context, based on a well-defined algebraic algorithm (described by a bracket operation defined in Section 2), one establishes exact identities among inter-level stringy scattering amplitudes.

The basic idea behind high-energy symmetry as envisioned by Gross et al. [6] is to view string theory as a higher-spin gauge theory with spontaneous symmetry breakdown. Here all higher-level stringy excitations gain their masses through a higher-spin generalization of Higgs mechanism [5]. Furthermore, if we combine both the master formula [9] of tree-level stringy amplitudes for all transverse-polarized highest spin states at given mass levels, together with the linear relations among leading high-energy scattering amplitudes, these patterns strongly suggest an underlying structure of string theory as that of equivalent theorem in the electroweak theory [10]. The fact that we are able to deduce linear relations among the stringy scattering amplitudes [9] provides further evidences that the high-energy symmetry is a reflection of global symmetry associated with the would-be Goldstone particles in the unbroken phase of string theory.

In contrast, the advantages of the identities of Moore are: (1) the derivation of the identities is based on a clear algebraic structure of symmetry, some subsectors of the bracket states and their bracket relations, based on the bracket operation, can be described in explicit mathematical frameworks, e.g. [11]. (2) While there are infinitely many exact relations one can write down based on bracket algebra, these are not totally independent identities. In fact, one of the special features of these exact symmetry relations is that, it almost realizes a "bootstrap" scenario which allows us to derive infinite many scattering amplitudes among massive stringy excitations based on the Veneziano amplitude. (3) These are exact identities among stringy amplitudes, no special kinematic limits are taken (either high-energy [12] or Regge limits [14]).

On the contrary, to display the contents of these exact identities, especially in terms of all explicit kinematics (momenta, polarizations) are definitely not a trivial task. There are several immediate issues which demand special efforts. For instance: (1) What are the physical characteristics (momenta, polarizations) of a state generated by bracket computation (referred to as a bracket state henceforth)? (2) From the structure of bracket computation (to be reviewed in Section 2), it is clear that the exact identities generally relate stringy excited states at different levels. In order to view all stringy states and their scattering amplitudes as representations of a huge symmetry underlying string theory, it is natural to ask if bracket states generate full stringy spectrum? (3) Note that due to the "dressing" of the deformer to various seed operators (to be defined in Section 2), the stringy amplitudes as related by the exact symmetry identities in general have different kinematic configurations. Specifically, all amplitudes involved in a given symmetry identity describe different scattering processes, in which participant particles may have different spins and momenta. In application to four-point scattering amplitudes, for example, while we know that the explicit form of any Lorentz invariant four-point amplitudes must be a function of two Mandelstam variables (s, t), there is no guarantee that all four four-point amplitudes appearing in an exact symmetry identity share the same set of Mandelstam variables.

In view of these, the symmetry identities as derived from the bracket algebra not only are generic inter-level symmetry relations but also connect amplitudes with different kinematic configurations. Clearly, these relations are based on a well-defined infinite-dimensional symmetry algebra and may cover a wider energy region as compared with, say, high-energy symmetries à la Gross. Nevertheless, from a physicist's point of view, if we believe that all scattering amplitudes form representations of the grand symmetry of string theory, one would like to have explicit actions on the scattering amplitudes as explicit functions of Mandelstam variables. Indeed, while neither symmetry relations mentioned above cover complete patterns of the stringy amplitudes, it is still of interest and importance to see if we can make connections between these two approaches. To achieve this goal, we need to pin down the explicit kinematic dependence of the exact identities and study their high-energy behaviors. In this paper, we begin the first exploration of such a connection/unification based on a couple of case studies. We make a detailed comparison of the spectrum of stringy scattering states as generated from bracket algebra and identify a new kinematic basis for the decomposition of the polarization tensors. Most importantly, through the choice of proper basis of the string state Fock space (Verma module), we obtain much-simplified representations of the exact identities which allow us to extract high-energy limits easily. Though we have worked out two specific cases, they already provide a couple of essential and generic features of these exact/high-energy relations. We therefore believe that the study worked out here will be a good starting point toward more general understanding of Moore's relation.

This paper is organized as follows: we first give a brief review of Moore's derivation of exact identities among string amplitudes in Section 2. Then we discuss the condition of conformal invariance on the bracket states and study their spectrum in Section 3. In order to examine the explicit kinematic dependence of the stringy amplitudes, we give a detailed study of the 4-point kinematic configuration in Section 4. Here we also construct a new basis set for the helicity vector/tensor (q-orthonormal basis) which leads to improved expressions of exact identities. In Section 5, we investigate how the physical bracket states are related to the conventional positive-norm states as well as light-cone like physical states based on Del Giudice, Di Vecchia, and Fubini (DDF) operators [15,1] (referred to as DDF states). It is also discussed that the derivations of the exact symmetry identities of the stringy tree amplitudes with explicit kinematic dependence. Two explicit cases are used to illustrate our idea. Finally, based on the explicit constructions, we study the high-energy expansions of the exact identities and compare them with previous work in Section 6. Section 7 consists of summary and the discussion of future directions related to this work.

To streamline our discussion, we only use simple examples in the main text for various explanations. Some technical details and further illustrative examples are collected in the appendices for reference. Appendix A aims at supplying discussion about necessary and sufficient conditions to make bracket operators conformally invariant. We give a simple explanation and useful formulas of DDF states in Appendix B. Finally, we discuss some subtleties regarding the choice of reference kinematic variables in the study of high-energy limits of Moore's exact identities in Appendix C.

2. A brief review of G.W. Moore's derivation

2.1. Outline of the basic idea

Let us first begin with a brief review of the argument of Moore [8]. Throughout this paper we use convention $X^\mu(w)X^\nu(z) \sim -2\alpha'\eta^{\mu\nu}\log(w-z)$ for string world-sheet propagators. Starting with dimension 1 chiral currents $J(q,w)$ (referred to as the deformer) and $V(k,z)$ (referred to as the seed operator) which carry the momenta q^μ and k^μ respectively, we define a new operator by

$$V^{\mathrm{br}}(\tilde{k}, z) = \big\{\mathcal{J}(q), V(k, z)\big\} \equiv \oint_z \frac{dw}{2\pi i} J(q, w) V(k, z), \tag{2.1}$$

where $\mathcal{J}(q)$ is the integrated operator of the current $J(q, w)$, and $\tilde{k} = k + q$ is a deformed momentum associated with this new operator which we call a bracket operator. This expression is well-defined when $\mathcal{J}(q)$ and $V(k, z)$ are mutually local, $2\alpha' q \cdot k \in \mathbb{Z}$. When $J(q, w)$ and $V(k, z)$ are primary, it is easy to see that $V^{\mathrm{br}}(\tilde{k}, z)$ is also primary and defines a physical vertex operator. We shall revisit physical state conditions later, and temporarily we assume that both $V(k, z)$ and $V^{\mathrm{br}}(\tilde{k}, z)$ are primary.

Now we look at four-point tree-level amplitudes of bosonic open string theory,

$$\mathcal{A}\big[\{\mathcal{V}_i(k_i)\}\big] = \int_0^1 dx \big\langle V_1(k_1, x) V_2(k_2, 0) V_3(k_3, 1) V_4(k_4, \infty)\big\rangle, \tag{2.2}$$

where $\mathcal{V}_i(k_i)$ are again integrated vertex operators. On the right hand side, we did not write explicitly the ghost part which should be understood in a standard way. It should be noted that the string scattering amplitude includes integration over the other domains, $-\infty < x < 0$ and $1 < x < \infty$, and also the contribution from the different ordering of the vertex operators. However, the relations among the scattering amplitudes we deal with in this paper are already manifest in this part, as we will see, so we concentrate on this part of the scattering amplitudes. The scattering amplitude is given as a function of independent momentum invariants, $k_i \cdot k_j$, which we choose as the standard Mandelstam variables,

$$s = -(k_1 + k_2)^2, \qquad t = -(k_1 + k_3)^2, \tag{2.3}$$

for the four-point scattering amplitudes, as well as an independent set of polarization invariants, such as $\zeta_i \cdot k_j$ or $\zeta_i \cdot \zeta_j$.

Now let us turn to unintegrated correlation functions with deformed operators. We may consider the second operator at $z = 0$ to be deformed by the action of $\mathcal{J}(q)$ operator,

$$\big\langle V_1(k_1, x)\big\{\mathcal{J}(q), V_2(k_2, 0)\big\} V_3(k_3, 1) V_4(k_4, \infty)\big\rangle. \tag{2.4}$$

It should be noted that the momentum conservation condition now includes q^μ,

$$q^\mu + \sum_{i=1}^{4} k_i^\mu = 0. \tag{2.5}$$

By deforming the contour and integrating x from 0 to 1, we obtain a relation among scattering amplitudes,

$$\begin{aligned} 0 = {} & \mathcal{A}\big[\mathcal{V}_1(k_1)\mathcal{V}_2^{\mathrm{br}}(\tilde{k}_2)\mathcal{V}_3(k_3)\mathcal{V}_4(k_4)\big] \\ & + (-1)^{2\alpha' q\cdot k_1}\mathcal{A}\big[\mathcal{V}_1^{\mathrm{br}}(\tilde{k}_1)\mathcal{V}_2(k_2)\mathcal{V}_3(k_3)\mathcal{V}_4(k_4)\big] \\ & + (-1)^{2\alpha' q\cdot (k_1+k_3)}\mathcal{A}\big[\mathcal{V}_1(k_1)\mathcal{V}_2(k_2)\mathcal{V}_3^{\mathrm{br}}(\tilde{k}_3)\mathcal{V}_4(k_4)\big] \\ & + (-1)^{2\alpha' q\cdot (k_1+k_3+k_4)}\mathcal{A}\big[\mathcal{V}_1(k_1)\mathcal{V}_2(k_2)\mathcal{V}_3(k_3)\mathcal{V}_4^{\mathrm{br}}(\tilde{k}_4)\big]. \end{aligned} \tag{2.6}$$

Recall that the position of the unintegrated vertex operator $V_1(k_1, x)$ is $0 < x < 1 < \infty$. For this procedure to be well-defined, all $2\alpha' q \cdot k_i$ have to be integral, while each pair of k_i and k_j does not need to be mutually local. In [8], this relation has been employed to derive functional equations for scattering amplitudes, which turn out to suffice for determining tachyon scattering amplitudes up to a constant. This is an intriguing result, but in this article we revisit these relations from the viewpoint of the standard scattering amplitudes in the center-of-momentum frame and their linear relations at high energy.

To make this point clearer, we look closer to the deformation of vertex operators and also recall a (fixed-angle) high-energy limit in the string scattering amplitudes. For four-point scattering amplitudes, we may prepare momenta k_i and an extra momentum q^μ to satisfy the following

$$k_i^2 = -m_i^2, \qquad q^2 = -m_q^2, \qquad 2\alpha' q \cdot k_i = n_i \quad (n_i \in \mathbb{Z}), \tag{2.7}$$

where the momentum conservation condition (2.5) is also imposed. These conditions lead to a consistency condition,

$$\sum_{i=1}^{4} n_i = 2\alpha' m_q^2. \tag{2.8}$$

The deformed momenta satisfy mass-shell conditions,

$$\alpha' \tilde{k}_i^2 = \alpha'(k_i + q)^2 = -\alpha' m_i^2 + n_i - \alpha' m_q^2 \equiv -\alpha' \tilde{m}_i^2, \tag{2.9}$$

where $\alpha' \tilde{m}_i^2$ are again integers. Therefore the level of the vertex operator $V_i(k)$ is shifted by $\alpha' m_q^2 - n_i$. It is easy to see from the bracket computation that if the deformed mass $\alpha' \tilde{m}_i^2 \le -2$, the deformed operator identically vanishes. The Mandelstam variables for the physical momenta in the second amplitude in (2.6) are defined as

$$\begin{aligned} s &\equiv -(\tilde{k}_1 + k_2)^2 = -(k_1 + k_2)^2 - \frac{1}{2\alpha'}(n_1 + n_2) + m_q^2, \\ t &\equiv -(\tilde{k}_1 + k_3)^2 = -(k_1 + k_3)^2 - \frac{1}{2\alpha'}(n_1 + n_3) + m_q^2, \end{aligned} \tag{2.10}$$

Following similar definitions, we obtain the relations between Mandelstam variables in various scattering amplitudes related by a exact symmetry identity.

Now we look at, for example, the second amplitude in (2.6),

$$\mathcal{A}\big[\mathcal{V}_1^{\mathrm{br}}(\tilde{k}_1)\mathcal{V}_2(k_2)\mathcal{V}_3(k_3)\mathcal{V}_4(k_4)\big]. \tag{2.11}$$

If we take a high-energy limit,[1] $\alpha' s \to \infty$ with t/s fixed, then each component of each momentum also goes to infinity; for example, $\tilde{k}_1^0, |\vec{\tilde{k}}_1| \to \infty$, where $\vec{\tilde{k}}_1$ is the spatial part of 26-momentum $\tilde{k}_1^\mu$. The same is true for the other momenta. On the other hand, the inner products of q^μ with these momenta, and also itself, are all constant. Therefore, each component of q^μ is $\mathcal{O}(1)$ or less. This means that the deformation due to q^μ becomes negligible in the high-energy limit, and we can obtain a high-energy relation among the usual scattering amplitudes with the same external momenta. The purpose of this article is to make this observation more precise, and demonstrate how the high-energy relations are obtained by use of a couple of concrete examples. We shall examine the bracket relation in terms of conventional scattering amplitudes, and also explore the relation to the amplitudes based on Del Giudice, Di Vecchia and Fubini (DDF)

[1] In the high-energy regime, the string scattering amplitudes are extremely soft and damped exponentially. In this paper, we compare the high-energy limit of the amplitudes up to a common exponentially damping part. See (5.6) in Section 5.1.

operators. The DDF operators are spectrum generating operators in string theory, and play an important role, for example, in proving the no-ghost theorem. The DDF states spanned by the action of the DDF operators thus form a convenient basis of the positive norm states. The DDF amplitudes, associated with these DDF states, have an advantage that the patterns of the energy hierarchy are much more transparent than those of conventional amplitudes. They therefore prove to be a particularly convenient basis when we discuss high-energy asymptotic relations among scattering amplitudes [13,16,17], as we will briefly explain in Section 6.1.

2.2. Mass and level parameters in our case studies

In the following sections, we investigate the properties of the bracket states and kinematics. We shall first write down the expressions of vertex operators

$$V_{(3)}(k,z;\zeta)=:\left[\frac{-i\zeta_{\mu\nu\rho}}{(2\alpha')^{3/2}}\partial X^\mu\partial X^\nu\partial X^\rho-\frac{\zeta_{\mu;\nu}}{2\alpha'}\partial^2X^\mu\partial X^\nu+\frac{i\zeta_\mu\partial^3X^\mu}{2\sqrt{2\alpha'}}\right]e^{ik\cdot X}:(z), \quad (2.12)$$

$$V_{(2)}(k,z;\zeta)=:\left(\frac{-\zeta_{\mu\nu}}{2\alpha'}\partial X^\mu\partial X^\nu+\frac{i}{\sqrt{2\alpha'}}\zeta_\mu\partial^2X^\mu\right)e^{ik\cdot X}:(z), \quad (2.13)$$

$$V_{(1)}(k,z;\zeta)=\frac{i\zeta\cdot\partial X}{\sqrt{2\alpha'}}e^{ik\cdot X}(z), \qquad V_{(0)}(k,z)=:e^{ik\cdot X}:(z), \quad (2.14)$$

where the subscript of $V_{(\ell)}(k_i)$, (ℓ), denotes the level of the vertex operator; (0) is for a tachyon, (1) for a massless state, and so on. In the argument of $V_{(n)}(k,z;\zeta)$, ζ schematically stands for the set of polarization tensors. A deformer at level n is represented by $J_{(n)}(q,w)=V_{(n)}(q,w;\zeta_q)$, and its polarization tensors are usually denoted as ζ_q otherwise specified. As in (2.1), bracket operators are written with the superscript "br", $V^{\rm br}_{(n)}(\tilde{k},z;\tilde{\zeta})$. The deformation of bracket operation appears as a special form of the polarization tensors (as well as the shift of the momentum by q), as we are about to see.

We will mainly consider the following example,

$$m_1^2=m_q^2=0, \qquad m_2^2=m_3^2=m_4^2=-1/\alpha', \qquad n_1=n_2=-1, \qquad n_3=n_4=1,$$

which implies $\tilde{m}_1^2=1/\alpha'$ and $\tilde{m}_2^2=0$. It also gives $\tilde{n}_1=2\alpha'\tilde{k}_1\cdot q=-1$ and $\tilde{n}_2=2\alpha'\tilde{k}_2\cdot q=-1$. Namely, we prepare the following deformer operator $J_{(1)}(q,w)$ and seed operators, $V_{(1)}(k_1,z;\zeta_1)$ and $V_{(0)}(k_i,z)$ $(i=2,3,4)$. This choice of the parameters leads to

$$\mathcal{A}\big[\mathcal{V}^{\rm br}_{(2)}(\tilde{k}_1)\mathcal{V}_{(0)}(k_2)\mathcal{V}_{(0)}(k_3)\mathcal{V}_{(0)}(k_4)\big]=\mathcal{A}\big[\mathcal{V}_{(1)}(k_1)\mathcal{V}^{\rm br}_{(1)}(\tilde{k}_2)\mathcal{V}_{(0)}(k_3)\mathcal{V}_{(0)}(k_4)\big]. \quad (2.15)$$

Since $\tilde{m}_3^2=\tilde{m}_4^2=-2/\alpha'$, the corresponding operators identically vanish, and the relation involves only these two amplitudes. The explicit expressions of the polarization tensors of the bracket operators, $V^{\rm br}_{(2)}(\tilde{k}_1,z;\zeta^{(2)})$ and $V^{\rm br}_{(1)}(\tilde{k}_2,z;\zeta_R)$ in terms of the seed and the deformer are (for simplicity, $\alpha'=1/2$ in these expressions)

$$\begin{aligned}\zeta^{(2)}_{\mu\nu}(\zeta_1,\zeta_q)&=(\zeta_q\cdot k_1)q_{(\mu}\zeta_{1\nu)}-(\zeta_1\cdot q)q_{(\mu}\zeta_{q\nu)}+\zeta_{q(\mu}\zeta_{1\nu)}\\&\quad+\frac{1}{2}\big((\zeta_q\cdot\zeta_1)-(\zeta_q\cdot k_1)(\zeta_1\cdot q)\big)q_\mu q_\nu,\end{aligned} \quad (2.16)$$

$$\zeta^{(2)}_\mu(\zeta_1,\zeta_q)=-(\zeta_1\cdot q)\zeta_{q\mu}+\frac{1}{2}\big((\zeta_q\cdot\zeta_1)-(\zeta_q\cdot k_1)(\zeta_1\cdot q)\big)q_\mu, \quad (2.17)$$

$$\zeta_{R\mu}(\zeta_q)=(\zeta_q\cdot k_2)q_\mu+\zeta_{q\mu}. \quad (2.18)$$

In the relation, we call the left hand side $\mathcal{A}[\tilde{2}000]$ amplitude and the right hand side $\mathcal{A}[1\tilde{1}00]$ by using the sequences of the levels. The tilde for the level number stands for deformed (bracket) operators. In later sections, we frequently refer to this example as "Case study I: $\mathcal{A}[\tilde{2}000] = \mathcal{A}[1\tilde{1}00]$". The Mandelstam variables for $\mathcal{A}[\tilde{2}000]$ side are defined by $s_{[\tilde{2}000]} = -(\tilde{k}_1 + k_2)^2$ and $t_{[\tilde{2}000]} = -(\tilde{k}_1 + k_3)^2$. On the other hand, on $\mathcal{A}[1\tilde{1}00]$ side, they are given as $s_{[1\tilde{1}00]} = -(k_1 + \tilde{k}_2)^2$ and $t_{[1\tilde{1}00]} = -(k_1 + k_3)^2$. Using the mass-shell conditions and the values of $q \cdot k_i$, one can find that these two variables are equivalent, $s_{[\tilde{2}000]} = s_{[1\tilde{1}00]}$ and $t_{[\tilde{2}000]} = t_{[1\tilde{1}00]}$. We therefore simply write them as s and t, and the relation between the amplitudes is understood as the relation of functions of these s and t, $\mathcal{A}[\tilde{2}000](s,t) = \mathcal{A}[1\tilde{1}00](s,t)$.

In another example we will consider, we prepare a massive deformer operator $J_{(2)}(q,w)$, and the same set of seed operators, $V_{(1)}(k_1, z; \zeta_1)$ and $V_{(0)}(k_i, z)$. With the following choice of the parameters,

$$m_1^2 = 0, \qquad m_q^2 = 1/\alpha',$$
$$m_2^2 = m_3^2 = m_4^2 = -1/\alpha', \qquad n_1 = n_2 = -1, \qquad n_3 = n_4 = 2,$$

we obtain the following relation,

$$\mathcal{A}\big[\mathcal{V}_{(3)}^{\mathrm{br}}(\tilde{k}_1)\mathcal{V}_{(0)}(k_2)\mathcal{V}_{(0)}(k_3)\mathcal{V}_{(0)}(k_4)\big] = \mathcal{A}\big[\mathcal{V}_1(k_1)\mathcal{V}_{(2)}^{\mathrm{br}}(\tilde{k}_2)\mathcal{V}_{(0)}(k_3)\mathcal{V}_{(0)}(k_4)\big]. \tag{2.19}$$

This example will be referred to as "Case study II: $\mathcal{A}[\tilde{3}000] = \mathcal{A}[1\tilde{2}00]$". Note that $\tilde{m}_1^2 = 2/\alpha'$, $\tilde{n}_1 = -3$, $\tilde{m}_2^2 = 1/\alpha'$ and $\tilde{n}_2 = -3$. The explicit forms of the polarization tensors of $V_{(3)}^{\mathrm{br}}(\tilde{k}_1, z; \zeta^{(3)})$ and $V_{(2)}^{\mathrm{br}}(\tilde{k}_2, z; \zeta_R)$ are spelled out as (again $\alpha' = 1/2$)

$$\begin{aligned}\zeta^{(3)}_{\mu\nu\rho}(\zeta_q, \tilde{\zeta}_q; \zeta_1) = {} & \zeta_{q(\mu\nu}\zeta_{1\rho)} + 2q_{(\mu}\zeta_{1\nu}(\zeta_q \cdot k_1)_{\rho)} - (\zeta_1 \cdot q)\big[\zeta_{q(\mu\nu}q_{\rho)} + (\zeta_q \cdot k_1)_{(\mu}q_\nu q_{\rho)}\big] \\ & + q_{(\mu}q_\nu(\zeta_q \cdot \zeta_1)_{\rho)} + \frac{1}{2}\Xi_1 q_{(\mu}q_\nu\zeta_{1\rho)} - \frac{1}{6}\Xi_2 q_\mu q_\nu q_\rho,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\zeta^{(3)}_{\mu;\nu}(\zeta_q, \tilde{\zeta}_q; \zeta_1) = {} & \tilde{\zeta}_{q\mu}\zeta_{1\nu} + 2(\zeta_q \cdot k_1)_\mu\zeta_{1\nu} \\ & - (\zeta_1 \cdot q)\big[2\zeta_{q\mu\nu} + 2(\zeta_q \cdot k_1)_\mu q_\nu + q_\mu(\zeta_q \cdot k_1)_\nu + \tilde{\zeta}_{q\mu}q_\nu\big] \\ & + 2(\zeta_q \cdot \zeta_1)_\mu q_\nu + q_\mu(\zeta_q \cdot \zeta_1)_\nu + \frac{1}{2}\Xi_1 q_\mu\zeta_{1\nu} - \frac{1}{2}\Xi_2 q_\mu q_\nu,\end{aligned} \tag{2.21}$$

$$\zeta^{(3)}_\mu(\zeta_q, \tilde{\zeta}_q; \zeta_1) = 2(\zeta_q \cdot \zeta_1)_\mu - 2(\zeta_1 \cdot q)(\zeta_q \cdot k_1)_\mu - 2(\zeta_1 \cdot q)\tilde{\zeta}_{q\mu} - \frac{1}{3}\Xi_2 q_\mu, \tag{2.22}$$

$$\Xi_1 = (k_1 \cdot \zeta_q \cdot k_1) - (\tilde{\zeta}_q \cdot k_1),$$
$$\Xi_2 = (k_1 \cdot \zeta_q \cdot k_1)(\zeta_1 \cdot q) - (\tilde{\zeta}_q \cdot k_1)(\zeta_1 \cdot q) - 2(\zeta_1 \cdot \zeta_q \cdot k_1) + 2(\tilde{\zeta}_q \cdot \zeta_1),$$

$$\zeta_{R\mu\nu}(\zeta_q, \tilde{\zeta}_q) = \zeta_{q\mu\nu} + 2q_{(\mu}\zeta_{q\nu)\rho}k_2^\rho + \frac{1}{2}(k_2 \cdot \zeta_q \cdot k_2 - \tilde{\zeta}_q \cdot k_2)q_\mu q_\nu, \tag{2.23}$$

$$\zeta_{R\mu}(\zeta_q, \tilde{\zeta}_q) = \tilde{\zeta}_{q\mu} + 2\zeta_{q\mu\nu}k_2^\nu + \frac{1}{2}(k_2 \cdot \zeta_q \cdot k_2 - \tilde{\zeta}_q \cdot k_2)q_\mu. \tag{2.24}$$

Here, the polarization tensors of the deformer are represented as $\zeta_{q\mu\nu}$ and $\tilde{\zeta}_{q\mu}$ for distinction. One can check that the Mandelstam variables of both hands sides coincide also in this case, and we write them as s and t.

3. The bracket operators and the spectrum analysis

As we have explained in Section 2, the basic idea underlying the derivation of exact identities among n-point scattering amplitudes is to deform the contour of the bracket operator in a null $n+1$ point scattering amplitude into separate "dressing" of the n individual seed vertex operators. The bracket algebra leads to a relation among n n-point scattering amplitudes, where each amplitude includes one deformed operator and other $n-1$ seed operators. While it is natural to demand that all seed operators are conformal invariant, the nature of the bracket operator requires some explanations. In particular, we will examine the following questions in this section:

- What are necessary and sufficient conditions for the bracket operators to be conformal invariant?
- Do bracket operators at a fixed level generate the complete positive-norm spectrum?

3.1. Conformal invariance of the bracket operators

In order for the relation (2.6) to make sense as a relation among string scattering amplitudes, the deformed operator V^{br} has to be a decent vertex operator. If $J(q,w)$ is a primary operator of dimension 1, the integrated one, $\mathcal{J}(q)$, is a dimension zero operator and commutes with the Virasoro generators. Therefore if a seed operator $V(k,z)$ is also a dimension 1 primary operator, the resultant bracket operator will be a dimension 1 primary operator. Let $\mathcal{J}(q)$ and $\mathcal{V}(k)$ be the integrated operators,

$$\mathcal{J}(q)=\oint\frac{dw}{2\pi i}J(q,w),\qquad \mathcal{V}(k)=\oint\frac{dz}{2\pi i}V(k,z).\tag{3.1}$$

The state constructed by the action of V^{br} is written, by state–operator correspondence through the action of the commutator on the momentum vacuum, as $[\mathcal{J}(q),\mathcal{V}(k)]|0;0\rangle$. The physical state condition is

$$0=\big[L_n,\big[\mathcal{J}(q),\mathcal{V}\big](k)\big]|0;0\rangle=\big(\big[\mathcal{J}(q),[L_n,\mathcal{V}](k)\big]-\big[\mathcal{V}(k),\big[L_n,\mathcal{J}(q)\big]\big]\big)|0;0\rangle,$$

for $n\geq 1$ where we have used Jacobi's identity. L_n represents a Virasoro generator. Therefore it is easy to see that a sufficient condition for V^{br} to be a physical vertex operator is both $J(q,w)$ and $V(k,z)$ being physical. The question is what are necessary conditions. Let us take the bracket operator in $[\tilde{2}000]$ amplitude as an example. The bracket states corresponding to $V^{\text{br}}_{(2)}(\tilde{k},z)$ is (here k represents k_1 in the example)

$$\Big[(\zeta_q\cdot k)(q\cdot\alpha_{-1})(\zeta\cdot\alpha_{-1})-(\zeta\cdot q)(q\cdot\alpha_{-1})(\zeta_q\cdot\alpha_{-1})+(\zeta_q\cdot\alpha_{-1})(\zeta\cdot\alpha_{-1})\\ -(\zeta\cdot q)\zeta_q\cdot\alpha_{-2}+\frac{1}{2}\big((\zeta_q\cdot\zeta)-(\zeta_q\cdot k)(\zeta\cdot q)\big)\big((q\cdot\alpha_{-1})^2+q\cdot\alpha_{-2}\big)\Big]|0;\tilde{k}\rangle,\tag{3.2}$$

and the physical state conditions are

$$0=(\zeta\cdot q)(\zeta_q\cdot q),\\ 0=(\zeta_q\cdot q)\zeta_\mu+(\zeta\cdot k)\zeta_{q\mu}+\big[(\zeta\cdot k)(\zeta_q\cdot k)-(\zeta\cdot q)(\zeta_q\cdot q)\big]q_\mu.\tag{3.3}$$

The first condition requires $\zeta\cdot q=0$ or $\zeta_q\cdot q=0$. When $\zeta_q\cdot q=0$, the second condition says $\zeta\cdot k=0$ or $\zeta_q^\mu=-(\zeta_q\cdot k)q^\mu$. It is easy to see that when $\zeta_q\propto q$, the bracket state (3.2) identically vanishes. So a sensible condition is $\zeta \cdot k = 0$. On the other hand, if we take $\zeta \cdot q = 0$, the second condition is

$$(\zeta_q \cdot q)\zeta^\mu + (\zeta \cdot k)\zeta_q^\mu + (\zeta \cdot k)(\zeta_q \cdot k)q^\mu = 0. \tag{3.4}$$

Contraction with q_μ leads to $(\zeta \cdot k)(\zeta_q \cdot q) = 0$. $\zeta_q \cdot q = 0$ coincides with the previous choice, while with $\zeta \cdot k = 0$, (3.4) implies $\zeta_q \cdot q = 0$ unless $\zeta^\mu = 0$ identically. Therefore, the physical state conditions for the bracket state lead to the conditions, $\zeta \cdot k = \zeta_q \cdot q = 0$, which are nothing but the physical state conditions for each J_q and $V(k)$. So in this case, the sufficient conditions are also the necessary conditions.

However this may not be a general feature. Indeed, in the case of $\alpha' q^2 = -1$, we find physical bracket states generated by a deformer operator with an unphysical choice of polarizations. The details of this example are presented in Appendix A. However, such physical states seem quite special, and in the following discussion we confine ourselves in considering physical bracket states generated by physical deformer and seed operators.

3.2. *Spectrum analysis of the bracket states*

As seen, possible physical states obtained through the bracket operator is governed by the physical polarizations for the deformer operator J_q and the seed operator.

In the previous example, there are 25 choices for each ζ_μ and $\zeta_{q\mu}$. It is easy to see that the bracket state (3.2) are symmetric under the exchange of ζ and ζ_q when physical; As seen, $\zeta_q = q$ makes (3.2) trivially vanish, while it is not difficult to check that $\zeta = k$ gives a null state. Since $q^2 = k^2 = 0$ and $k \cdot q = -1$, k and q are linearly independent, which implies $\zeta \cdot q = \zeta_q \cdot k = 0$ for physical states of positive norm, and the statement follows. Both ζ_μ and $\zeta_{q\mu}$ are transverse to both k and q, and then this choice of seed and deformer operators generates at most 300 physical states, while the total number of positive norm states at level 2 is 324.

This counting will be more vividly illustrated by considering the simplest case; namely, both seed and deformer operators are tachyons, $J_{(0)}(q, w)$ and $V_{(0)}(k, z)$ with $q^2 = k^2 = 2$ $(\alpha' = 1/2)$. The bracket operator $\{\mathcal{J}_{(0)}(q), V_{(0)}(k, z)\}$ is not trivial for $q \cdot k \le -1$, and a first few choices of $q \cdot k$ lead to

$$\begin{aligned}
&: e^{i\tilde{k}\cdot X} : \quad (q \cdot k = -1), \qquad i\zeta_1 \cdot \partial X e^{i\tilde{k}\cdot X} \quad (q \cdot k = -2),\\
&: \left[-\zeta_{2\mu\nu}\partial X^\mu \partial X^\nu + i\zeta_{2\mu}\partial^2 X^\mu\right] e^{i\tilde{k}\cdot X} : \quad (q \cdot k = -3), \quad \cdots
\end{aligned} \tag{3.5}$$

where $\zeta_{1\mu} = q_\mu$, $\zeta_{2\mu\nu} = q_\mu q_\nu/2$, and $\zeta_{2\mu} = q_\mu/2$. These polarization tensors satisfy the physical state conditions and then the bracket operators are physical. In this case, we have only one state at each level.

In general, the number of physical states of a given bracket operator is restricted by the numbers of physical states of the seed and deformer operators and is not much larger than the product of these two numbers.[2] However, as seen from construction, in order to generate a bracket operator at a given level, there are infinitely many possible choices of seed and deformer operators with $q \cdot k$ suitably chosen. Therefore, missing physical states from a choice of seed and deformer operators will be obtained from another choice. These two different choices are in general involved

[2] When necessary conditions agree with sufficient ones, the product gives an upper bound. If not, there can be some extra physical states, but the number of them does not seem so large. See Appendix A.

in different sets of exact relations. Through a possible overlap of states, scattering amplitudes are related to one another in a complicated way and then are highly constrained.

We have observed that Moore's relation is quite powerful to relate infinitely many scattering amplitudes in a very nontrivial way, and these relations hopefully provide some trails of stringy symmetries. As stated in the introduction, we will carry out a first concrete analysis by use of a couple of specific cases. When we come to the consideration in massive inter-level relations, there appear another complication in the choice of momenta and also physical polarizations. In the following sections, we consider the simplest choice of the physical bracket operators and physical states; namely, the ones from physical seed and deformer operators, and the corresponding states. The more systematic analysis will be reserved for future study.

4. Kinematics of the four-point amplitudes

4.1. Kinematic configuration

In this section, we shall give explicit solutions of the kinematic configuration both in the rest frame (of the first particle) and the center-of-momentum frame. All components of seed/bracket momenta can be expressed as functions of the Mandelstam variables s and t, and this will help us in constructing various polarization vectors needed for higher-spin amplitudes.

We start with the kinematic configuration of the scattering processes in the rest frame of the first massive particle ($\tilde{m}_1^2 \neq 0$). For the sake of convenience, we take $\alpha' = 1/2$ in the following discussion. The following setup is the most economical ansatz which is compatible with the momentum conservation.

$$q = (c_0, c_1, c_2, c_3, \vec{0}), \tag{4.1}$$

$$\tilde{k}_1 = k_1 + q = \big(\tilde{k}_1^0, 0, 0, 0, \vec{0}\big), \tag{4.2}$$

$$k_2 = \big(k_2^0, k_2^1, 0, 0, \vec{0}\big), \tag{4.3}$$

$$k_3 = \big(k_3^0, k_3^1, k_3^2, 0, \vec{0}\big), \tag{4.4}$$

$$k_4 = \big(k_4^0, k_4^1, k_4^2, 0, \vec{0}\big) = -\tilde{k}_1 - k_2 - k_3. \tag{4.5}$$

In this rest-frame configuration, we can embed the seed and bracket momenta into a $(1+3)$-dimensional space–time, while the relevant physical momentum($\tilde{k}_1, k_2 \sim k_4$) are confined within the $(1+2)$-dimensional scattering plane. Aside from the fourth momentum k_4 which is fixed by momentum conservation, we have ten unknown components to be solved from the five on-shell conditions ($\tilde{k}_1^2 = -\tilde{m}_1^2$, $k_i = -m_i^2$ $(i = 2, 3, 4)$, and $q^2 = m_q^2$) and the three level number constraints ($\tilde{n}_1 = \tilde{k}_1 \cdot q = n_1 - m_q^2$, $n_2 = k_2 \cdot q$, and $n_3 = k_3 \cdot q$ (n_4 condition is trivial due to the consistency condition (2.8) when momentum conservation is satisfied)). Hence it is natural to expect that we can solve all momenta in terms of two Mandelstam variables, $s = -(\tilde{k}_1 + k_2)^2$ and $t = -(\tilde{k}_1 + k_3)^2$. We also define $\tilde{s} = \tilde{k}_1 \cdot k_2$ and $\tilde{t} = \tilde{k}_1 \cdot k_3$ for convenience.

Through some algebraic manipulations, we find

$$\tilde{k}_1^0 = \tilde{m}_1, \tag{4.6}$$

$$k_2^0 = \frac{\tilde{s}}{2\tilde{m}_1}, \qquad k_2^1 = \frac{\delta_1\sqrt{K_1(s)}}{2\tilde{m}_1}, \tag{4.7}$$

$$k_3^0 = \frac{\tilde{t}}{2\tilde{m}_1}, \qquad k_3^1 = \frac{\delta_1 K_3(s,t)}{2\tilde{m}_1\sqrt{K_1(s)}}, \qquad k_3^2 = \frac{\delta_2}{2\tilde{m}_1}\sqrt{\frac{K_1(s)K_2(t) - [K_3(s,t)]^2}{K_1(s)}}, \tag{4.8}$$

where we have defined $K_1(s) = \tilde{s}^2 - 4\tilde{m}_1^2 m_2^2$, $K_2(t) = \tilde{t}^2 - 4\tilde{m}_1^2 m_3^2$, and $K_3(s,t) = 2\tilde{m}_1^2(\tilde{s} + \tilde{t} + \tilde{m}_1^2 + m_2^2 + m_3^2 - m_4^2) + \tilde{s}\tilde{t}$ to make the equations brief. $\delta_i = \pm 1$ $(i = 1, 2, 3)$ are introduced for the sign ambiguity. On the other hand, the components of the bracket momenta q are:

$$c_0 = -\frac{\tilde{n}_1}{\tilde{m}_1}, \qquad c_1 = \delta_1 \frac{2n_2\tilde{m}_1^2 - \tilde{s}\tilde{n}_1}{\tilde{m}_1\sqrt{K_1(s)}}, \tag{4.9}$$

$$c_2 = \frac{\delta_2}{\tilde{m}_1} \frac{K_1(s)(2\tilde{m}_1^2 n_3 - \tilde{t}\tilde{n}_1) - (2n_2\tilde{m}_1^2 - \tilde{s}\tilde{n}_1)K_3(s,t)}{\sqrt{K_1(s)[K_1(s)K_2(t) - (K_3(s,t))^2]}}, \tag{4.10}$$

$$c_3 = \delta_3\sqrt{-m_q^2 + c_0^2 - c_1^2 - c_2^2}. \tag{4.11}$$

Note that if we demand that all physical momenta have real components, then the items inside the square root should be positive. Hence, we have the following inequalities,

$$K_1(s) \geq 0, \qquad K_1(s)K_2(t) \geq \big(K_3(s,t)\big)^2, \qquad c_0^2 \geq m_q^2 + c_1^2 + c_2^2. \tag{4.12}$$

Having obtained the expressions of various momenta in the rest frame we can derive the kinematic configuration in the center-of-momentum (CM) frame, $\tilde{k}_1^{1'} + k_2^{1'} = 0$, by boosting along x^1 direction with velocity $\beta = K_1(s)/(\tilde{s} + 2\tilde{m}_1^2)$. For $\delta_{1,2,3} = 1$ choice, we find (assuming $s > 0$)

$$\tilde{k}_1^{(\mathrm{CM})} = \frac{1}{2\sqrt{s}}\big(s + \tilde{m}_1^2 - m_2^2, -\sqrt{K_1(s)}, 0, 0, \vec{0}\big), \tag{4.13}$$

$$k_2^{(\mathrm{CM})} = \frac{1}{2\sqrt{s}}\big(s - \tilde{m}_1^2 + m_2^2, \sqrt{K_1(s)}, 0, 0, \vec{0}\big), \tag{4.14}$$

$$k_3^{(\mathrm{CM})} = \left(-\frac{\tilde{s} + \tilde{m}_1^2 + m_2^2 + m_3^2 - m_4^2}{2\sqrt{s}}, k_3^{(\mathrm{CM})1}, k_3^2 0, \vec{0}\right), \tag{4.15}$$

$$q^{(\mathrm{CM})} = \left(-\frac{\tilde{n}_1 + n_2}{\sqrt{s}}, \frac{-\tilde{s}(\tilde{n}_1 - n_2) - 2m_2^2\tilde{n}_1 + 2\tilde{m}_1^2 n_2}{\sqrt{s}\sqrt{K_1(s)}}, c_2, c_3, \vec{0}\right) \tag{4.16}$$

where

$$k_3^{(\mathrm{CM})1} = \frac{1}{2\sqrt{s}\sqrt{K_1(s)}}\big[\tilde{s}^2 + 2\tilde{s}\tilde{t} + \tilde{s}\big(3\tilde{m}_1^2 + m_2^2 + m_3^2 - m_4^2\big) + 2\tilde{t}\big(\tilde{m}_1^2 + m_2^2\big) + 2\tilde{m}_1^2\big(\tilde{m}_1^2 + m_2^2 + m_3^2 - m_4^2\big)\big], \tag{4.17}$$

and k_3^2, c_2, and c_3 are the same as the rest frame configuration, (4.8), (4.10) and (4.11).

Our main interest is to examine the relations between high-energy symmetry à la Gross and the exact identities as derived from bracket algebra. In the case of fixed-angle high-energy scattering, we take $s, t \to \infty$, and keep t/s fixed, and consequently,

$$\tilde{s} = s + \mathcal{O}(1), \qquad \tilde{t} = t + \mathcal{O}(1) = -\frac{s}{2}(1 - \cos\theta_{CM}) + \mathcal{O}(1), \tag{4.18}$$

where θ_{CM} is the scattering angle in the CM frame. The leading-order expressions are given by (with the sign factors δ_i restored)

$$\tilde{k}_1^{(\mathrm{CM})} = \frac{\sqrt{s}}{2}(1, -\delta_1, 0, 0, \vec{0}), \qquad k_2^{(\mathrm{CM})} = \frac{\sqrt{s}}{2}(1, \delta_1, 0, 0, \vec{0}), \tag{4.19}$$

$$k_3^{(\mathrm{CM})} = \frac{\sqrt{s}}{2}\big(-1, \delta_1\cos\theta_{CM}, \delta_2\sqrt{1 - \cos^2\theta_{CM}}, 0, \vec{0}\big), \tag{4.20}$$

$$q^{(\mathrm{CM})} = \frac{-1}{\sqrt{s}}\left(\tilde{n}_1 + n_2, \delta_1(\tilde{n}_1 - n_2), -\delta_2\frac{(n_2-\tilde{n}_1)(1-\cos\theta_{CM}) + 2n_3 + 2\tilde{n}_1}{\sqrt{1-\cos^2\theta_{CM}}},\right.$$
$$\left. -\delta_3\sqrt{s}\sqrt{-m_q^2}, \vec{0}\right). \tag{4.21}$$

One can see that δ_1 and δ_2 are responsible for covering all the kinematic range by use of this parametrization, while δ_3 has no physical importance. Note that the third spatial component of the momentum q becomes pure imaginary in this limit, when $m_q^2 > 0$. However, all physical momenta are real.

4.2. Complex momenta and Lorentz transformations

As seen in (4.19)–(4.21), in the high-energy limit $s \to \infty$, the momenta $\tilde{k}_1$, k_2, k_3, and k_4 posses real components for generic choices of masses and $q \cdot k_i$, while q will develop a complex component when it corresponds to a massive state. One uses $\tilde{k}_1$, k_2, k_3, and k_4 as the momenta for external particles to calculate a scattering amplitudes, and the amplitude is regarded as a physical scattering process in a high energy regime. In the calculation of scattering amplitudes, q appears only through polarization tensors for bracket states.

Moore's prescription relates a set of scattering amplitudes in which different operators are deformed, and each amplitude carries different sets of momenta. For examples discussed in the paper, one of them has $\tilde{k}_1 = k_1 + q$, $k_2 = k_2$, k_3, and k_4 and the other k_1, $\tilde{k}_2 = k_2 + q$, k_3, and k_4. Thus, k_1 and $\tilde{k}_2$ may become complex in the high-energy limit. In view of Moore's relation as an identity among analytic functions of momenta and polarization invariants, it is not a problem. However, one may be worried about whether the relation is understood as a relation among *physical* amplitudes, at least in an asymptotic regime of our main interest. The latter momentum set is characterized by the masses $m_1, \tilde{m}_2, m_3, m_4, m_q$ and the integers $n_1, \tilde{n}_2, n_3, n_4$. Since the general formulas (4.13)–(4.16) defines real external momenta for the given set of the parameters, we may work with these momenta to compute physical scattering amplitudes. Since the amplitude is a function of Lorentz invariants, such as $k_i \cdot k_j$ or $\zeta \cdot k_i$, it defines an equivalent amplitude if the invariants are the same. This condition is satisfied if there is a "Lorentz transformation" $SO(1,25;\mathbb{C})$ (or $SO(1,3;\mathbb{C})$ in practice) that relates these two configurations. For the kinematic configuration in the case of $\mathcal{A}[\tilde{3}000] = \mathcal{A}[1\tilde{2}00]$, we have shown it by explicitly constructing the transformation matrix, and we conclude that the exact symmetry identities indeed relate various physical scattering amplitudes with real momenta.

4.3. Scattering and q-orthonormal helicity bases for polarizations

We first present a general discussion of constructing a new orthonormal basis which is suitable for the study of Moore's relation, based on the helicity representation with respect to the first particle. Assume that $\tilde{k}_1$ is a momentum for a massive state. Let e^P, e^L, e^T, e^I, and e^{J_i} be the helicity vectors with respect to $\tilde{k}_1$ in the CM frame. Here, $e^P \propto \tilde{k}_1$ is the momentum direction of the first particle. e^L is the longitudinal vector, namely a unit vector parallel to $\tilde{k}_1$ on the scattering plane. e^T is the transverse vector lying on the scattering plane. e^I is one of the other transverse vectors which has an overlap with q. e^{J_i} $(i = 1, \cdots, 22)$ are the rest of the transverse vectors, chosen so that they are orthogonal to all the momenta in question. The completely transverse vectors e^{J_i} are not relevant for the discussion here, and we neglect them for the time being.

Since these vectors serve a natural basis for polarization tensors when we discuss scattering amplitudes in the CM frame, we call them the scattering helicity basis. They are also convenient basis to analyze the physical state conditions for the massive first particle [21]. However, when the first particle corresponds to a bracket operator as for our examples, the physical state conditions are more neatly written down by use of a basis regarding the deformation momentum q as we shall see. $\tilde{k}_1$ and q have the following expression on the scattering helicity basis,

$$\tilde{k}_1 = \sqrt{\tilde{m}_1^2} e^P, \qquad q = c_P e^P + c_L e^L + c_T e^T + c_I e^I, \tag{4.22}$$

where c_P is determined by the condition $q \cdot \tilde{k}_1 = \tilde{n}_1$ as $c_P = -\tilde{n}_1/\sqrt{\tilde{m}_1^2}$, while the other coefficients depend on the choice of k_2, k_3, and k_4, as we have just seen. We are now going to define a new set of orthonormal basis vectors with which q takes the following simple form, $q = c_P e^P + c_Q e^Q$, with e^Q being a unit vector defined simply by $c_Q e^Q = c_L e^L + c_T e^T + c_I e^I$ and $c_Q^2 = \sqrt{c_L^2 + c_T^2 + c_I^2} = c_P^2 - m_q^2$. In the subspace spanned by e^L, e^T and e^I, we define another two unit vectors orthogonal to e^Q; we choose e^{T_q} to be purely spatial and e^{I_q} is the orthogonal complement to e^Q and e^{T_q} in this subspace. The sign ambiguity has no physical importance. e^{T_q}, e^{I_q}, and e^Q, together with e^P, form a new orthonormal basis which we call the q-orthonormal basis. On this basis, k_1 and q are represented as

$$k_1 = \left(\sqrt{\tilde{m}_1^2} - c_P\right) e^P - c_Q e^Q, \qquad q = c_P e^P + c_Q e^Q, \tag{4.23}$$

and then the physical state conditions for the seed operator $V(k_1, z)$ and the deformer $J(q, z)$ are written down by use of "longitudinal-like" e^Q and "transverse" unit vectors e^{T_q} and e^{I_q} (and also completely transverse vectors e^{J_i}). In this basis, e^P is not the momentum direction of q and then e^Q is not really longitudinal. However, it turns out to be convenient to keep e^P dependence explicitly, since e^P is directly related to decoupling states from bracket operators.

In summary, we have defined a new set of the unit orthogonal vectors

$$e^{A'} = \sum_{a'=L,T,I} C^{A'}{}_{a'} e^{a'}, \tag{4.24}$$

where $A' = T_q, I_q, Q$ and the explicit form of the transformation matrix $C^{A'}_{a'}$ is

$$C^{T_q}{}_L = 0, \qquad C^{T_q}{}_T = \frac{c_I}{\sqrt{c_T^2 + c_I^2}}, \qquad C^{T_q}{}_I = \frac{-c_T}{\sqrt{c_T^2 + c_I^2}}, \tag{4.25}$$

$$C^{I_q}{}_L = \frac{\sqrt{c_T^2 + c_I^2}}{c_Q}, \qquad C^{I_q}{}_T = \frac{-c_L c_T}{c_Q \sqrt{c_T^2 + c_I^2}}, \qquad C^{I_q}{}_I = \frac{-c_L c_I}{c_Q \sqrt{c_T^2 + c_I^2}}, \tag{4.26}$$

$$C^Q{}_L = \frac{c_L}{c_Q}, \qquad C^Q{}_T = \frac{c_T}{c_Q}, \qquad C^Q{}_I = \frac{c_I}{c_Q}. \tag{4.27}$$

Since the both (e^L, e^T, e^I) and (e^{I_q}, e^{T_q}, e^Q) are orthonormal with the positive metric, the transformation matrix is orthogonal, namely

$$\sum_{a'=L,T,I} C^{A'}{}_{a'} C^{B'}{}_{a'} = \delta^{A'B'}, \qquad \sum_{A'=I_q,T_q,Q} C^{A'}{}_{a'} C^{A'}{}_{b'} = \delta_{a'b'}, \tag{4.28}$$

where $A', B' = I_q, T_q, Q$ and $a', b' = L, T, I$. In later sections we shall see the advantage of this new basis to represent usual scattering amplitudes in the CM frame and also DDF amplitudes.

4.4. Kinematics for case studies

Based on the general discussion so far, we write down the explicit kinematic configurations for the examine we examine in this paper, for reference.

4.4.1. Case study I: $\mathcal{A}[\tilde{2}000] = \mathcal{A}[1\tilde{1}00]$

In this case, the scattering helicity basis with respect to $\tilde{k}_1$ ($\tilde{m}_1^2 = 2$) is given by

$$e^P = \frac{1}{2\sqrt{2s}}\left(s+4, -\sqrt{s^2+16}, 0, 0\right), \tag{4.29}$$

$$e^L = \frac{1}{2\sqrt{2s}}\left(\sqrt{s^2+16}, -(s+4), 0, 0\right), \tag{4.30}$$

$$e^T = (0,0,1,0), \qquad e^I = (0,0,0,1). \tag{4.31}$$

The momenta in $\mathcal{A}[\tilde{2}000]$ side are obtained by the general formulas (4.13)–(4.16) and the auxiliary vector q is represented in this basis as

$$q = \frac{1}{\sqrt{2}}e^P - \frac{s-4}{\sqrt{2}\sqrt{s^2+16}}e^L + \frac{4(s+2t+4)}{\sqrt{s^2+16}\sqrt{f_1(s,t)}}e^T + \frac{2\sqrt{f_2(s,t)}}{\sqrt{f_1(s,t)}}e^I, \tag{4.32}$$

where $f_1(s,t) = 32 - st(s+t+4)$ and $f_2(s,t) = -4 - t(s+t+4)$. The basis vectors of the q-orthonormal basis are obtained from the transformation formulas (4.24)–(4.27) in the previous subsection.

For the right hand side (RHS), $\mathcal{A}[1\tilde{1}00]$, we need to prepare another scattering helicity basis with respect to k_1 and $\tilde{k}_2 = k_2 + q$. Let $e^{P_1}, e^{P_2}, e^{T_R}, e^{I_R}$, and e^{J_i} be basis vectors in question. e^{P_1} and e^{P_2} are momentum polarization with respect to k_1 and $\tilde{k}_2$ respectively. Since they are null, there are no L-directions. e^{T_R} is on the RHS scattering plane (now spanned by $\vec{k}_1$ and $\vec{k}_3$) and orthogonal to k_1. e^{I_R} is a unit vector perpendicular to the RHS scattering plane. The purely transverse directions e^{J_i} are common on both hands sides, and we use the same basis vectors. By use of q-orthonormal basis, they are represented as

$$e^{P_1} = \frac{1}{\sqrt{2}}e^P - \frac{1}{\sqrt{2}}e^Q, \qquad e^{P_2} = \frac{s+2}{2\sqrt{2}}e^P + \frac{s-2}{2\sqrt{2}}e^Q - \sqrt{s}e^{I_q}, \tag{4.33}$$

$$e^{T_R} = e^{T_q}, \qquad e^{I_R} = -\frac{\sqrt{2}}{\sqrt{s}}e^P + \frac{\sqrt{2}}{\sqrt{s}}e^Q + e^{I_q}. \tag{4.34}$$

It is straightforward to check that e^{I_R} is orthogonal to e^{P_1} and e^{P_2}.

4.4.2. Case study II: $\mathcal{A}[\tilde{3}000] = \mathcal{A}[1\tilde{2}00]$

The momenta for the case are obtained from the general formulas (4.13)–(4.16). The helicity basis with respect to $\tilde{k}_1$ is

$$e^P = \frac{1}{\sqrt{\tilde{m}_1^2}}\tilde{k}_1 = \frac{1}{4\sqrt{s}}\left(s+6, -\sqrt{F_3(s)}, 0, 0\right), \tag{4.35}$$

$$e^L = \frac{1}{4\sqrt{s}}\left(\sqrt{F_3(s)}, -(s+6), 0, 0\right), \tag{4.36}$$

$$e^T = (0,0,1,0), \qquad e^I = (0,0,0,1), \tag{4.37}$$

and the coefficients of q in this basis are obtained by $c_A = \pm e^A_\mu q^\mu$ for $A = P, L, T, I$ (the negative sign is for e^P). Through c_A one can generate q-orthonormal basis easily. The helicity basis with respect to k_1 for $\mathcal{A}[1\tilde{2}00]$ side is constructed in a similar way to the previous example, in the q-orthonormal basis, as

$$
\begin{aligned}
e^{P_2} &= \frac{1}{\sqrt{2}}\tilde{k}_2 = -\frac{\sqrt{2}(s+4)}{8}e^P - \frac{3\sqrt{2}(s-4)}{8}e^Q + \frac{\sqrt{F_2(s)}}{2}e^{I_q},\\
e^{L_2} &= \frac{-\sqrt{2}(s^2+2s-16)}{8(s-2)}e^P + \frac{\sqrt{2}(3s^2-18s+32)}{8(s-2)}e^Q - \frac{\sqrt{F_2(s)}}{2}e^{I_q},\\
e^{T_R} &= \frac{-(s-4)(s+2t+2)}{2(s-2)\sqrt{F_5(s,t)}}e^P + \frac{F_6(s,t)}{2(s-2)\sqrt{F_5(s,t)}}e^Q + \frac{F_6(s,t)}{\sqrt{2F_2(s)F_5(s,t)}}e^{I_q}\\
&\quad + \frac{-F_7(s,t)\sqrt{F_1(s,t)}}{F_4(s,t)\sqrt{2F_2(s)F_5(s,t)}}e^{T_q},\\
e^{I_R} &= \frac{\sqrt{F_1(s,t)}}{2\sqrt{F_5(s,t)}}\left(e^P - e^Q\right) + \frac{F_6(s,t)}{\sqrt{2F_2(s)F_5(s,t)}}e^{T_q} + \frac{F_7(s,t)\sqrt{F_1(s,t)}}{F_4(s,t)\sqrt{2F_2(s)F_5(s,t)}}e^{I_q},
\end{aligned}
$$

where $e^{P_1} = k_1 = 2e^P - q$. e^{L_2} is the longitudinal unit vector for $\tilde{k}_2$, and

$$
\begin{aligned}
&F_1(s,t) = 2s^2t - s^2 - 12st + 2st^2 + 16s - 96 - 16t^2 - 32t,\\
&F_2(s) = -s^2 + 10s - 20, \qquad F_3(s) = s^2 - 4s + 36,\\
&F_4(s,t) = 72 - st(s+t+2), \qquad F_5(s,t) = -st(s+t+2) - 8s + 8,\\
&F_6(s,t) = s^2 - 2s + 2st - 8 - 8t,\\
&F_7(s,t) = s^3t + s^2t^2 - 2st^2 - 4st - 72s + 144.
\end{aligned}
$$

5. Exact identities in various bases

In this section, we discuss the relation of physical bracket states with standard positive norm/DDF states. As seen in Section 3.1, a bracket operator is physical when a deformer and a seed operators on which the bracket operator is based are physical, and physical state conditions for the deformer and seed operators are simply solved by use of the q-orthonormal basis introduced in Section 4. As we will see, Moore's relation takes a simple form when it is represented in terms of amplitudes with q-orthonormal polarizations. Especially, the coefficients in the relations are found to be t-independent. However, the q-orthonormal basis is constructed with respect to a deformation momentum q which does not show up in physical momenta for scattering amplitudes and then it does not respect certain physical symmetries. We therefore want to represent exact relations in a physical basis — usual helicity basis with respect to a momentum for an external particle. Further transformation to a DDF basis is advantageous when we discuss high-energy symmetries as explained in Section 6.1.

The translation involves energy dependent transformation coefficients connecting different bases and the expressions of the coefficients are fixed by bracket operation and the choice of q. As we will see, the high-energy expansions of these coefficients provide proportional constants of high-energy linear relations.

5.1. Exact identities for Case study I

In this subsection, we discuss the bracket states that appear in $\mathcal{A}[\tilde{2}000] = \mathcal{A}[1\tilde{1}00]$ relation and their decompositions into scattering helicity bases.

Since we are interested in scattering amplitudes for physical processes, we need to impose physical state conditions on bracket states. As discussed in Section 3, the bracket operator is automatically physical for a physical choice of the seed polarization ζ_1 and the deformer polarization ζ_q. Physical conditions for the deformer and the seed operators are met easily by use of q-orthonormal basis introduced in Section 4 as

$$\zeta_1 = e^A, \qquad \zeta_q = e^B, \quad A, B = T_q, I_q, J_i. \tag{5.1}$$

By plugging (5.1) into (2.16), (2.17), and (2.18), the polarization tensors of the bracket states are written as

$$\zeta^{(2)}_{\mu\nu}(\zeta_1, \zeta_q) = e^A_{(\mu} e^B_{\nu)} + \frac{\delta^{AB}}{2} q_\mu q_\nu, \qquad \zeta^{(2)}_{\mu}(\zeta_1, \zeta_q) = \frac{\delta^{AB}}{2} q_\mu,$$
$$\zeta_{R\mu}(\zeta_q) = e^B_\mu + k^B_2 q_\mu. \tag{5.2}$$

The two polarization tensors on the first line are for the level 2 bracket operator that appears in $\mathcal{A}[\tilde{2}000]$ calculation, while the last one is for the level 1 bracket operator on $\mathcal{A}[1\tilde{1}00]$ side. We may write the bracket operators of these choices of the polarization tensors as $V^{\mathrm{br}AB}_{(2)}(\tilde{k}_1, x)$ and $V^{\mathrm{br}B}_{(1)}(\tilde{k}_2, 1)$. We are interested in the four-point amplitudes with these operators inserted,

$$\mathcal{T}^{AB}_{\mathrm{br}[\tilde{2}000]} = \int_0^1 dx \big\langle V^{\mathrm{br}AB}_{(2)}(\tilde{k}_1, x) V_{(0)}(k_2, 0) V_{(0)}(k_3, 1) V_{(0)}(k_4, \infty)\big\rangle, \tag{5.3}$$

$$\mathcal{T}^{A|B}_{\mathrm{br}[1\tilde{1}00]} = \int_0^1 dx \big\langle V^{A}_{(1)}(k_1, x) V^{\mathrm{br}B}_{(1)}(\tilde{k}_2, 1) V_{(0)}(k_3, 1) V_{(0)}(k_4, \infty)\big\rangle, \tag{5.4}$$

where the vertical line in the superscript of the second amplitude denotes the separation of the first and the second particles. In terms of these "bracket amplitudes," Moore's relation trivially reads

$$\mathcal{T}^{AB}_{\mathrm{br}[\tilde{2}000]} = \mathcal{T}^{A|B}_{\mathrm{br}[1\tilde{1}00]}. \tag{5.5}$$

Since the bracket operators are vertex operators with specific forms of the polarization tensors, we may write Moore's relation in terms of the amplitudes associated with conventional scattering amplitudes such as

$$\int_0^1 dx \big\langle V_{(2)}(\tilde{k}_1, x; \zeta) V_{(0)}(k_2, 0) V_{(0)}(k_3, 1) V_{(0)}(k_4, \infty)\big\rangle$$
$$= F_{s-t}\big[\zeta_{\mu\nu}\mathcal{T}^{\mu\nu}_{[2000]} + \zeta_\mu \mathcal{T}^{\mu}_{[2000]}\big], \tag{5.6}$$

where

$$F_{s-t} = \frac{\Gamma(-\alpha s' - 1)\Gamma(-\alpha' t - 1)}{\Gamma(\alpha' u + 2)}, \tag{5.7}$$

is "Veneziano-like" part which is responsible for the soft behavior in the high-energy regime. We take out this factor from scattering amplitudes as a common factor, and then $\mathcal{T}^{\mu\nu}_{[2000]}$ and $\mathcal{T}^{\mu}_{[2000]}$ are the rest of "polynomial" part. In the same manner we define $\mathcal{T}^{\mu|\nu}_{[1100]}$, and Moore's relation is written in the following form,

$$\mathcal{T}^{AB}_{[2000]} + \frac{\delta^{AB}}{2}\left(\mathcal{T}^{qq}_{[2000]} + \mathcal{T}^{q}_{[2000]}\right) = \mathcal{T}^{A|B}_{[1100]} + k_2^B \mathcal{T}^{A|q}_{[1100]}, \tag{5.8}$$

where $\mathcal{T}^{AB}_{[2000]} = e^A_\mu e^B_\nu \mathcal{T}^{\mu\nu}_{[2000]}$, $\mathcal{T}^{q}_{[2000]} = q_\mu \mathcal{T}^{\mu}_{[2000]}$, and so on. Since $q = \frac{1}{\sqrt{2}}(e^P + e^Q)$, the second terms of the both sides contain e^P components. The amplitudes with this component are related to vanishing amplitudes due to the decoupling of zero norm states. After dropping such trivial part, we can write the both hands sides in terms of the q-transverse polarizations as

$$\begin{aligned}
&\mathcal{T}^{AB}_{[2000]} - \frac{\delta^{AB}}{20}\left(-4\mathcal{T}^{QQ}_{[2000]} + \mathcal{T}^{I_q I_q}_{[2000]} + \mathcal{T}^{T_q T_q}_{[2000]} + \sum_{i=1}^{22} \mathcal{T}^{J_i J_i}_{[2000]}\right) \\
&\quad = \mathcal{T}^{A|B}_{[1100]} + \frac{2k_2^B}{s+2}\left(\sqrt{s}\,\mathcal{T}^{A|I_q}_{[1100]} + \sqrt{2}\,\mathcal{T}^{A|Q}_{[1100]}\right),
\end{aligned} \tag{5.9}$$

where $A, B = T_q, I_q, J_i$. This equality is exact and holds for arbitrary s and t. The coefficients in the equality are almost just constants and even non-constant coefficients are simple functions of s, since $k_2^{I_q} = -\sqrt{s}$, $k_2^Q = \frac{s-4}{2\sqrt{2}}$, and $k_2^{T_q} = k_2^{J_i} = 0$. Especially, the coefficients are t (therefore the scattering angle) independent. There appear five independent relations with respect to the choice of A and B. This is first our observation; the exact identity relation takes a particularly simple form with projection onto q-orthonormal basis. Hence, the deformation momentum q also provides a natural frame to describe the exact identity.

However, when we look at the relation as a relation among physical scattering amplitudes, q does not explicitly appear as a momentum of external particles but is implicitly encoded in a specific form of deformed polarization tensors. Therefore transverse projections with respect to q does not have manifest physical significance. We thus rewrite the relation in terms of amplitudes in scattering helicity basis which is standard basis to describe scattering amplitudes in the CM frame.

5.1.1. Level 2 bracket state in $\mathcal{A}[\tilde{2}000]$ amplitude

We first write bracket states in terms of scattering helicity states, with zero norm states dropped. The level 2 bracket state corresponding to $V^{\text{br}}_{(2)}(\tilde{k}_1, z; \zeta^{(2)})$ is rewritten as[3]

$$\begin{aligned}
\left|\zeta_1 = e^A, \zeta_q = e^B; \tilde{k}_1\right\rangle_{\text{br}} &= \left[\alpha^{AB}_{-1} + \frac{\delta^{AB}}{2}\left(\alpha^{qq}_{-1} + \alpha^{q}_{-2}\right)\right]|0; \tilde{k}_1\rangle \\
&= \left[\left(C^{(A}{}_a C^{B)}{}_b + \frac{\delta^{AB}}{4} C^Q{}_a C^Q{}_b\right)\alpha^{ab}_{-1} + 2\left(C^{(A}{}_a C^{B)}{}_L + \frac{\delta^{AB}}{4} C^Q{}_a C^Q{}_L\right)\alpha^{aL}_{-1}\right. \\
&\quad + \left(C^A{}_L C^B{}_L + \frac{\delta^{AB}}{4} C^Q{}_L C^Q{}_L\right)\alpha^{LL}_{-1}
\end{aligned}$$

[3] $|0; \tilde{k}_1\rangle$ is the tachyon state with momentum $\tilde{k}_1$.

$$+\frac{\sqrt{2}C^Q{}_a\delta^{AB}}{4}\left(\alpha^a_{-2}+\sqrt{2}\alpha^{aP}_{-1}\right)+\frac{\sqrt{2}C^Q{}_L\delta^{AB}}{4}\left(\alpha^L_{-2}+\sqrt{2}\alpha^{LP}_{-1}\right)$$
$$+\frac{\delta^{AB}}{20}\left(5\alpha^{PP}_{-1}+5\sqrt{2}\alpha^P_{-2}\right)\Bigg]|0;\tilde{k}_1\rangle, \tag{5.10}$$

where $A, B = T_q, I_q, J_i$ and $a, b = T, I, J_i$. We summarize a product of oscillators of the same level as $\alpha^{ab}_{-1}\equiv\alpha^a_{-1}\alpha^b_{-1}$, and $\alpha^q_{-1}=q\cdot\alpha_{-1}$. $C^A{}_a$ is the transformation matrix defined in (4.24). In the second from the last line, these two specific linear combinations are proportional to zero norm states of this level. The explicit forms of zero norm states are summarized in Appendix B.2. In the last line, we consider the difference with one of the zero norm state as

$$\left(5\alpha^{PP}_{-1}+5\sqrt{2}\alpha^P_{-2}\right)|0;\tilde{k}_1\rangle=-\left(\alpha^{LL}_{-1}+\sum_a\alpha^{aa}_{-1}\right)|0;\tilde{k}_1\rangle+|ZN_1;\tilde{k}_1\rangle.$$

By dropping all the zero norm state parts, we find that the bracket state (5.10) can be written in terms of positive norm states, up to zero norm states, as

$$(5.10)=\left(\sum_{a',b'}G^{AB}_{a'b'}\alpha^{a'b'}_{-1}+G^{AB}\sum_{a'}\alpha^{a'a'}_{-1}\right)|0;\tilde{k}_1\rangle, \tag{5.11}$$

where $a', b' = L, T, I, J_i$, namely the longitudinal direction and the transverse directions. The coefficients are

$$G^{AB}_{a'b'}=\left(C^AC^B+\frac{\delta^{AB}}{4}C^QC^Q\right)_{(a'b')},\qquad G^{AB}=-\frac{\delta^{AB}}{20}. \tag{5.12}$$

$(C^AC^B)_{(a'b')}$ is the symmetrization, $(C^AC^B)_{(a'b')}\equiv\frac{1}{2}(C^A_{a'}C^B_{b'}+C^A_{b'}C^B_{a'})$.

5.1.2. *Level 1 bracket state in* $\mathcal{A}[1\tilde{1}00]$ *amplitude*

We now move on to the $\tilde{m}_2^2=0$ bracket state. The state is fairly simple,

$$\left|\zeta_q=e^B;\tilde{k}_2\right\rangle_{\rm br}=\left(\alpha^B_{-1}+k_2^B\alpha^q_{-1}\right)|0;\tilde{k}_2\rangle. \tag{5.13}$$

Note that e^B is transverse to q but not to k_2. We have chosen the polarization tensor of the deformer $\zeta_{q\mu}$ so that the polarization tensor associated with this bracket state satisfies the transversality condition with respect to $\tilde{k}_2$,

$$\tilde{k}_2\cdot\zeta_R=0,\qquad \zeta_{R\mu}\left(\zeta_q=e^B\right)=e^B_\mu+k_2^Bq_\mu. \tag{5.14}$$

Therefore, this bracket states can be rewritten as a linear combination of the momentum and the transverse oscillators,

$$(5.13)=\left(\tilde{G}^B_{P_2}\alpha^{P_2}_{-1}+\sum_{\tilde{a}}\tilde{G}^B_{\tilde{a}}\alpha^{\tilde{a}}_{-1}\right)|0;\tilde{k}_2\rangle, \tag{5.15}$$

where $\tilde{a}=T_R, I_R, J_i$, the transverse directions with respect to k_1 and $\tilde{k}_2$, and the coefficients are determined to be

$$\tilde{G}^B_{P_2}=-e^{P_2}\cdot\zeta_R,\qquad \tilde{G}^B_{\tilde{a}}=e^{\tilde{a}}\cdot\zeta_R. \tag{5.16}$$

Here, we put the tilde for coefficients on the right-hand side (namely $\mathcal{A}[1\tilde{1}00]$) states for distinction. For the first particle associated with the undeformed seed operator $V^\mu_{(1)}(k_1)$, we simply apply the transformation matrix $C^A{}_{\tilde{a}}$ to rotate q-orthonormal directions to k_1 transverse directions.

5.1.3. Exact identity in terms of standard scattering amplitudes

We can now write down Moore's identity (5.5) in terms of the amplitudes in the scattering helicity basis as

$$\sum_{a',b'} G^{AB}_{a'b'}\mathcal{T}^{a'b'}_{[2000]} + G^{AB}\sum_{a'}\mathcal{T}^{a'a'}_{[2000]} = \sum_{\tilde a,\tilde b} C^{A}{}_{\tilde a}\tilde G^{B}_{\tilde b}\mathcal{T}^{\tilde a|\tilde b}_{[1100]}. \tag{5.17}$$

where $a', b' = L, T, I, J_i$ and $\tilde a, \tilde b = T_R, I_R, J_i$. On the left hand side, the amplitudes with a completely transverse polarization (e^I or e^{J_i}) are trivially zero, while on the right hand side, completely transverse directions should appear in a pairwise way for the amplitude to be non-vanishing. Therefore, the summation can be made explicit as

$$\begin{aligned}&\left(G^{AB}_{TT} + G^{AB}\right)\mathcal{T}^{TT}_{[2000]} + 2G^{AB}_{LT}\mathcal{T}^{LT}_{[2000]} + \left(G^{AB}_{LL} + G^{AB}\right)\mathcal{T}^{LL}_{[2000]}\\ &\quad = C^{A}{}_{T_R}\tilde G^{B}_{T_R}\mathcal{T}^{T_R|T_R}_{[1100]} + C^{A}{}_{I_R}\tilde G^{B}_{I_R}\mathcal{T}^{I_R|I_R}_{[1100]} + \sum_i C^{A}{}_{J_i}\tilde G^{B}_{J_i}\mathcal{T}^{J_i|J_i}_{[1100]}.\end{aligned} \tag{5.18}$$

In this expression, the coefficients G, C, and $\tilde G$ have nontrivial s and t dependence to make the equality hold. As we shall see in Section 6, the high-energy expansion of these coefficients gives proportional constants of linear relations in the fixed-angle high-energy limit.

5.1.4. The exact identity in terms of DDF amplitudes

As explained in B.2, the bracket state (5.10) can further be transformed into the sum of the DDF states, up to zero norm states, as

$$(5.10) = \sum_{a,b} D^{AB}_{ab}|ab;\tilde k_1\rangle_{\text{DDF}} + \sum_a D^{AB}_{a}|a;\tilde k_1\rangle_{\text{DDF}} + D^{AB}\sum_a |aa;\tilde k_1\rangle_{\text{DDF}},$$

where $a, b = T, I, J_i$, the transverse directions, and $|\cdots;\tilde k_1\rangle_{\text{DDF}}$ are the DDF states given by the action of DDF raising operators A^{a}_{-n} on a tachyonic vacuum. The coefficients are given as

$$D^{AB}_{ab} = G^{AB}_{ab},\qquad D^{AB}_{a} = -\sqrt{2}G^{AB}_{La},\qquad D^{AB} = \frac{1}{4}\left(G^{AB}_{LL} + 5G^{AB}\right). \tag{5.19}$$

For the massless bracket state Since there is no distinction between positive norm states and DDF states for massless states, in $\mathcal{A}[1\tilde 100]$ side we use (5.12) to write down Moore's relation in terms of DDF amplitudes,

$$\begin{aligned}&\sum_{a,b} D^{AB}_{ab}\mathcal{T}^{ab}_{\text{DDF}[2000]} + \sum_a D^{AB}_{a}\mathcal{T}^{a}_{\text{DDF}[2000]} + D^{AB}\sum_b \mathcal{T}^{bb}_{\text{DDF}[2000]}\\ &\quad = \sum_{\tilde a,\tilde b} C^{A}{}_{\tilde a}\tilde G^{B}_{\tilde b}\mathcal{T}^{\tilde a|\tilde b}_{[1100]},\end{aligned} \tag{5.20}$$

where $a, b = T, I, J_i$ and $\mathcal{T}^{ab}_{\text{DDF}[2000]}$ is defined as an amplitude with one DDF state $|ab;\tilde k_1\rangle_{\text{DDF}}$ and three tachyons. The others are understood in the same manner. One can again use the non-vanishing conditions for the choice of completely transverse directions, to make summation explicit as

$$\begin{aligned}&D^{AB}_{TT}\mathcal{T}^{TT}_{\text{DDF}[2000]} + D^{AB}_{II}\mathcal{T}^{II}_{\text{DDF}[2000]} + \sum_i D^{AB}_{J_iJ_i}\mathcal{T}^{J_iJ_i}_{\text{DDF}[2000]} + D^{AB}_{T}\mathcal{T}^{T}_{\text{DDF}[2000]}\\ &\quad + D^{AB}\left(\mathcal{T}^{TT}_{\text{DDF}[2000]} + \mathcal{T}^{II}_{\text{DDF}[2000]} + 22\mathcal{T}^{JJ}_{\text{DDF}[2000]}\right)\end{aligned}$$

$$= C^{A}{}_{T_R}\tilde{G}^{B}_{T_R}\mathcal{T}^{T_R|T_R}_{[1100]} + C^{A}{}_{I_R}\tilde{G}^{B}_{I_R}\mathcal{T}^{I_R|I_R}_{[1100]} + \sum_i C^{A}{}_{J_i}\tilde{G}^{B}_{J_i}\mathcal{T}^{J_i|J_i}_{[1100]}, \tag{5.21}$$

where, for the third term in the second line, all J_i $(i = 1, \cdots, 22)$ gives the same result, and then we can take J as a representative of J_i and multiply 22.

5.2. *Exact identities for Case study II*

Here we discuss the bracket states that appear in the discussion of $\mathcal{A}[\tilde{3}000] = \mathcal{A}[1\tilde{2}00]$ relation, presented in Section 2. As in the previous case, we will decompose the relevant bracket states in terms of standard positive norm states as well as DDF states. The discussion is parallel to the previous subsection, and the readers who do not need the details can skip to Section 6.

We again need to prepare polarization tensors of the seed and the deformer operators to make the corresponding bracket states physical. We set both seed and deformer operators to be physical. For the seed operator, we impose

$$\zeta_{1\mu} = e^{C}_{\mu}, \quad e^{C} \in e^{T_q}, e^{I_q}, e^{J_i}. \tag{5.22}$$

The polarization tensors of the deformer operator that satisfy the physical state conditions are

$$\begin{cases} \text{(I)}: & \zeta_{q\mu\nu} = e^{AB}_{(\mu\nu)} - \frac{\delta^{AB}}{24}E_{\mu\nu}, \\ \text{(II)}: & \zeta_{q\mu\nu} = 2\sqrt{2}e^{L_q A}_{(\mu\nu)}, \\ \text{(III)}: & \zeta_{q\mu\nu} = 24 e^{L_q L_q}_{\mu\nu} - E_{\mu\nu}, \end{cases} \qquad \left(E_{\mu\nu} \equiv \sum_{D=T_q, I_q, J_i} e^{D}_{\mu}e^{D}_{\nu}\right) \tag{5.23}$$

where $A, B = T_q, I_q, J_i$ and $\tilde{\zeta}_{q\mu} = 0$ for all the cases. $e^{L_q} = (e^P + 3e^Q)/2\sqrt{2}$ is the longitudinal polarization with respect to q; namely, $e^{P_q} = q/\sqrt{2}$, e^{L_q}, e^{T_q}, e^{I_q}, and e^{J_i} form a helicity basis with respect to q. In this case, the bracket state polarization tensors depend on different numbers of q-transverse directions, and we will call these three cases Choice (I), (II), and (III) respectively. By plugging these in (2.20)–(2.24), one obtains the polarization tensors for the bracket operators (the explicit forms are given later), and by using them the exact bracket relations for a given set of A, B, C are simply written as

$$\begin{aligned} \mathcal{T}^{ABC}_{\text{br}[\tilde{3}000]} &= \mathcal{T}^{C|AB}_{\text{br}[1\tilde{2}00]}, \\ \mathcal{T}^{AC}_{\text{br}[\tilde{3}000]} &= \mathcal{T}^{C|A}_{\text{br}[1\tilde{2}00]}, \\ \mathcal{T}^{C}_{\text{br}[\tilde{3}000]} &= \mathcal{T}^{C}_{\text{br}[1\tilde{2}00]}, \end{aligned} \tag{5.24}$$

for Choice (I), (II), and (III), respectively. As in the previous example, these relations can be represented as a relation among scattering amplitudes in q-orthonormal basis, and one can check that again the coefficients are very simple t-independent ones. We do not spell out them here and directly move on to scattering helicity basis expressions.

5.2.1. *Level 3 bracket state in $\mathcal{A}[\tilde{3}000]$ amplitude*

We start with Choice (I). With this choice, the polarization tensors for the bracket states are

$$\zeta^{(3)}_{\mu\nu\rho} = e^{ABC}_{(\mu\nu\rho)} + \frac{1}{2}q_{(\mu}q_{\nu}\left(\delta^{AC}e^{B} + \delta^{BC}e^{A}\right)_{\rho)} - \frac{\delta^{AB}}{24}\left[E_{(\mu\nu}e^{C}_{\rho)} + q_{(\mu}q_{\nu}e^{C}_{\rho)}\right], \tag{5.25}$$

$$\zeta^{(3)}_{\mu;\nu} = \left(\delta^{AC}e^{B} + \delta^{BC}e^{A}\right)_{\mu}q_{\nu} + \frac{1}{2}q_{\mu}\left(\delta^{AC}e^{B} + \delta^{BC}e^{A}\right)_{\nu} - \frac{\delta^{AB}}{24}\left[2e^{C}_{\mu}q_{\nu} + q_{\mu}e^{C}_{\nu}\right], \tag{5.26}$$

$$\zeta^{(3)}_{\mu} = \left(\delta^{AC} e^{B} + \delta^{BC} e^{A}\right)_{\mu} - \frac{2\delta^{AB}}{24} e^{C}_{\mu}. \tag{5.27}$$

The state corresponding to $V^{\mathrm{br}}_{(3)}(\tilde{k}_1, z; \zeta^{(3)})$ is rewritten in terms of scattering helicity basis as before,

$$\begin{aligned}
&\left|\zeta_q = e^{AB} - \frac{\delta^{AB}}{24} E, \tilde{\zeta}_q = 0, \zeta_1 = e^{C}; \tilde{k}_1\right\rangle_{\mathrm{br}} \\
&= \Bigg[\alpha^{ABC}_{-1} + \delta^{AC}\left(\frac{1}{2}\alpha^{qqB}_{-1} + \alpha^{B}_{-2}\alpha^{q}_{-1} + \frac{1}{2}\alpha^{q}_{-2}\alpha^{B}_{-1} + \alpha^{B}_{-3}\right) + (A \leftrightarrow B) \\
&\quad - \frac{\delta^{AB}}{24}\left(\sum_{D}\alpha^{DDC}_{-1} + \alpha^{qqC}_{-1} + 2\alpha^{C}_{-2}\alpha^{q}_{-1} + \alpha^{q}_{-2}\alpha^{C}_{-1} + 2\alpha^{C}_{-3}\right)\Bigg]|0; \tilde{k}_1\rangle \\
&= \Bigg[\sum_{a',b',c'} G^{ABC}_{a'b'c'}\alpha^{a'b'c'}_{-1} + \sum_{a',b'} G^{ABC}_{[a'b']}\alpha^{[a'}_{-2}\alpha^{b']}_{-1}\Bigg]|0; \tilde{k}_1\rangle + (\text{zero norm states}),
\end{aligned} \tag{5.28}$$

where $a', b' = L, T, I, J_i$, namely the longitudinal and the transverse directions with respect to $\tilde{k}_1$. The coefficients are

$$\begin{aligned}
G^{ABC}_{a'b'c'} = \Bigg[&C^{A}C^{B}C^{C} + \frac{\delta^{AC}}{24}\left(2C^{Q}C^{Q}C^{B} - \sum_{D}C^{D}C^{D}C^{B}\right) + (A \leftrightarrow B) \\
&- \frac{\delta^{AB}}{288}\left(2C^{Q}C^{Q}C^{C} + 11\sum_{D}C^{D}C^{D}C^{C}\right)\Bigg]_{(a'b'c')},
\end{aligned} \tag{5.29}$$

$$G^{ABC}_{[a'b']} = \left[-\frac{\delta^{AC}}{4}C^{Q}C^{B} + (A \leftrightarrow B) + \frac{\delta^{AB}}{48}C^{Q}C^{C}\right]_{[a'b']}. \tag{5.30}$$

${C^{A}}_{a} = e^{A} \cdot e^{a}$ is the transformation matrix from q-basis to $\tilde{k}_1$-basis as before.

For Choice (II) and (III), we can repeat the same procedure and just display the results here. For Choice (II), the polarization tensors are

$$\zeta^{(3)}_{\mu\nu\rho} = -2e^{(P-Q)AC}_{(\mu\nu\rho)} + \frac{\delta^{AC}}{3}\left(9e^{PPQ} + 6e^{PQQ} + e^{QQQ}\right)_{(\mu\nu\rho)}, \tag{5.31}$$

$$\zeta^{(3)}_{\mu;\nu} = -2e^{A}_{(\mu}e^{C}_{\nu)} - 2e^{A}_{[\mu}e^{C}_{\nu]} + 2\delta^{AC}\left(3e^{P}_{(\mu}e^{Q}_{\nu)} - e^{P}_{[\mu}e^{Q}_{\nu]} + e^{Q}_{\mu}e^{Q}_{\nu}\right), \tag{5.32}$$

$$\zeta^{(3)}_{\mu} = \frac{8\delta^{AC}}{3}e^{Q}_{\mu}, \tag{5.33}$$

and the bracket state is decomposed in the same way as (5.28) with $G^{ABC}_{a'b'c'}$ and $G^{ABC}_{[a'b']}$ replaced with

$$G^{AC}_{a'b'c'} = \left(2C^{A}C^{C}C^{Q} + \frac{2\delta^{AC}}{9}C^{Q}C^{Q}C^{Q} - \frac{\delta^{AC}}{9}\sum_{D}C^{D}C^{D}C^{Q}\right)_{(a'b'c')}, \tag{5.34}$$

$$G^{AC}_{[a'b']} = -2{C^{A}}_{[a'}{C^{C}}_{b']}. \tag{5.35}$$

For Choice (III), the polarization tensors are

$$\zeta^{(3)}_{\mu\nu\rho} = \frac{1}{4}\left(\left[-15e^{PP} - 138e^{PQ} + 41e^{QQ}\right]e^{C}\right)_{(\mu\nu\rho)} - \sum_{D}e^{DDC}_{(\mu\nu\rho)}, \tag{5.36}$$

$$\zeta^{(3)}_{\mu;\nu} = -\frac{1}{2} e^{(9P+67Q)}_{\mu} e^{C}_{\nu} - e^{C}_{\mu} e^{(3P+Q)}_{\nu}, \tag{5.37}$$

$$\zeta^{(3)}_{\mu} = -2 e^{C}_{\mu}, \tag{5.38}$$

and the decomposition is carried out with

$$G^{C}_{a'b'c'} = \frac{5}{12}\left(26 C^Q C^Q C^C - \sum_D C^D C^D C^C\right)_{a'b'c'}, \quad G^{C}_{[a'b']} = -\frac{65}{2} C^Q{}_{[a'} C^C_{b']}. \tag{5.39}$$

5.2.2. Level 2 bracket state in $\mathcal{A}[1\tilde{2}00]$ amplitude

We move on to the bracket state in $\mathcal{A}[1\tilde{2}00]$ side. We first investigate the Choice (I) case. The polarization tensors are

$$\begin{aligned}\zeta_{R\mu\nu} &= e^{AB}_{(\mu\nu)} + q_{(\mu}\left(k_2^A e^B + k_2^B e^A\right)_{\nu)} + \frac{k_2^A k_2^B}{2} q_\mu q_\nu \\ &\quad - \frac{\delta^{AB}}{24}\left[E_{\mu\nu} + 2\sum_D q_{(\mu} e^D_{\nu)} k_2^D + \frac{\sum_D (k_2^D)^2}{2} q_\mu q_\nu\right],\end{aligned} \tag{5.40}$$

$$\zeta_{R\mu} = \left(k_2^A e^B + k_2^B e^A\right)_\mu + \frac{k_2^A k_2^B}{2} q_\mu - \frac{\delta^{AB}}{24}\left[2\sum_D e^D_\mu k_2^D + \frac{\sum_D (k_2^D)^2}{2} q_\mu\right]. \tag{5.41}$$

The state corresponding to $V^{\mathrm{br}}_{(2)}(\tilde{k}_2)$ is

$$\begin{aligned}&\left|\zeta_q = e^{AB} - \frac{\delta^{AB}}{24} E, \tilde{\zeta}_q = 0; \tilde{k}_2\right\rangle_{\mathrm{br}} \\ &= \Bigg[\alpha^{AB}_{-1} + \left(k_2^A \alpha^{Bq}_{-1} + k_2^A \alpha^B_{-2} + (A \leftrightarrow B)\right) + \frac{k_2^A k_2^B}{2}\left(\alpha^{qq}_{-1} + \alpha^q_{-2}\right) \\ &\quad - \frac{\delta^{AB}}{24}\left(\sum_D \alpha^{DD}_{-1} + 2\sum_D k_2^D\left(\alpha^{Dq}_{-1} + \alpha^D_{-2}\right) + \frac{\sum_D (k_2^D)^2}{2}\left(\alpha^{qq}_{-1} + \alpha^q_{-2}\right)\right)\Bigg]|0; \tilde{k}_2\rangle.\end{aligned} \tag{5.42}$$

We decompose this state by use of the helicity states with respect to k_1, introduced in Section 4.4,[4] as

$$(5.42) = \left[\sum_{\tilde{a}',\tilde{b}'} \tilde{G}^{AB}_{\tilde{a}'\tilde{b}'} \alpha^{\tilde{a}'\tilde{b}'}_{-1} + \tilde{G} \sum_{\tilde{a}'} \alpha^{\tilde{a}'\tilde{a}'}_{-1}\right]|0; \tilde{k}_2\rangle + (\text{zero norm states}), \tag{5.43}$$

where $\tilde{a}', \tilde{b}' = L_2, T_R, I_R, J_i$, namely the longitudinal and the transverse directions with respect to k_1. The coefficients are

$$\tilde{G}^{AB}_{\tilde{a}'\tilde{b}'} = \left\{\left(C^A C^B - \frac{\delta^{AB}}{24}\sum_D C^D C^D\right) + 2\left(k_2^{(A} C^{B)} - \frac{\delta^{AB}}{24}\sum_D k_2^D C^D\right)c\right.$$

[4] In the k_1 helicity basis, there exist another null vector, may be called $e^{\tilde{L}_1}$, which satisfies $e^{P_1} \cdot e^{\tilde{L}_1} = 1$ and is transverse to the other basis vectors. In the discussion of the physical amplitudes, $e^{\tilde{L}_1}$ turns out to be irrelevant to our analysis.

$$+\frac{1}{2}\left(k_2^A k_2^B - \frac{\delta^{AB}}{24}\sum_D (k_2^D)^2\right)cc\bigg\}_{(\tilde{a}'\tilde{b}')}, \tag{5.44}$$

$$\tilde{G}^{AB} = \frac{1}{20}\left(k_2^A k_2^B - \frac{\delta^{AB}}{24}\sum_D (k_2^D)^2\right). \tag{5.45}$$

Here, $C^A{}_{\tilde{a}'}$ is defined by $e^A \cdot e^{\tilde{a}'}$ and we do not list up the explicit components here. The lower case $c_{\tilde{a}'}$ is the $e^{\tilde{a}'}$ component of q.

For Choice (II), the polarization tensors are

$$\zeta_{R\mu\nu} = 2\sqrt{2}\left(e^{L_q A}_{(\mu\nu)} + k_2^{L_q} q_{(\mu}e^A_{\nu)} + k_2^A q_{(\mu}e^{L_q}_{\nu)} + \frac{k_2^{L_q}k_2^A}{2} q_\mu q_\nu\right), \tag{5.46}$$

$$\zeta_{R\mu} = 2\sqrt{2}\left(k_2^{L_q} e^A_\mu + k_2^A e^{L_q}_\mu + \frac{k_2^{L_q}k_2^A}{2} q_\mu\right), \tag{5.47}$$

and the corresponding bracket state is decomposed as in (5.43) with

$$\tilde{G}^A_{\tilde{a}'\tilde{b}'} = 2\sqrt{2}\left(C^{L_q}C^A + k_2^{L_q}C^A c + k_2^A C^{L_q} c + \frac{k_2^A k_2^{L_q}}{2} cc\right)_{(\tilde{a}'\tilde{b}')}, \tag{5.48}$$

$$\tilde{G}^A = \frac{2\sqrt{2}k_2^A k_2^{L_q}}{20}. \tag{5.49}$$

For Choice (III), the polarization tensors are

$$\zeta_{R\mu\nu} = 24 e^{L_q L_q}_{(\mu\nu)} - E_{(\mu\nu)} + 48 k_2^{L_q} q_{(\mu}e^{L_q}_{\nu)} - 2\sum_D k_2^D q_{(\mu}e^D_{\nu)} + \frac{24(k_2^{L_q})^2 - \sum_D (k_2^D)^2}{2} q_\mu q_\nu, \tag{5.50}$$

$$\zeta_{R\mu} = 48 k_2^{L_q} e^{L_q}_\mu - 2\sum_D k_2^D e^D_\mu + \frac{24(k_2^{L_q})^2 - \sum_D (k_2^D)^2}{2} q_\mu, \tag{5.51}$$

and the coefficients for the decomposition of the bracket state are

$$\tilde{G}_{\tilde{a}'\tilde{b}'} = \left(24 C^{L_q}C^{L_q} - \sum_D C^D C^D + 48 k_2^{L_q} C^{L_q} c - 2\sum_D k_2^D C^D c + \frac{24(k_2^{L_q})^2 - \sum_D (k_2^D)^2}{2} cc\right)_{(\tilde{a}'\tilde{b}')}, \tag{5.52}$$

$$\tilde{G} = \frac{24(k_2^{L_q})^2 - \sum_D (k_2^D)^2}{20}. \tag{5.53}$$

5.2.3. The exact identity in terms of the standard scattering amplitudes

Now we can write down the exact identity relation in terms of $\mathcal{T}^{\mu\nu\rho}_{[3000]}$, $\mathcal{T}^{\mu;\nu}_{[3000]}$, and $\mathcal{T}^{\mu}_{[3000]}$ amplitudes which are "polynomial pieces" of the scattering amplitudes with $V_{(3)}(\tilde{k}_1, x; \zeta)$ and three tachyons insertion. For right hand side, the amplitude pieces are denoted as $\mathcal{T}^{\mu|\nu\rho}_{[1200]}$ and $\mathcal{T}^{\mu|\nu}_{[1200]}$, which come from a $V_{(1)}(k_1, x; \zeta)$, $V_{(2)}(\tilde{k}_2, 0; \zeta)$ and two tachyons amplitude. Like the previous example, the vertical line in the superscript separates an index from the first particle from ones of the second. With these pieces of the amplitudes, the exact identity is given as

$$\sum_{a',b',c'} G^{ABC}_{a'b'c'} \mathcal{T}^{a'b'c'}_{[3000]} + \sum_{a',b'} G^{ABC}_{[a'b']} \mathcal{T}^{[a';b']}_{[3000]}$$
$$= \sum_{\tilde{c}} C^{C}{}_{\tilde{c}} \left(\sum_{\tilde{a}',\tilde{b}'} \tilde{G}^{AB}_{\tilde{a}'\tilde{b}'} \mathcal{T}^{\tilde{c}|\tilde{a}'\tilde{b}'}_{[1200]} + \tilde{G}^{AB} \sum_{\tilde{a}'} \mathcal{T}^{\tilde{c}|\tilde{a}'\tilde{a}'}_{[1200]} \right),$$

for Choice (I), where the indices are $a', b', c' = L, T, I, J_i$, $\tilde{a}', \tilde{b}' = L_2, T_R, I_R, J_i$, and $\tilde{c} = T_R, I_R, J_i$. Here, $[a; b]$ represents the anti-symmetrization of the indices and the symmetrized ones are missing since they appear only as a part of decoupling amplitudes. There are also similar relations for Choice (II) and (III), where G coefficients are replaced with the ones in (5.35), (5.48) and (5.48), (5.53) respectively. By dropping trivially vanishing amplitudes summation is made explicit as, for Choice (I), (B and AB indices will be missing for Choice (II) and (III) respectively)

$$G^{ABC}_{TTT} \mathcal{T}^{TTT}_{[3000]} + 3G^{ABC}_{LTT} \mathcal{T}^{LTT}_{[3000]} + 3G^{ABC}_{LLT} \mathcal{T}^{LLT}_{[3000]} + G^{ABC}_{TTT} \mathcal{T}^{TTT}_{[3000]} + 2G^{ABC}_{[TL]} \mathcal{T}^{[T;L]}_{[3000]}$$
$$= C^{C}{}_{T_R} \big((\tilde{G}^{AB}_{T_R T_R} + \tilde{G}^{AB}) \mathcal{T}^{T_R|T_R T_R}_{[1200]} + 2\tilde{G}^{AB}_{L_2 T_R} \mathcal{T}^{T_R|L_2 T_R}_{[1200]} + (\tilde{G}^{AB}_{L_2 L_2} + \tilde{G}^{AB}) \mathcal{T}^{T_R|L_2 L_2}_{[1200]} \big)$$
$$+ C^{C}{}_{I_R} \big(\tilde{G}^{AB}_{T_R I_R} \mathcal{T}^{I_R|T_R I_R}_{[1200]} + \tilde{G}^{AB}_{L_2 I_R} \mathcal{T}^{I_R|L_2 I_R}_{[1200]} \big)$$
$$+ \sum_{i=1}^{22} C^{C}{}_{J_i} \big(\tilde{G}^{AB}_{T_R J_i} \mathcal{T}^{J_i|T_R J_i}_{[1200]} + \tilde{G}^{AB}_{L_2 J_i} \mathcal{T}^{J_i|L_2 J_i}_{[1200]} \big). \tag{5.54}$$

5.2.4. The exact identity in terms of DDF amplitudes

As before, we rewrite the level 3 and 2 bracket states for Choice (I) in terms of DDF amplitudes as, up to zero norm states,

$$(5.30) = \sum_{a,b,c} D^{ABC}_{abc} |abc\rangle_{\text{DDF}} + \sum_{a,b} \big(D^{ABC}_{(ab)} \big|(a;b)\big\rangle_{\text{DDF}} + D^{ABC}_{[ab]} \big|[a;b]\big\rangle_{\text{DDF}} \big)$$
$$+ \sum_{a} D^{ABC}_{1a} |a\rangle_{\text{DDF}} + \sum_{a,b} D^{ABC}_{2a} |abb\rangle_{\text{DDF}} + D^{ABC} \sum_{b} |b;b\rangle_{\text{DDF}},$$
$$(5.43) = \sum_{\tilde{a},\tilde{b}} \tilde{D}^{AB}_{\tilde{a}\tilde{b}} |\tilde{a}\tilde{b}\rangle_{\text{DDF}} + \sum_{\tilde{a}} \tilde{D}^{AB}_{\tilde{a}} |\tilde{a}\rangle_{\text{DDF}} + \tilde{D}^{AB} \sum_{\tilde{a}} |\tilde{a}\tilde{a}\rangle_{\text{DDF}},$$

where $a, b, c = T, I, J_i$ and $\tilde{a}, \tilde{b} = T_R, I_R, J_i$. The coefficients are determined by general consideration in B.2 as

$$D^{ABC}_{abc} = G^{ABC}_{abc}, \qquad D^{ABC}_{(ab)} = -3G^{ABC}_{Lab},$$
$$D^{ABC}_{[ab]} = G^{ABC}_{[ab]}, \qquad D^{ABC} = -\frac{1}{2} G^{ABC}_{LLL},$$
$$D^{ABC}_{1a} = \frac{1}{4} \big(9G^{ABC}_{LLa} + 2G^{ABC}_{[La]} \big), \qquad D^{ABC}_{2a} = \frac{1}{8} \big(3G^{ABC}_{LLa} - 2G^{ABC}_{[La]} \big), \tag{5.55}$$

and

$$\tilde{D}^{AB}_{\tilde{a}\tilde{b}} = \tilde{G}^{AB}_{(\tilde{a}\tilde{b})}, \qquad \tilde{D}^{AB}_{\tilde{a}} = -\sqrt{2} \tilde{G}^{AB}_{L_2 \tilde{a}}, \qquad \tilde{D}^{AB} = \frac{1}{4} \big(\tilde{G}^{AB}_{L_2 L_2} + 5\tilde{G}^{AB} \big), \tag{5.56}$$

For Choice (II) and (III), we simply replace G coefficients in the D coefficients with the corresponding ones.

By use of the DDF amplitudes that correspond to these states, the exact identities are expressed, after taking the trivially vanishing amplitudes into account, as

$$\begin{aligned}
&\big(D^{ABC}_{TTT}+D^{ABC}_{2T}\big)\mathcal{T}^{TTT}_{\text{DDF}[3000]}+\big(3D^{ABC}_{TII}+D^{ABC}_{2T}\big)\mathcal{T}^{TII}_{\text{DDF}[3000]}\\
&\quad+\sum_{i=1}^{22}\big(3D^{ABC}_{TJ_iJ_i}+D^{ABC}_{2T}\big)\mathcal{T}^{TJ_iJ_i}_{\text{DDF}[3000]}\\
&\quad+\big(D^{ABC}_{(T;T)}+D^{ABC}\big)\mathcal{T}^{(T;T)}_{\text{DDF}[3000]}+\big(D^{ABC}_{(I;I)}+D^{ABC}\big)\mathcal{T}^{(I;I)}_{\text{DDF}[3000]}\\
&\quad+\sum_{i=1}^{22}\big(D^{ABC}_{(J_i;J_i)}+D^{ABC}\big)\mathcal{T}^{(J_i;J_i)}_{\text{DDF}[3000]}+D^{ABC}_{T}\mathcal{T}^{T}_{\text{DDF}[3000]}\\
&=C^{C}{}_{T_R}\Bigg(\big(\tilde{D}^{AB}_{T_RT_R}+D^{AB}\big)\mathcal{T}^{T_R|T_RT_R}_{\text{DDF}[1200]}+\big(\tilde{D}^{AB}_{I_RI_R}+D^{AB}\big)\mathcal{T}^{T_R|I_RI_R}_{\text{DDF}[1200]}\\
&\quad+\sum_{i=1}^{22}\big(\tilde{D}^{AB}_{J_iJ_i}+D^{AB}\big)\mathcal{T}^{T_R|J_iJ_i}_{\text{DDF}[1200]}+\tilde{D}^{AB}_{T_R}\mathcal{T}^{T_R|T_R}_{[1200]}\Bigg)\\
&\quad+C^{C}{}_{I_R}\big(2\tilde{D}^{AB}_{T_RI_R}\mathcal{T}^{I_R|T_RI_R}_{\text{DDF}[1200]}+\tilde{D}^{AB}_{I_R}\mathcal{T}^{I_R|I_R}_{[1200]}\big)\\
&\quad+\sum_{i=1}^{22}C^{C}{}_{J_i}\big(2\tilde{D}^{AB}_{T_RJ_i}\mathcal{T}^{J_i|T_RJ_i}_{\text{DDF}[1200]}+\tilde{D}^{AB}_{J_i}\mathcal{T}^{J_i|J_i}_{[1200]}\big).
\end{aligned}\tag{5.57}$$

6. High-energy stringy symmetry v.s. exact identities from bracket algebra

In this section, we consider the high-energy expansion of bracket relations and examine how these relations constrain the asymptotic forms of scattering amplitudes.

The relations are, for example in the case of $\mathcal{A}[\tilde{2}000]=\mathcal{A}[1\tilde{1}00]$,

$$\sum_{a',b'}G^{AB}_{a'b'}\mathcal{T}^{a'b'}_{[2000]}+G^{AB}\sum_{a'}\mathcal{T}^{a'a'}_{[2000]}=\sum_{\tilde{a},\tilde{b}}C^{A}{}_{\tilde{a}}\tilde{G}^{B}_{\tilde{b}}\mathcal{T}^{\tilde{a}|\tilde{b}}_{[1100]},$$

for standard scattering amplitudes. We expand the transformation matrices, G and C, as well as the amplitudes under the $s\to\infty$ with $\hat{t}=t/s$ fixed limit. At each order of s, there will be relations among asymptotic amplitudes. We will explore how these relations “bootstrap” asymptotic amplitudes, and whether or not they reproduce known high-energy relations.

In this program, the transformation matrices $C^{A}{}_{a}$ (therefore, G and D) are regarded as inputs, since they are determined once we specify the momenta k_i and q. On the other hand, the amplitudes are considered to be unknowns which are to be determined. However, we need to supply information on the leading power of each amplitude,[5] such as $\mathcal{T}^{TT}_{[2000]}=\mathcal{O}(s^3)$. For both types of amplitudes, a “power counting rule” has been established [13,16,17] and it tells the relative power of a given amplitude with respect to the leading order power. Though it turns out that it

[5] As mentioned in Sections 2 and 5.1, we consider amplitudes up to a common exponential part (F_{s-t} in (5.7)), and mean the leading power by the leading power of the rest of “polynomial” parts.

is actually sufficient to know the relative powers to carry out the program, to make expressions concrete we employ our empirical knowledge on the orders of scattering amplitudes. We also need to use triviality of amplitudes, like completely transverse directions J_i must appear in a pairwise way for an amplitude to be non-vanishing. This fact reduces the number of independent unknowns. We have already taken this fact into account, for example, in (5.18) and (5.21).

6.1. High-energy linear relations from the decoupling of high-energy zero-norm states and saddle-point calculation

We are about to investigate how Moore's relations restrict amplitudes under the fixed-angle high-energy limit. In order to have a view on what kinds of relations we expect to see, we briefly review an approach based on the decoupling of (high-energy) zero-norm states and collect some known linear relations from [9,12,13]. We will also mention a couple of relations which are obtained by saddle-point calculation. In the following subsections, we examine which of these relations are extracted from exact relations by a high-energy expansion.

To illustrate the analysis, we take four point amplitudes with one level 2 state and three tachyons, $\mathcal{T}^{\mu\nu}_{[2000]}$, as an example. For simplicity, the helicity basis with respect to the momentum for the level 2 state is denoted as e^P, e^L, and e^T in this subsection. The completely transverse directions are irrelevant here. From the oscillator expressions of zero norm states, (B.9) and (B.10), one can immediately see that the amplitudes obey the following relations,

$$5\mathcal{T}^{PP}_{[2000]} + \mathcal{T}^{LL}_{[2000]} + \mathcal{T}^{TT}_{[2000]} + 5\sqrt{2}\mathcal{T}^{P}_{[2000]} = 0, \qquad \sqrt{2}\mathcal{T}^{PL}_{[2000]} + \mathcal{T}^{P}_{[2000]} = 0. \tag{6.1}$$

In the high-energy limit, the masses are negligible and e^P approximates to e^L (as explicitly seen from (4.29) and (4.30)). By taking a linear combination, one finds that in a linear combination $\mathcal{T}^{TT}_{[2000]} - 4\mathcal{T}^{LL}_{[2000]}$ the leading order part vanishes, since this combination approximates a zero-norm state in the high-energy limit. One thus obtains a linear relation in the high-energy limit,

$$\mathcal{T}^{TT}_{[2000]} = 4\mathcal{T}^{LL}_{[2000]}. \tag{6.2}$$

Actually, in order to come to this conclusion, one need to be sure that these two are indeed of leading order. As mentioned in the previous subsection, a "power counting rule" of [9,13] tells the relative power of an amplitude with a set of helicity projections compared to the leading order power, and the amplitudes in (6.2) are indeed the leading order ones. Thus this is a high-energy linear relation of this amplitude. The same argument leads to a linear relation for $\mathcal{T}^{\mu\nu\rho}_{[3000]}$ [12],

$$\mathcal{T}^{TTT}_{[3000]} : \mathcal{T}^{LLT}_{[3000]} : \mathcal{T}^{(L;T)}_{[3000]} : \mathcal{T}^{[L;T]}_{[3000]} = 8 : 1 : -1 : -1, \tag{6.3}$$

where $\mathcal{T}^{(L;T)}_{[3000]}$ and $\mathcal{T}^{[L;T]}_{[3000]}$ are symmetric and anti-symmetric combinations of the indices in an amplitude that corresponds to $\alpha^L_{-2}\alpha^T_{-1}$.

Such high-energy linear relations based on the decoupling of zero-norm states should hold in very general circumstances. The same relations should hold for other choices of extra vertex operators (three tachyons in the current examples) and also for all orders in perturbation theory. We thus take these relations as symmetry identities in the high-energy limit. On the other hand, this argument is on a state-level analysis and is confined in a set of states at a fixed level. The decoupling of high-energy zero-norm states does not give any inter-level relation, but we expect that there appear several inter-level relations as well, as all the mass levels are degenerate in the high-energy limit. Among many possible inter-level linear relations, we may be interested in the following relations among all T-polarized amplitudes,

$$\mathcal{T}^{TT}_{[2000]} = \mathcal{T}^{T|T}_{[1100]}, \qquad \mathcal{T}^{TTT}_{[3000]} = \mathcal{T}^{T|TT}_{[1200]}, \tag{6.4}$$

at the leading order. Here, all the states are generated only by the level 1 oscillator α^T_{-1} with T-polarization (with respect to the momentum of the state on which it acts) and the total level on the both hands sides are the same. As long as these two conditions are met, at the leading order, the same kind of relations hold in general; for example, $\mathcal{T}^{TTTT}_{[4000]} = \mathcal{T}^{TTT|T}_{[3100]} = \mathcal{T}^{TT|T|T}_{[2110]} = \mathcal{T}^{T|T|T|T}_{[1111]}$ and so on. These relations can be derived through direct calculation by use of the saddle-point approximation [9]. We will come back to this partonic behavior of scattering amplitudes in Section 7, but in this section, we check if this kind of relation is also obtained through Moore's relations.

In general, the appearance of such leading order relations implies that it is possible to choose another basis of physical states such that there exists a unique state in the basis at the leading order and all the other states are of subleading. Such basis has been found and discussed in [13,16] and called DDF gauge, where positive norm physical states are spanned by DDF operators. The corresponding amplitudes are DDF amplitudes, such as $\mathcal{T}^{TTT}_{\mathrm{DDF}[3000]}$. In this gauge, the leading energy dependence of an amplitude is determined by the number of T indices. For example, at level 3, $\mathcal{T}^{TTT}_{\mathrm{DDF}[3000]}$ generated by $(A^T_{-1})^3$ starts with the highest power in s. $\mathcal{T}^{T;T}_{\mathrm{DDF}[3000]}$ is at the next-to-leading order, and $\mathcal{T}^{T}_{\mathrm{DDF}[3000]}$ and $\mathcal{T}^{TII}_{\mathrm{DDF}[3000]}$ are further sub-leading. Therefore, in this gauge, leading order linear relations become trivial, and we can concentrate on inter-level relations like (6.4) as well as *subleading* relations among DDF amplitudes. Actually, we can develop a systematic high-energy expansion [17], and observe several interesting relations connecting amplitudes of different leading energy dependence. Among amplitudes generated only by A^T_{-1} and A^I_{-1}, it is found

$$\mathcal{T}^{(T)^{n-m}(I)^m}_{\mathrm{DDF}[n000]} = \mathcal{T}^{(T)^{n-m-2}(I)^{m+2}}_{\mathrm{DDF}[n000]}\left(\frac{-2s}{m+1} + \mathcal{O}\left(s^0\right)\right). \tag{6.5}$$

Namely, up to subleading corrections, the following relations obey:

$$\mathcal{T}^{TT}_{\mathrm{DDF}[2000]} = -2s\mathcal{T}^{II}_{\mathrm{DDF}[2000]}, \qquad \mathcal{T}^{TTTT}_{\mathrm{DDF}[4000]} = -2s\mathcal{T}^{TTII}_{\mathrm{DDF}[4000]} = \frac{4s^2}{3}\mathcal{T}^{IIII}_{\mathrm{DDF}[4000]}. \tag{6.6}$$

Since Moore's relation is exact, we should be able to obtain such inter-level and inter-energy-level relations from it. As a preliminary trial, we shall derive a first few nontrivial relations among DDF amplitudes in the following subsections.

6.2. *High-energy expansions of the scattering amplitudes:* $\mathcal{A}[\tilde{2}000] = \mathcal{A}[1\tilde{1}00]$

As explained in the beginning of this section, we take the large-s expansions of the amplitudes and the coefficients of the exact relation (5.18), with $\hat{t} = t/s$ fixed, as

$$\mathcal{T}^{TT}_{[2000]} = \mathcal{T}^{TT}_{[2000](3)}s^3 + \mathcal{T}^{TT}_{[2000](2)}s^2 + \cdots, \tag{6.7}$$

$$G^{T_qT_q}_{TT} = G^{T_qT_q}_{TT(0)} + G^{T_qT_q}_{TT(-1)}s^{-1} + \cdots, \tag{6.8}$$

and so on. Here, on the right hand side, factors like $\mathcal{T}^{TT}_{[2000](n)}$ denote coefficients of s^n and are in general functions of $\hat{t}$, for example, $\mathcal{T}^{TT}_{[2000](3)} = \frac{-\hat{t}(1+\hat{t})}{4}$.[6]

[6] Explicit expressions of amplitudes are found in the preprint version (v2) of the manuscript. You may obtain all the other coefficients by use of them.

In the bracket relation (5.18), by collecting the terms at the same order in s, we can find several relations among these expansion coefficients. For example, for $(A,B)=(T_q,T_q)$ choice, the coefficient of s^3 reads

$$0=\left(G^{T_qT_q}_{TT(0)}+G^{T_qT_q}_{(0)}\right)\mathcal{T}^{TT}_{[2000](3)}+\left(G^{T_qT_q}_{LL(0)}+G^{T_qT_q}_{(0)}\right)\mathcal{T}^{LL}_{[2000](3)} \\ -C^{T_q}{}_{T_R(0)}\tilde{G}^{T_q}_{T_R(0)}\mathcal{T}^{T_R|T_R}_{[1100](3)}. \tag{6.9}$$

After evaluating C and G coefficients by use of their asymptotic forms, it leads to asymptotic relations among the leading order part of scattering amplitudes.

Before going further, we point out that the coefficient function for the right hand side, $C^A{}_{\tilde{a}}\tilde{G}^B_{\tilde{b}}$ is constant and almost diagonal. The amplitudes $\mathcal{T}^{\tilde{a}|\tilde{b}}_{[1100]}$ are non-zero only for $(\tilde{a},\tilde{b})=(T_R,T_R)$, $(T_R,\tilde{L}_2)$, (I_R,I_R), and (J_i,J_i). Here $e^{\tilde{L}_2}$ is one of the basis vector for $\tilde{k}_2$ helicity basis, $k_2\cdot e^{\tilde{L}_2}=1$. For these $\tilde{a},\tilde{b}$, non-vanishing coefficients are

$$C^{I_q}{}_{I_R}\tilde{G}^{I_q}_{I_R}=-1,\qquad C^{T_q}{}_{T_R}\tilde{G}^{T_q}_{T_R}=1,\qquad C^{J_i}{}_{J_i}\tilde{G}^{J_i}_{J_i}=1\quad(\text{no sum for } i),$$

and the unphysical projection onto $e^{\tilde{L}_2}$ does not appear. Therefore, on the right hand side, the coefficients can be regarded as constants and the asymptotic expansion only involves the expansion of the amplitudes.

To first two orders with $(A,B)=(T_q,T_q)$, (I_q,I_q), (J,J), and (T_q,I_q), the relations are

$$\mathcal{O}(s^3):\quad 0=\frac{19}{20}\mathcal{T}^{TT}_{[2000](3)}+\frac{1}{5}\mathcal{T}^{LL}_{[2000](3)}-\mathcal{T}^{T_R|T_R}_{[1100](3)}, \tag{6.10}$$

$$0=\mathcal{T}^{TT}_{[2000](3)}-4\mathcal{T}^{LL}_{[2000](3)}, \tag{6.11}$$

$$\mathcal{O}(s^2):\quad 0=\frac{19}{20}\mathcal{T}^{TT}_{[2000](2)}-2\mathcal{T}^{LL}_{[2000](3)}+\frac{1}{5}\mathcal{T}^{LL}_{[2000](2)}-\mathcal{T}^{T_R|T_R}_{[1100](2)}, \tag{6.12}$$

$$0=-\frac{1}{20}\mathcal{T}^{TT}_{[2000](2)}+6\mathcal{T}^{LL}_{[2000](3)}+\frac{1}{5}\mathcal{T}^{LL}_{[2000](2)}+\mathcal{T}^{I_R|I_R}_{[1100](2)}, \tag{6.13}$$

$$0=-\frac{1}{20}\mathcal{T}^{TT}_{[2000](2)}-2\mathcal{T}^{LL}_{[2000](3)}+\frac{1}{5}\mathcal{T}^{LL}_{[2000](2)}-\mathcal{T}^{J|J}_{[1100](2)}, \tag{6.14}$$

$$0=(2\hat{t}+1)\mathcal{T}^{TT}_{[2000](3)}+\sqrt{-2\hat{t}(1+\hat{t})}\,\mathcal{T}^{TL}_{[2000](5/2)}, \tag{6.15}$$

$$\mathcal{O}(s):\quad 0=-2\left(4\hat{t}^2+6\hat{t}+1\right)\mathcal{T}^{TT}_{[2000](3)}+\left(2\hat{t}^2+3\hat{t}+1\right)\mathcal{T}^{TT}_{[2000](2)} \\ +\sqrt{-2\hat{t}(1+\hat{t})}(1+\hat{t})\mathcal{T}^{TL}_{[2000](3/2)}. \tag{6.16}$$

From (6.10) and (6.11), it is easy to obtain the following linear relations

$$\mathcal{T}^{TT}_{[2000](3)}=4\mathcal{T}^{LL}_{[2000](3)},\qquad \mathcal{T}^{TT}_{[2000](3)}=\mathcal{T}^{T_R|T_R}_{[1100](3)}. \tag{6.17}$$

The first one is the linear relation (6.2) derived from the decoupling of zero-norm states, while the second one is an inter-level relation (6.4).

On $\mathcal{A}[1\tilde{1}00]$ side, I_R and J_i have the same geometrical meaning; namely, both represent completely transverse directions to the scattering plane. Therefore, replacing (J_i,J_i) with (I_R,I_R) does not change the amplitudes. More explicitly, $\mathcal{T}^{I_R|I_R}_{[1100]}=\mathcal{T}^{J|J}_{[1100]}$ holds for all orders in s. By use of this rotational symmetry for (6.12), (6.13), and (6.14), we find further relations,

$$4\mathcal{T}^{LL}_{[2000](3)}+\mathcal{T}^{I_R|I_R}_{[1100](2)}=0,\qquad \mathcal{T}^{TT}_{[2000](2)}-\mathcal{T}^{T_R|T_R}_{[1100](2)}+\mathcal{T}^{I_R|I_R}_{[1100](2)}=0. \tag{6.18}$$

The rest of the relations (6.15) and (6.16) provide some angle-dependent relations. As already mentioned, we have used the explicit s dependence of each amplitudes. The power counting rule [13] provides the information on relative power of each amplitudes. Since the relation is homogeneous, these relative powers of the amplitudes suffice to determine high-energy linear relations; some of them have been derived by use of the decoupling of high-energy zero norm states [12,13], and we also find extra subleading relations and inter-level relations.

6.3. High-energy expansions of the DDF amplitudes: $\mathcal{A}[\tilde{2}000] = \mathcal{A}[1\tilde{1}00]$

We move on to exact relations among DDF amplitudes (5.21). Again, we expand all the elements that appear in (5.21) with respect to its s dependence, $\mathcal{T}^{TT}_{\mathrm{DDF}[2000]} = \mathcal{T}^{TT}_{\mathrm{DDF}[2000]\,(3)} s^3 + \cdots$, $D^{I_q I_q}_{II} = D^{I_q I_q}_{II\,(0)} + D^{I_q I_q}_{II\,(-1)} s^{-1} + \cdots$, and so on. We then organize the exact relations with respect to its s dependence as before, with D coefficients evaluated, as[7]

$$\mathcal{O}(s^3): \qquad \mathcal{T}^{TT}_{\mathrm{DDF}[2000]\,(3)} = \mathcal{T}^{T_R|T_R}_{[1100](3)}, \tag{6.19}$$

$$\mathcal{O}(s^2): \qquad \mathcal{T}^{TT}_{\mathrm{DDF}[2000]\,(2)} - \frac{1}{2}\mathcal{T}^{TT}_{\mathrm{DDF}[2000]\,(3)} = \mathcal{T}^{T_R|T_R}_{[1100]\,(2)}, \tag{6.20}$$

$$\mathcal{T}^{JJ}_{\mathrm{DDF}[2000]\,(2)} - \frac{1}{2}\mathcal{T}^{TT}_{\mathrm{DDF}[2000]\,(3)} = \mathcal{T}^{J|J}_{[1100]\,(2)}, \tag{6.21}$$

$$\mathcal{T}^{II}_{\mathrm{DDF}[2000]\,(2)} + \frac{3}{2}\mathcal{T}^{TT}_{\mathrm{DDF}[2000]\,(3)} = -\mathcal{T}^{I_R|I_R}_{[1100]\,(2)}, \tag{6.22}$$

$$\frac{4\hat{t}+2}{\sqrt{-\hat{t}(\hat{t}+1)}}\mathcal{T}^{TT}_{\mathrm{DDF}[2000]\,(3)} - 2\mathcal{T}^{T}_{\mathrm{DDF}[2000]\,(5/2)} = 0, \tag{6.23}$$

where J represents one of 22 J_i, and the $\mathcal{O}(s^3)$ relation is for $(A,B) = (T_q,T_q)$ while at $\mathcal{O}(s^2)$ the relations are for $(A,B) = (T_q,T_q), (J,J), (I_q,I_q)$ and (T_q,I_q) in order. By using the fact that $\mathcal{T}^{JJ}_{\mathrm{DDF}[2000]} = \mathcal{T}^{II}_{\mathrm{DDF}[2000]}$ and $\mathcal{T}^{J|J}_{[1100]} = \mathcal{T}^{I_R|I_R}_{[1100]}$, which follows from the rotational symmetry in the directions transverse to the scattering plane, one can further derive

$$2\mathcal{T}^{II}_{\mathrm{DDF}[2000]\,(2)} = \mathcal{T}^{I_R|I_R}_{[1100]\,(2)}. \tag{6.24}$$

The leading order relation (6.19) is a DDF-amplitude counter part of the relation (6.4). (6.22), combined with (6.24), gives a known DDF amplitude relation (6.6).

At order $\mathcal{O}(s)$, the relations involve $\hat{t}$, then the scattering angle. For example, for $A = B = I_q$, one finds (we omit the subscript DDF[2000] and [1100] for simplicity)

$$\begin{aligned}-2\hat{t}(\hat{t}+1)\mathcal{T}^{I_R|I_R}_{(1)} &= 12(2\hat{t}+1)\sqrt{\hat{t}+1}\sqrt{-\hat{t}}\,\mathcal{T}^{T}_{(5/2)} - 8(2\hat{t}+1)^2\mathcal{T}^{TT}_{(3)} \\ &\quad + \hat{t}(\hat{t}+1)\big(-9\mathcal{T}^{II}_{(2)} + 2\mathcal{T}^{II}_{(1)} + 3\mathcal{T}^{TT}_{(2)} + 66\mathcal{T}^{JJ}_{(2)}\big).\end{aligned}$$

By taking some linear combinations of four relations at this order, we can derive some $\hat{t}$-independent relations, for example,

$$\mathcal{T}^{T_R|T_R}_{(1)} - \mathcal{T}^{I_R|I_R}_{(1)} + 2\mathcal{T}^{J|J}_{(1)} = \mathcal{T}^{TT}_{(1)} + \mathcal{T}^{II}_{(1)} + 2\mathcal{T}^{JJ}_{(1)}. \tag{6.25}$$

Again, by employing an obvious symmetry, $I_R \to J$ and $J \to I$ on both hands sides, we have

7 Recall that the right hand side, $\mathcal{A}[1\tilde{1}00]$ side, is the same as the previous case.

$$\mathcal{T}_{(1)}^{T_R|T_R}+\mathcal{T}_{(1)}^{I_R|I_R}=\mathcal{T}_{(1)}^{TT}+3\mathcal{T}_{(1)}^{II}. \tag{6.26}$$

In summary, by using the power counting rule of the DDF amplitudes, the bracket relation provides some of the known asymptotic relations, such as (6.19), and also several subleading relations.

6.4. High-energy expansions of the scattering amplitudes: $\mathcal{A}[\tilde{3}000]=\mathcal{A}[1\tilde{2}00]$

As in the case of $\mathcal{A}[\tilde{2}000]=\mathcal{A}[1\tilde{1}00]$ relation, we expand the amplitudes and the G matrices in the bracket relation (5.54) to derive the high-energy relations. In this case, there are many relations for each Choice (I), (II), and (III) with various choices of A, B, and C.

In the case of Choice (I), for $(A,B,C)=(T_q,T_q,T_q)$, the first two leading order relations read

$$\begin{aligned}
\mathcal{O}(s^{13/2}):\quad & 0=C^{T_q}{}_{T(0)}\big(\tilde{G}^{T_qT_q}_{T_RT_R(2)}\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}+\tilde{G}^{T_qT_q}_{L_2L_2(2)}\mathcal{T}^{T_R|L_2L_2}_{[1200](9/2)}\big),\\
\mathcal{O}(s^{11/2}):\quad & 0=C^{T_q}{}_{T_R(0)}\tilde{G}^{T_qT_q}_{T_RL_2(3/2)}\mathcal{T}^{T_R|T_RL_2}_{[1200](4)}+C^{T_q}{}_{T_R(0)}\tilde{G}^{T_qT_q}_{T_RT_R(2)}\mathcal{T}^{T_R|T_RT_R}_{[1200](7/2)}\\
&\quad+C^{T_q}{}_{T(0)}\tilde{G}^{T_qT_q}_{L_2L_2(2)}\mathcal{T}^{T_R|L_2L_2}_{[1200](7/2)}+C^{T_q}{}_{T_R(-1)}\tilde{G}^{T_qT_q}_{L_2L_2(2)}\mathcal{T}^{T_R|L_2L_2}_{[1200](9/2)}\\
&\quad+\big(C^{T_q}{}_{T_R(-1)}G^{T_qT_q}_{(2)}+C^{T_q}{}_{T_R(0)}\tilde{G}^{T_qT_q}_{(1)}+C^{T_q}{}_{T_R(0)}\tilde{G}^{T_qT_q}_{T_RT_R(1)}\big)\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}.
\end{aligned}$$

Interestingly, unlike the previous case, some of $\tilde{G}$ coefficients on the right hand side have positive power in s, while the left hand side coefficients do not. Thus, first few leading order relations in s only involve amplitudes from $\mathcal{A}[1\tilde{2}00]$. We now write down the leading order relations, with C, G and $\tilde{G}$ coefficients evaluated, for various choices of (A,B,C) as

$$0=4\mathcal{T}^{T_R|L_2L_2}_{[1200](9/2)}-\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}, \tag{6.27}$$

$$0=\sqrt{2}\mathcal{T}^{T_R|T_RL_2}_{[1200](4)}-\frac{(2\hat{t}+1)\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}, \tag{6.28}$$

$$\begin{aligned}
0=&-3\mathcal{T}^{[L;T]}_{[3000](9/2)}+\frac{1}{6}\mathcal{T}^{TTT}_{[3000](9/2)}+\frac{83}{3}\mathcal{T}^{TLL}_{[3000](9/2)}+2\frac{\sqrt{2}(2\hat{t}+1)\mathcal{T}^{T_R|T_RL_2}_{[1200](4)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}\\
&+4\mathcal{T}^{I_R|T_RI_R}_{[1200](7/2)}+2\frac{(2\hat{t}+1)^2\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}}{(\hat{t}+1)\hat{t}},
\end{aligned} \tag{6.29}$$

$$0=18\mathcal{T}^{[L;T]}_{[3000](9/2)}-\mathcal{T}^{TTT}_{[3000](9/2)}+26\mathcal{T}^{TLL}_{[3000](9/2)}, \tag{6.30}$$

$$\begin{aligned}
0=&-2\mathcal{T}^{[L;T]}_{[3000](9/2)}+6\mathcal{T}^{TLL}_{[3000](9/2)}+\frac{1}{2}\frac{\sqrt{2}(2\hat{t}+1)\mathcal{T}^{T_R|T_RL_2}_{[1200](4)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}+\mathcal{T}^{I_R|T_RI_R}_{[1200](7/2)}\\
&+\frac{1}{2}\frac{(2\hat{t}+1)^2\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}}{\hat{t}(\hat{t}+1)},
\end{aligned} \tag{6.31}$$

$$\begin{aligned}
0=&-\frac{(2\hat{t}+1)\mathcal{T}^{[L;T]}_{[3000](9/2)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}+\frac{2}{3}\mathcal{T}^{TTL}_{[3000](4)}-\frac{52}{3}\mathcal{T}^{LLL}_{[3000](4)}+\frac{1}{2}\frac{(2\hat{t}+1)\mathcal{T}^{TTT}_{[3000](9/2)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}\\
&-37\frac{(2\hat{t}+1)\mathcal{T}^{TLL}_{[3000](9/2)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}+4\sqrt{2}\mathcal{T}^{J|JL_2}_{[1200](3)}-4\frac{(2\hat{t}+1)\mathcal{T}^{J|T_RJ}_{[1200](7/2)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}},
\end{aligned} \tag{6.32}$$

$$0=\frac{1}{6}\mathcal{T}^{TTL}_{[3000](4)}-\frac{13}{3}\mathcal{T}^{LLL}_{[3000](4)}+\frac{1}{9}\frac{(2\hat{t}+1)\mathcal{T}^{TTT}_{[3000](9/2)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}-\frac{80}{9}\frac{(2\hat{t}+1)\mathcal{T}^{TLL}_{[3000](9/2)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}$$
$$+\sqrt{2}\mathcal{T}^{J|JL_2}_{[1200](3)}-\frac{(2\hat{t}+1)\mathcal{T}^{J|T_RJ}_{[1200](7/2)}}{\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}, \tag{6.33}$$

where J stands for one of J_i.

From (6.27)–(6.28), we find

$$\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}=4\mathcal{T}^{T_R|L_2L_2}_{[1200](9/2)}=\frac{\sqrt{2}\sqrt{-\hat{t}}\sqrt{\hat{t}+1}}{2\hat{t}+1}\mathcal{T}^{T_R|T_RL_2}_{[1200](4)}. \tag{6.34}$$

The first equality is the linear relation at level 2 we have seen before, (6.2). Using this relation, (6.29)–(6.31) lead to

$$\mathcal{T}^{TTT}_{[3000](9/2)}=8\mathcal{T}^{TLL}_{[3000](9/2)}=-8\mathcal{T}^{[L;T]}_{[3000](9/2)}=-\mathcal{T}^{I_R|T_RI_R}_{[1200](7/2)}. \tag{6.35}$$

The first three relations are indeed the linear relation (6.3) obtained by use of the decoupling of zero-norm states. Since $\alpha^{(L}_{-2}\alpha^{T)}_{-1}|0;\tilde{k}_1\rangle$ appears in zero-norm states, $\mathcal{T}^{(L;T)}_{[3000]}$ is missing in our computation here.

So far, one can see that the leading order amplitudes, $\mathcal{T}^{TTT}_{[3000](9/2)}$ and $\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}$, can be regarded as basic ones to represent the rest of amplitudes. The final two relations, (6.32) and (6.33), give a single relation

$$0=\mathcal{T}^{TTL}_{[3000](4)}-26\mathcal{T}^{LLL}_{[3000](4)}+6\sqrt{2}\mathcal{T}^{J|JL_2}_{[1200](3)}, \tag{6.36}$$

after using the rotational symmetry, $\mathcal{T}^{J|T_RJ}_{[1200]}=\mathcal{T}^{I_R|T_RI_R}_{[1200]}$. It involves other leading order amplitudes. By direct computation, one can check that these three coefficients are proportional to $\hat{t}(\hat{t}+1)(2\hat{t}+1)$, and then they are actually proportional to one another.

The next-to-leading order relations are very complicated in general. Among those, the highest order one at $\mathcal{O}(s^{11/2})$ is found to provide a simple relation

$$0=10\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}-\mathcal{T}^{T_R|T_RT_R}_{[1200](7/2)}+4\mathcal{T}^{T_R|L_2L_2}_{[1200](7/2)}. \tag{6.37}$$

Explicit evaluation shows that these three are not proportional to one another, as a function of $\hat{t}$, unlike the previous example. Other relations are more involved and contain other subleading order amplitudes. We do not present explicit forms of them here.

The leading order relations (6.34) and (6.35) reproduce some of the leading order relations observed in [12,13]. The relation between all T-polarized amplitudes, $\mathcal{T}^{TTT}_{[3000](9/2)}=\mathcal{T}^{T_R|T_RT_R}_{[1200](9/2)}$, is not obtained from Moore's relations, up to this order. If we supply it, the leading order relations up to $\mathcal{O}(s^{9/2})$, (6.34) and (6.35), are written by one of them.

6.5. High-energy expansions of the DDF amplitudes: $\mathcal{A}[\tilde{3000}]=\mathcal{A}[\tilde{1200}]$

Finally, we consider asymptotic relations among DDF amplitudes, (5.57). As before, we expand the DDF amplitudes and the D matrices, We list the leading order ones from Choice (I), (II), and (III), with the same structure ones collected, as

$$0=\frac{5\hat{t}^2+5\hat{t}+1}{(-\hat{t})(1+\hat{t})}\mathcal{T}^{T_R|T_RT_R}_{\mathrm{DDF}[1200](9/2)}+\frac{2\hat{t}+1}{\sqrt{(-\hat{t})(1+\hat{t})}}\mathcal{T}^{T_R|T_R}_{\mathrm{DDF}[1200](4)} - 2\mathcal{T}^{T_R|I_RI_R}_{\mathrm{DDF}[1200](7/2)}, \tag{6.38}$$

$$0=\frac{(2\hat{t}+1)(5\hat{t}^2+5\hat{t}+1)}{(-\hat{t})^{3/2}(1+\hat{t})^{3/2}}\mathcal{T}^{T_R|T_RT_R}_{\mathrm{DDF}[1200](9/2)}+\frac{(2\hat{t}+1)^2}{(-\hat{t})(1+\hat{t})}\mathcal{T}^{T_R|T_R}_{\mathrm{DDF}[1200](4)} - \frac{2(2\hat{t}+1)}{\sqrt{(-\hat{t})(1+\hat{t})}}\left(2\mathcal{T}^{I_R|T_RI_R}_{\mathrm{DDF}[1200](7/2)}+\mathcal{T}^{T_R|I_RI_R}_{\mathrm{DDF}[1200](7/2)}\right)-2\mathcal{T}^{I_R|I_R}_{\mathrm{DDF}[1200](3)}, \tag{6.39}$$

$$0=\frac{2\hat{t}+1}{\sqrt{(-\hat{t})(1+\hat{t})}}\mathcal{T}^{T_R|T_RT_R}_{\mathrm{DDF}[1200](9/2)}+\mathcal{T}^{T_R|T_R}_{\mathrm{DDF}[1200](4)}, \tag{6.40}$$

$$0=-2\mathcal{T}^{TTT}_{\mathrm{DDF}[3000](9/2)}+\frac{(2\hat{t}+1)^2}{(-\hat{t})(1+\hat{t})}\mathcal{T}^{T_R|T_RT_R}_{\mathrm{DDF}[1200](9/2)}+\frac{2\hat{t}+1}{\sqrt{(-\hat{t})(1+\hat{t})}}\mathcal{T}^{T_R|T_R}_{\mathrm{DDF}[1200](4)} - 2\mathcal{T}^{I_R|T_RI_R}_{\mathrm{DDF}[1200](7/2)}, \tag{6.41}$$

$$0=\left(D^{T_qJJ}_{(0)}+D^{T_qJJ}_{TTT(0)}\right)\mathcal{T}^{TTT}_{\mathrm{DDF}[3000](9/2)}, \tag{6.42}$$

$$0=-\frac{2\hat{t}+1}{\sqrt{(-\hat{t})(1+\hat{t})}}\left(\mathcal{T}^{TTT}_{\mathrm{DDF}[3000](9/2)}+\mathcal{T}^{J_i|TJ_i}_{\mathrm{DDF}[1200](7/2)}\right) + 2\mathcal{T}^{T;T}_{\mathrm{DDF}[3000](4)}-\mathcal{T}^{J_i|J_i}_{\mathrm{DDF}[1200](3)}. \tag{6.43}$$

For (6.42), the coefficient vanishes once the explicit expression is plugged in. So this relation does not give any constraint on the amplitudes. The other relations impose nontrivial relations among DDF amplitudes of a fixed order of s. So far, there appear nine amplitudes and there are five relations. The first four relations are simplified as

$$\mathcal{T}^{T_R|T_R}_{\mathrm{DDF}[1200](4)}=-\frac{2\hat{t}+1}{\sqrt{(-\hat{t})(1+\hat{t})}}\mathcal{T}^{T_R|T_RT_R}_{\mathrm{DDF}[1200](9/2)}, \tag{6.44}$$

$$\mathcal{T}^{I_R|I_R}_{\mathrm{DDF}[1200](3)}=\frac{-2(2\hat{t}+1)}{\sqrt{(-\hat{t})(1+\hat{t})}}\mathcal{T}^{I_R|T_RI_R}_{\mathrm{DDF}[1200](7/2)}, \tag{6.45}$$

$$\mathcal{T}^{T_R|I_RI_R}_{\mathrm{DDF}[1200](7/2)}=-\frac{1}{2}\mathcal{T}^{T_R|T_RT_R}_{\mathrm{DDF}[1200](9/2)},\qquad \mathcal{T}^{I_R|T_RI_R}_{\mathrm{DDF}[1200](7/2)}=-\mathcal{T}^{TTT}_{\mathrm{DDF}[3000](9/2)}. \tag{6.46}$$

The relations in the last line are the ones mentioned in Section 6.1. Together with the last relation involving J_i index, one can see that the subleading amplitudes are related to the higher order part of the amplitudes. By use of the rotational symmetry, $I_R\to J$, one further finds

$$\mathcal{T}^{I_R|I_R}_{\mathrm{DDF}[1200](3)}=2\mathcal{T}^{T;T}_{\mathrm{DDF}[3000](4)}. \tag{6.47}$$

Thus, these amplitudes are represented by two of the leading order parts, say $\mathcal{T}^{T_R|T_RT_R}_{\mathrm{DDF}[1200](9/2)}$ and $\mathcal{T}^{TTT}_{\mathrm{DDF}[3000](9/2)}$. The next order relations involve new leading order part (like $\mathcal{T}^{T}_{\mathrm{DDF}[3000](7/2)}$) or subleading parts of the amplitudes. In this way, the exact bracket relations provide constraints on the high-energy expansions of the DDF amplitudes, and relate them in a complicated manner. The relation between the leading order parts of the amplitudes of T-projection,

$$\mathcal{T}^{TTT}_{\mathrm{DDF}[3000](9/2)}=\mathcal{T}^{T_R|T_RT_R}_{\mathrm{DDF}[1200](9/2)}, \tag{6.48}$$

is not obtained also in this case. If we assume this relation, one can see that the higher order amplitudes can be written in terms of one of the leading order parts.

7. Summary and conclusion

In this paper, through a detailed study of bracket state spectrum and a new definition of q-orthonormal helicity basis, we re-examine the exact symmetry identities in bosonic open string theory as first derived by G.W. Moore. Based on two illustrative case studies, we are able to spell out the concrete kinematic contents of the symmetry relations among tree-level four-point amplitudes of low-lying stringy excitations. These relations are also recast into identities among conventional and DDF amplitudes, where the participant states in the scattering amplitudes in each basis are more familiar and form well-defined representations of other symmetry groups. In so doing, we can also connect and compare with other well-known symmetry patterns of scattering amplitudes in string theory, in particular, the high-energy symmetry as advocated by D.J. Gross. We show that, under certain presumptions, part of high-energy symmetry (especially the linear relations as derived from decoupling of the high-energy zero-norm states) can be extracted from a high-energy expansion of Moore's symmetry relations. Furthermore, we can detect some of the energy hierarchy of the DDF amplitudes and give new inter-level and subleading relations from a high-energy expansion point of view.

To summarize, our findings in this paper are:

(1) While Moore's exact symmetry identities lead to infinitely many strong constrains among inter-level amplitudes, their high-energy limits do not have complete overlap with the high-energy linear relations as derived from decoupling of high-energy zero-norm states [9,12]. To make connections and comparisons with the high-energy symmetry of Gross, we need to make high-energy expansions of both transformation matrices and the relevant scattering amplitudes. Only if we obtain a closed set of linear equations among leading components of scattering amplitudes with various physical polarizations, we can derive some constraints among leading amplitudes at the same mass level. Given that the transformation matrices among stringy state bases are energy-dependent, the mixing of different components (organized by powers of s) of various stringy scattering amplitudes is unavoidable and the existence of closed relations is definitely non-trivial. For instance, (6.36) as derived from the order s^4 relations in $\mathcal{A}[\tilde{3}000] = \mathcal{A}[1\tilde{2}00]$, is a constraint among amplitudes, $\mathcal{T}^{TTL}_{[3000](4)}$, $\mathcal{T}^{LLL}_{[3000](4)}$, and $\mathcal{T}^{J|JL_2}_{[1200](3)}$, but it is not strong enough to give the linear relation as shown in [9,12]. It is, however, not a problem of Moore's relation. As we have seen, there are infinitely many different ways to realize bracket operators at a given level, and they are related to different sets of amplitudes through Moore's relation. Therefore, the missing link observed here should be connected if we include further sets of relations. The failure here means that our simple example is not a sufficient set to obtain all the known relations. Note that the successful case of $\mathcal{A}[\tilde{3}000] = \mathcal{A}[1\tilde{2}00]$, (6.35), where the linear relation among $\mathcal{T}^{TTT}_{[3000](9/2)}$, $\mathcal{T}^{TLL}_{[3000](9/2)}$, and $\mathcal{T}^{[L,T]}_{[3000](9/2)}$ can be derived from the high-energy expansions of Moore's exact identities, in our opinion, should be taken as an accident.

(2) In our previous studies of the symmetry patterns of stringy scattering amplitudes [17], we have observed a special feature of the all-transversely-polarized (with respect to each own momentum) scattering amplitudes. The high-energy limits of this class of stringy scattering amplitudes demonstrate some partonic behaviors which we call string bit pictures. For instance, (6.48) provides one simple example, where as long as the total level numbers and total spins of scattering states are the same ($3 + 0 + 0 + 0 = 1 + 2 + 0 + 0$ in this case), the scattering amplitudes are always equal (up to sign). For $\mathcal{A}[\tilde{3}000] = \mathcal{A}[1\tilde{2}00]$ example, such an inter-level

symmetry pattern was only observed by explicit evaluations of stringy scattering amplitudes. While this inter-level symmetry pattern is derived by Moore's exact symmetry in the case of $\mathcal{A}[\tilde{2}000] = \mathcal{A}[1\tilde{1}00]$, due to the small number of the amplitudes, it is unlikely deducible for higher level amplitudes. The use of such a sting bit picture helps in simplifying high-energy relation in Moore's exact symmetry identities. Together with the rotational invariance (trading I (or I_R) polarizations into J direction), we are able to represent all leading-components (up to order $s^{9/2}$) of stringy scattering amplitudes in the $\mathcal{A}[\tilde{3}000] = \mathcal{A}[1\tilde{2}00]$ relation by a single amplitude $\mathcal{T}^{TTT}_{\mathrm{DDF}[3000](9/2)}$.

(3) In our study, it is clear that due to the energy dependence of the transformation matrices, the sub-leading components of different energy orders in various stringy scattering amplitudes will get mixed in the high-energy relations from Moore's identities. In order to make a systematic expansion and compare various components we need to choose a set of reference kinematic variables, which are s and $\hat{t} \equiv t/s$ in our two case studies. The fact that we have chosen special scattering processes such that, in each case, both sides of the Moore's exact identity share the same Mandelstam variables, is simply for the sake of convenience. In general, it is not clear that if there are many sets of Mandelstam variables $\{(s_a, t_a)^4_{a=1}\}$, how to choose the best reference kinematic variables. In addition, since each amplitude defines its own scattering plane and generally we have to compare scattering processes at different scattering angles. One has to be careful in making conclusions about the sub-leading patterns of the high-energy relations in Moore's exact identities. We give a brief illustration of subtleties about the choice of reference kinematic variables in Appendix C.

Aside from these symmetry relations and the connections among different approaches which are realized by scattering amplitudes as functions of kinematic variables, there are other issues worth further exploration:

- The algebra of bracket commutator and the origin of symmetry breakdown in string theory. This is a question of fundamental importance in string theory. One can easily imagine the bracket algebra relating different stringy excitation should be some kind of residue symmetry after spontaneous breakdown of certain (presumably infinite-dimensional) symmetry. See discussion in [18]. One of the original motivations studying the high-energy symmetry is to make an analogy of equivalence theorem in electroweak theory. It is not clear if we can view these exact relations (which apparently look kinematic dependent) as a non-linear realization of the broken symmetry.
- Similarity between constructions of DDF states and that of bracket states, especially in the q-orthonormal base, is of interest. As demonstrated in [19], the algebraic structure among bracket states may be understood as a kind of Kac–Moody algebra, at least partially.
- Following the previous line of thought, one might wish to prove other kinematic limits these symmetry relations e.g. Regge limits ([14]) to see if we can obtain other useful patterns or connect with different approaches.
- We can make further generalizations by choosing different space–time backgrounds, adding supersymmetry, or studying string theory at finite temperature. The study of loop amplitudes may require special efforts.
- Similar structure and patterns can be captured in the exactly solvable string theory [20]. In either minimal string models or matrix models one may be able to give more mathematical insight to this ultimate question.

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Appendix A. More on bracket states and the physical state conditions

In this subsection, we present an interesting example in which a deformer operator that does not satisfy physical state conditions generates physical bracket operators, by using bracket operators in $\mathcal{A}[\tilde{3}000] = \mathcal{A}[1\tilde{2}00]$ relation. In this appendix, we take $\alpha' = 1/2$ for simplicity. The physical state conditions for the level 2 bracket operator $V^{\text{br}}_{(2)}(\tilde{k}_2, z)$ are reduced to the conditions on the deformer polarization tensors as

$$0 = [\zeta_q \cdot q + \tilde{\zeta}_q]_\mu + q_\mu k_2 \cdot [\zeta_q \cdot q + \tilde{\zeta}_q], \tag{A.1}$$

$$0 = \zeta_{q\mu\nu}\eta^{\mu\nu} + 2\tilde{\zeta}_q \cdot q + 6k_2 \cdot [\zeta_q \cdot q + \tilde{\zeta}_q], \tag{A.2}$$

with $q^2 = -2$ and $k_2 \cdot q = -1$. From $q \cdot k_2 = -1$, we have a solution of (A.1),

$$\zeta_{q\mu\nu}q^\nu + \tilde{\zeta}_{q\mu} = cq_\mu, \quad c \in \mathbb{C}. \tag{A.3}$$

It is easy to see that $\zeta_{q\mu\nu}q^\nu + \tilde{\zeta}_{q\mu}$ does not have a component transverse to q, and this is a general solution with one parameter c. Plugging this into (A.2), we find

$$\zeta_{q\mu\nu}\left(\eta^{\mu\nu} - 2q^\mu q^\nu\right) = 10c. \tag{A.4}$$

When $c = 0$, $\zeta_{q\mu\nu}$ and $\tilde{\zeta}_{q\mu}$ satisfy the physical state conditions for $J_{(2)}(q, w)$. Thus, c measures a failure of those conditions. We are interested in whether the conditions (A.3) and (A.4) allow a solution with $c \neq 0$. To solve these conditions, we introduce a helicity basis with respect to q; $e^{P_q} = q/\sqrt{2}$ and transverse orthonormal vectors e^{T_i} $(i = 1, \cdots, 25)$. When $c \neq 0$, solutions are

$$c = \frac{1}{10}: \quad \zeta_{q\mu\nu} = e^{T_i}_\mu e^{T_i}_\nu, \quad \tilde{\zeta}_{q\mu} = \frac{\sqrt{2}}{10}e^{P_q}_\mu \quad (i = 1, \cdots, 25, \text{no sum for } i), \tag{A.5}$$

$$c = -\frac{1}{2}: \quad \zeta_{q\mu\nu} = e^{P_q}_\mu e^{P_q}_\nu, \quad \tilde{\zeta}_{q\mu} = \frac{1}{\sqrt{2}}e^{P_q}_\mu. \tag{A.6}$$

The second choice leads to $V^{\text{br}}_{(2)} = 0$ identically, while for $c = 1/10$ one can check that it indeed gives nonvanishing bracket operators. Together with the solutions with $c = 0$, these complete the conditions for the level 2 bracket operator to be physical.

We consider the physical state condition of the level 3 bracket operator $V^{\text{br}}_{(3)}(\tilde{k}_1, z)$. In this case, three polarization tensors provide three physical state conditions for $\zeta_{1\mu}, \zeta_{q\mu\nu}, \tilde{\zeta}_{q\mu}$. These conditions turn out to be too complicated to find a general solution. However, curiously, if we take $V^{\text{br}}_{(2)}$ case solution (A.3) and the physical state condition for the seed operator $\zeta_1 \cdot k_1 = 0$ as an ansatz, the physical state conditions lead to a solution which is exactly the same as (A.5) (with the same $c = 1/10$). It should not be accidental since these two operators are related through Moore's relation and they would define physical amplitudes for the same choice of $J_{(2)}$. It is not a complete analysis, and there might be other physical choices for $V^{\text{br}}_{(3)}$, but we do not pursue it further and stop here.

Thus, a lesson from these examples is that it is indeed possible to define physical bracket operators by using unphysical deformer operators and physical seed operators. Since an extra physical state in this example seems very special, in the main part, we concentrate on "standard" physical choices where both seed and deformer operators are physical.

Appendix B. DDF states

In this appendix, we summarize the basic facts on Del Giudice, Di Vecchia, and Fubini (DDF) operators [15] and corresponding states. Our treatment follows closely that of [1], and we will spell out explicit formulas which are used in the analysis in the main part.

B.1. Construction

Let $|0; p_0\rangle$ be a tachyonic ground state of the bosonic open string theory with $p_0^2 = 1/\alpha'$. We introduce a null vector k_0 which satisfies $k_0^2 = 0$ and $p_0 \cdot k_0 = 1/(2\alpha')$. It is straightforward to see that $p_{(N)} = p_0 - Nk_0$ satisfies the mass-shell condition of level N states, $p_{(N)}^2 = (1 - N)/\alpha'$. A convenient parametrization of these momenta are

$$p_0^\mu = \frac{1}{\sqrt{\alpha'}}(0, 0, \cdots, 1), \qquad k_0^\mu = \frac{1}{2\sqrt{\alpha'}}(-1, 0, \cdots, 1). \tag{B.1}$$

The DDF operator is defined as

$$A_n^\ell(nk_0) = \oint \frac{dz}{2\pi i} \frac{i\partial X^\ell(z)}{\sqrt{2\alpha'}} e^{ink_0\cdot X(z)}, \tag{B.2}$$

where $z = e^{i\tau}$, and ℓ refers to the transverse directions with respect to p_0 and k_0, $\ell = 1, \cdots, 24$. DDF states are defined by the action of A^ℓ_{-n} on the tachyonic ground state $|0; p_0\rangle$. Since $p_0 \cdot k_0 = 1/(2\alpha')$, the action of a DDF operator on the tachyonic ground state $|0; p_0\rangle$ is well-defined. A_n^i commutes with Virasoro operators L_n and their commutation relation $[A_n^\ell, A_m^k] = n\delta^{\ell k}\delta_{n+m}$ is the same as the standard transverse oscillators α^ℓ_{-n}. Thus the DDF states generates the whole positive norm physical states.

We are ready to write down DDF states in terms of standard oscillators α_n^μ. It is convenient to use the helicity basis where the inner product with k_0 is proportional to $L - P$ projection, $k_0 \cdot \alpha_{-n} \propto (e^L - e^P)_\mu \alpha^\mu_{-n} = \alpha^{(L-P)}_{-n}$ [13]. For the first few levels, the result is

$$\text{Level 1}: \quad |a; p_{(1)}\rangle_{\text{DDF}} \equiv A^a_{-1}|0; p_0\rangle = \alpha^a_{-1}|0; p_{(1)}\rangle, \tag{B.3}$$

$$\begin{aligned}\text{Level 2}: \quad |ab; p_{(2)}\rangle_{\text{DDF}} &\equiv A^a_{-1}A^b_{-1}|0; p_0\rangle \\ &= \left[\alpha^{ab}_{-1} + \delta^{ab}\left(-\frac{1}{2\sqrt{2}}\alpha^{(L-P)}_{-2} + \frac{1}{4}\alpha^{(L-P)}_{-1}\alpha^{(L-P)}_{-1}\right)\right]|0; p_{(2)}\rangle,\end{aligned} \tag{B.4}$$

$$|a; p_{(2)}\rangle_{\text{DDF}} \equiv A^a_{-2}|0; p_0\rangle = \left(\alpha^a_{-2} - \sqrt{2}\alpha^a_{-1}\alpha^{(L-P)}_{-1}\right)|0; p_{(2)}\rangle, \tag{B.5}$$

$$\begin{aligned}\text{Level 3}: \quad |abc; p_{(3)}\rangle_{\text{DDF}} &\equiv A^a_{-1}A^b_{-1}A^c_{-1}|0; p_0\rangle \\ &= \bigg[\alpha^{abc}_{-1} + \left(\delta^{ab}\alpha^c_{-1} + \delta^{bc}\alpha^a_{-1} + \delta^{ca}\alpha^i_{-1}\right) \\ &\quad \times \left(-\frac{1}{4}\alpha^{(L-P)}_{-2} + \frac{1}{8}\alpha^{(L-P)}_{-1}\alpha^{(L-P)}_{-1}\right)\bigg]|0; p_{(3)}\rangle,\end{aligned} \tag{B.6}$$

$$\begin{aligned}|a; b; p_{(3)}\rangle_{\text{DDF}} &\equiv A^a_{-2}A^b_{-1}|0; p_0\rangle \\ &= \bigg[\alpha^a_{-2}\alpha^b_{-1} - \alpha^{ab}_{-1}\alpha^{(L-P)}_{-1} \\ &\quad + \delta^{ab}\left(-\frac{1}{3}\alpha^{(L-P)}_{-3} + \frac{1}{2}\alpha^{(L-P)}_{-2}\alpha^{(L-P)}_{-1} - \frac{1}{6}\left(\alpha^{(L-P)}_{-1}\right)^3\right)\bigg]|0; p_{(3)}\rangle,\end{aligned} \tag{B.7}$$

$$\begin{aligned}|a; p_{(3)}\rangle_{\text{DDF}} &\equiv A^a_{-3}|0; p_0\rangle \\ &= \left[\alpha^a_{-3} - \frac{3}{2}\alpha^a_{-2}\alpha^{(L-P)}_{-1} + \alpha^a_{-1}\left(-\frac{3}{4}\alpha^{(L-P)}_{-2} + \frac{9}{8}\left(\alpha^{(L-P)}_{-1}\right)^2\right)\right]|0; p_{(3)}\rangle.\end{aligned} \tag{B.8}$$

When it is obvious, the momenta may not be displayed and the states are expressed by its transverse indices. It should be noted that the definition of e^P and e^L depends on $p_{(N)}$ and they are different for each level. At level 1, there is no distinction between DDF states and usual massless states.

B.2. From positive norm states to DDF states

DDF states form a basis of physical states at a given level and then it is possible to rewrite a given physical state by use of them up to zero-norm states.[8] We utilize such decomposition to relate level 2 and 3 bracket states to DDF states.

B.2.1. Level 2

First, we list the zero-norm states at level 2,

$$|\text{ZN}_1\rangle = \left(5\alpha^{PP}_{-1} + \alpha^{LL}_{-1} + \sum_{a=1,\cdots,24}\alpha^{aa}_{-1} + 5\sqrt{2}\alpha^P_{-2}\right)|0; k\rangle, \tag{B.9}$$

$$|\text{ZN}_2\rangle = \left(\sqrt{2}\alpha^{PL}_{-1} + \alpha^L_{-2}\right)|0; k\rangle, \tag{B.10}$$

$$|\text{ZN}^a_3\rangle = \left(\sqrt{2}\alpha^{Pa}_{-1} + \alpha^a_{-2}\right)|0; k\rangle, \tag{B.11}$$

where e^P and e^L represent the momentum and the longitudinal helicity with respect to the momentum k. e^a $(a = 1, \cdots, 24)$ represents the transverse directions with respect to k. We now consider a decomposition of the following level 2 physical positive norm state,

$$\left[\sum_{a',b'} G_{a'b'}\alpha^{a'b'}_{-1} + G\sum_{a'}\alpha^{a'a'}_{-1}\right]|0; k\rangle, \tag{B.12}$$

where a', b' indices run over L, a, the transverse directions together with the longitudinal directions, and the physical state conditions imply $\sum_a G_{aa} + G_{LL} + 25G = 0$. This state can indeed be written in terms of the DDF states, up to zero norm states, as

[8] The structure of zero-norm states in terms of the helicity basis is discussed in [21].

$$(\text{B.12}) = \sum_{a,b} D_{ab}|ab;k\rangle_{\text{DDF}} + \sum_{a} D_a|a;k\rangle_{\text{DDF}} + D\sum_{a}|aa;k\rangle_{\text{DDF}}, \tag{B.13}$$

$$D_{ab} = G_{ab}, \qquad D_a = -\sqrt{2}G_{La}, \qquad D = \frac{1}{4}(G_{LL} + 5G). \tag{B.14}$$

B.2.2. Level 3

The zero-norm states of this level are

$$\left|ZN_1^{aa}\right\rangle = \left(\alpha_{-1}^{aaP} - \alpha_{-1}^{LLP} + \alpha_{-2}^{a}\alpha_{-1}^{a} - \alpha_{-2}^{L}\alpha_{-1}^{L}\right)|0;k\rangle, \tag{B.15}$$

$$\left|ZN_2^{a'b'}\right\rangle = \left(\alpha_{-1}^{a'b'P} + \alpha_{-2}^{(a'}\alpha_{-1}^{b')}\right)|0;k\rangle \quad (a' \neq b'), \tag{B.16}$$

$$\left|ZN_3^{a'}\right\rangle = \left(9\alpha_{-1}^{PPa'} + \alpha_{-1}^{LLa'} + \sum_b \alpha_{-1}^{a'bb} + 18\alpha_{-2}^{(P}\alpha_{-1}^{a')} + 6\alpha_{-3}^{a'}\right)|0;k\rangle, \tag{B.17}$$

$$\left|ZN_4^{a'}\right\rangle = \left(\alpha_{-1}^{LLa'} + \sum_b \alpha_{-1}^{a'bb} + 9\alpha_{-2}^{[P}\alpha_{-1}^{a']} - 3\alpha_{-3}^{a'}\right)|0;k\rangle, \tag{B.18}$$

$$|ZN_5\rangle = \left(25\alpha_{-1}^{PPP} + 75\alpha_{-2}^{P}\alpha_{-1}^{P} + 50\alpha_{-3}^{P} \quad + 9\sum_b\left(\alpha_{-1}^{bbP} + \alpha_{-2}^{b}\alpha_{-1}^{b}\right)\right)|0;k\rangle, \tag{B.19}$$

where $a, b, c = 1, \cdots, 24$ are transverse directions with respect to k and $a' = a, L$. The indices are not summed over otherwise explicitly displayed. As in the level 2 case, we consider a decomposition of the following physical positive norm state in terms of DDF states,

$$\left[\sum_{a',b',c'} G_{a'b'c'}\alpha_{-1}^{a'b'c'} + \sum_{a',b'} G_{[a'b']}\alpha_{-2}^{[a'}\alpha_{-1}^{b']}\right]|0;k\rangle, \tag{B.20}$$

with the physical state conditions, $\sum_b G_{abb} + G_{aLL} = 0$ and $\sum_b G_{Lbb} + G_{LLL} = 0$. After some algebra, we find that (B.20) can be rewritten as

$$\begin{aligned}&\sum_{a,b,c} D_{abc}|abc;k\rangle_{\text{DDF}} + \sum_{a,b} D_{(ab)}\big|(a;b);k\big\rangle_{\text{DDF}} + \sum_{a,b} D_{[ab]}\big|[a;b];k\big\rangle_{\text{DDF}}\\ &\quad + \sum_a D_{1a}|a;k\rangle_{\text{DDF}} + \sum_a D_{2a}\sum_b|abb;k\rangle_{\text{DDF}} + D\sum_b|b;b;k\rangle_{\text{DDF}},\end{aligned} \tag{B.21}$$

up to zero norm states, and

$$\begin{aligned}&D_{abc} = G_{abc}, \qquad D_{(ab)} = -3G_{Lab}, \qquad D_{[ab]} = G_{[ab]}, \qquad D = -\frac{1}{2}G_{LLL}\\ &D_{1a} = \frac{1}{4}(9G_{LLa} + 2G_{La}), \qquad D_{2a} = \frac{1}{8}(3G_{LLa} - 2G_{La}).\end{aligned} \tag{B.22}$$

Appendix C. High-energy expansion with a fixed scattering angle

In this paper, we focus on the high-energy limits ($s \to \infty$ with $\hat{t} = t/s$ fixed) of string scattering amplitudes. In the leading order, this limit corresponds to the fixed-angle high-energy limit of the amplitudes, but when we consider relation among subleading part of the amplitudes, there appears some difference. In this appendix, we discuss the high-energy asymptotic relation with the fixed scattering angle by taking $\mathcal{A}[\tilde{2}000] = \mathcal{A}[1\tilde{1}00]$ as an example.

Recall that Mandelstam variables s and t are common on both hands sides in this example. Due to the mass difference, t takes different forms, in terms of s and the scattering angles, on both hand sides as

$$t_{[\tilde{2}000]} = -\frac{s+4}{2} + \frac{\sqrt{(s^2+16)(s+8)}}{2\sqrt{s}}\cos\theta, \tag{C.1}$$

$$t_{[1\tilde{1}00]} = -\frac{s+4}{2} + \frac{\sqrt{s(s+8)}}{2}\cos\theta', \tag{C.2}$$

where θ is the scattering angle for $\mathcal{A}[\tilde{2}000]$ and θ' for $\mathcal{A}[1\tilde{1}00]$, and we have set $\alpha' = 1/2$. Since actually $t_{[\tilde{2}000]} = t_{[1\tilde{1}00]}$, these two angles are related as $\cos\theta' = \cos\theta(1 + 8s^{-2} - 32s^{-4} + \cdots)$. $\hat{t}$ and θ are related as $\hat{t} = -\sin^2\frac{\theta}{2} + \mathcal{O}(s^{-1})$ in the high-energy limit. At the leading order, the expressions are the same for θ and θ', but subleading corrections take different forms for them.

This poses a question on the high-energy expansions of string scattering amplitudes. We need to choose which angle to be fixed under the $s \to \infty$ limit. In this sense, a fixed-angle high-energy limit is ambiguous in the bracket relation. To proceed the analysis, we now choose θ to be fixed and expand the coefficients and the DDF amplitudes with θ fixed,

$$\mathcal{T}^{TT}_{\mathrm{DDF}[2000]} = \mathcal{T}^{TT}_{\theta[2000]\,(3)} s^3 + \cdots, \qquad D^{I_q I_q}_{II} = D^{I_q I_q}_{\theta II\,(0)} + D^{I_q I_q}_{\theta II\,(-1)} s^{-1} + \cdots, \tag{C.3}$$

where each coefficient is now a function of θ; for example, $\mathcal{T}^{TT}_{\theta[2000]\,(3)} = \frac{1}{16}\sin^2\theta$. It should be emphasized that, apart from the leading ones like $\mathcal{T}^{TT}_{\theta[2000]\,(3)}$ or $D^{I_q I_q}_{\theta II\,(0)}$, the coefficients are not simply obtained by identifying $\hat{t} = -\sin^2\frac{\theta}{2}$ in the fixed-$\hat{t}$ coefficients that appeared in Section 6.3. At $\mathcal{O}(s^3)$ and at $\mathcal{O}(s^2)$ with $(A, B) = (T_q, T_q), (I_R, I_R), (J, J)$ the relations take the same forms as (6.19)–(6.22). For $(A, B) = (T_q, I_q)$ at $\mathcal{O}(s^2)$, we find

$$2\mathcal{T}^{TT}_{\theta[2000](3)} = \tan\theta\,\mathcal{T}^{T}_{\theta[2000](5/2)}. \tag{C.4}$$

This relation involves only in leading amplitudes of $\mathcal{A}[\tilde{2}000]$ side and reproduces a known fixed-θ relation. At $\mathcal{O}(s)$, after simplifying by use of higher order relations and rotational symmetry, $(A, B) = (T_q, T_q), (I_q, I_q), (T_q, I_q)$ and (J, J) choices lead to, in order,

$$\begin{aligned}
-2\mathcal{T}^{T_R|T_R}_{\theta[1100](1)} &= 8\cot\theta\,\mathcal{T}^{T}_{\theta[2000](5/2)} + \mathcal{T}^{TT}_{\theta[2000](2)} + 19\mathcal{T}^{II}_{\theta[2000](2)} - 2\mathcal{T}^{TT}_{\theta[2000](1)},\\
2\mathcal{T}^{I_R|I_R}_{\theta[1100](1)} &= 8\cot\theta\,\mathcal{T}^{T}_{\theta[2000](5/2)} - 57\mathcal{T}^{II}_{\theta[2000](2)} - 2\mathcal{T}^{II}_{\theta[2000](1)} - 3\mathcal{T}^{TT}_{\theta[2000](2)},\\
0 &= \tan\theta\big(4\mathcal{T}^{T}_{\theta[2000](5/2)} + \mathcal{T}^{T}_{\theta[2000](3/2)}\big) - 2\mathcal{T}^{TT}_{\theta[2000](2)} + 2\mathcal{T}^{II}_{\theta[2000](2)},\\
2\mathcal{T}^{J|J}_{\theta[1100](1)} &= 8\cot\theta\,\mathcal{T}^{T}_{\theta[2000](5/2)} - 19\mathcal{T}^{II}_{\theta[2000](2)} + 2\mathcal{T}^{II}_{\theta[2000](1)} - \mathcal{T}^{TT}_{\theta[2000](2)}.
\end{aligned}$$

By solving them, one can find several relations among the amplitudes. However, for subleading amplitudes on $\mathcal{A}[1\tilde{1}00]$ side, such as $\mathcal{T}^{T_R|T_R}_{\theta[1100](1)}$, this expansion is not of direct physical relevance and it is not clear how useful these expressions are.

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