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Cartesian closed topological and monotopological hulls: A comparison

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Abstract

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The theory of the cartesian closed monotopological hull of a category can be developed in analogy to that of the cartesian closed topological hull. The main objective of this paper is to describe these hulls in terms of one another. On the local level, a comparison between the cartesian objects of a monotopological category and the cartesian objects of its MacNeille completion is carried out. The cartesian closed monotopological hulls of several well-known subcategories of **Top** are determined.

Keywords: (Mono)topological category, T_0-object, saturated topological category, MacNeille completion, cartesian object, power, cartesian closed (mono)topological hull.

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1. Introduction

It is well known that cartesian closedness is a useful property in connection with topological concepts. The fact that many important categories from topology do not have this convenience property has lead to the development of the theory of cartesian closed topological (CCT) hulls by Herrlich and Nel [12].

Categories of spaces satisfying a separation axiom—e.g., Hausdorff spaces—generally are not topological, but only monotopological. It is natural to ask whether cartesian closedness can be obtained by a process similar to the formation of a CCT hull, while retaining separation. The investigation of this question leads to the

* Most of the results in this paper are contained in a part of the author's doctoral dissertation [24].

concept of *cartesian closed monotopological* (CCMT) *hulls* [6, 24]: Given a concrete category **A** over **Set**, the CCMT hull of **A** is the least finally dense CCMT extension of **A**, i.e. every finally dense embedding of **A** into a CCMT category can be uniquely extended to the hull. (Since the notations are not quite consistent in the literature, we note that by a *monotopological category*, we understand a small-fibred, concrete category over **Set** with constant morphisms where point-separating structured sources have uniquely determined initial lifts; if a monotopological category is initially complete, then it is called *topological*, otherwise, *properly monotopological*. Embeddings and extensions are considered to be full and concrete.)

It is immediate from the definition that the CCMT hull of a category **A** is unique up to (concrete) isomorphism; if it exists—which is not ensured, see Theorem 4.1 and Remark 4.2—it will be denoted CCMTH **A**.

In analogy to [12], it can be shown that the CCMT hull of **A** can, alternatively, be described as a finally dense CCMT extension **B** of **A** which contains the function space objects $\mathbf{B}[X, Y]$, $X, Y \in \mathbf{A}$, as an initially mono-dense subclass, and that CCMTH **A** is obtained by formation of the surjective-reflective hull of $\{\mathbf{C}[X, Y] \mid X, Y \in \mathbf{A}\}$ in any finally dense CCMT extension **C** of **A**. In both cases it suffices that Y runs through an initially mono-dense subclass of **A**.

It is natural to ask for connections between the CCT and the CCMT hull of a category **A**. We will show that they can be described by one another.

An important tool for this comparison is the relationship between topological and monotopological categories via T_0-reflection and MacNeille completion, which is briefly examined in Section 2. It allows the characterization of the *cartesian objects* (i.e. those objects X for which the functor $- \times X$ has a right adjoint) of a monotopological category by the cartesian objects of its MacNeille completion, and vice versa (Section 3). These "local" results are applied, in Section 4, to obtain connections between CCT and CCMT hulls.

The paper concludes by giving the CCMT hulls of several well-known categories; in particular, it turns out that the important category of c-embedded convergence spaces is the CCMT hull of the Tychonoff spaces.

2. T_0-reflection and MacNeille completion

Recall that an object X of a topological category **B** is called a T_0-*object* of **B** if it does not have an indiscrete subspace with more than one point [18, 3.3.3]. If **B** is *saturated*, i.e. the class $T_0\mathbf{B}$ of T_0-objects is initially dense in **B**, then X is a T_0-object iff every initial source with domain X is a monosource (i.e. X is separated in the sense of [8, 1.3.8], [13, 1.5]). The T_0-objects of **B** form a quotient-reflective subcategory of **B** [18, 3.3.6]; in particular, $T_0\mathbf{B}$ is monotopological for any topological category **B**.

Conversely, every monotopological category **A** can be assigned a topological category: **A** has a *MacNeille completion* $\mathcal{M}\mathbf{A}$ (i.e. a least finally dense, initially complete extension), which is small-fibred by [1, 2.13]; it is easily seen that constant maps between $M\mathbf{A}$-objects are morphisms; hence $M\mathbf{A}$ is a topological category.

Generally, these processes are not "inverse" to each other: $M\mathrm{T}_0\mathbf{B} \subsetneq \mathbf{B}$ if $\mathrm{T}_0\mathbf{B}$ is topological (which is rare, but possible) or if $\mathbf{B}$ is not saturated, and $\mathrm{T}_0 M\mathbf{A} \subsetneq \mathbf{A}$ if $\mathbf{A}$ is topological. However, if $\mathbf{A}$ is properly monotopological, then $\mathbf{A} = \mathrm{T}_0 M\mathbf{A}$.

2.1. Theorem. *A properly monotopological category is the category of* T_0*-objects of its MacNeille completion.*

Proof. Let $\mathbf{A}$ be properly monotopological. Then $M\mathbf{A}$ contains $\mathbf{A}$ as an initially dense and finally dense subcategory. Finally dense embeddings preserve initial sources; consequently, $\mathbf{A}$ is closed under initial monosources in $M\mathbf{A}$, i.e. $\mathbf{A}$ is an epireflective subcategory of $M\mathbf{A}$. By [18, 3.3.6], we have $\mathbf{A} \subset \mathrm{T}_0 M\mathbf{A}$. Now let $R: M\mathbf{A} \to \mathbf{A}$ denote the reflector. Let $X \in \mathrm{T}_0 M\mathbf{A}$. Since $\mathbf{A}$ is initially dense in $M\mathbf{A}$, the reflection $\rho: X \to RX$ is initial; since X is a T_0-object, ρ is injective; and as an epimorphism, ρ is surjective: consequently, ρ is an isomorphism, which implies $X \in \mathbf{A}$. □

2.2. Remarks. (1) Not every topological category $\mathbf{B}$ is obtained as the MacNeille completion of a properly monotopological category $\mathbf{A}$: If $\mathbf{B} = M\mathbf{A}$, then $\mathbf{A} = \mathrm{T}_0 M\mathbf{A} = \mathrm{T}_0\mathbf{B}$ by Theorem 2.1, hence $\mathrm{T}_0\mathbf{B}$ must be properly monotopological and initially dense as well as finally dense in $\mathbf{B}$.

(2) The conditions on $\mathrm{T}_0\mathbf{B}$ given in (1) are actually redundant; for it can be shown that any properly monotopological category that is initially dense and epireflective in a topological category $\mathbf{B}$ is also finally dense in $\mathbf{B}$ [24, 1.2.3]. Hence, if $\mathbf{B}$ is saturated, $\mathrm{T}_0\mathbf{B}$ is properly monotopological iff $\mathrm{T}_0\mathbf{B}$ is finally dense in $\mathbf{B}$.

(3) While saturatedness of a topological category is inherited by its bireflective subcategories [18, 3.4.5(4)], this is not the case for the property of being the MacNeille completion of a properly monotopological category, as the example of quasiordered sets and sets with equivalence relations shows.

3. T_0-reflection and cartesian objects

Before we compare CCT and CCMT hulls, we will first investigate the relationship between cartesian objects in topological and monotopological categories. This will not only make the results on hulls more transparent and their proofs more elegant, but is interesting in its own right, as cartesian objects represent the "localized form of cartesian closedness".

Let us first recall some facts about cartesian objects in a monotopological category $\mathbf{A}$. For $X, Y \in \mathbf{A}$, denote by $\mathbf{A}_p(X, Y)$ the smallest object with underlying set $\mathbf{A}(X, Y) = \{f \mid f: X \to Y \text{ morphism}\}$ such that for each $W \in \mathbf{A}$ and each morphism

$h: W \times X \to Y$, the associated map $h^*: W \to \mathbf{A}_p(X, Y)$, $h^*(w)(x) = h(w, x)$, is also a morphism. (The order between **A**-objects with the same underlying set is given by the requirement that the identity map be a morphism.) The objects of the form $\mathbf{A}_p(X, Y)$ are called *powers*. Then an object $X \in \mathbf{A}$ is cartesian iff for every $Y \in \mathbf{A}$, the usual evaluation map $\mathrm{ev}: \mathbf{A}_p(X, Y) \times X \to Y$, $\mathrm{ev}(f, x) = f(x)$, is a morphism. Indeed, for X to be cartesian in **A**, it is sufficient that for some initially dense subclass **D** of **A**, all evaluation maps $\mathrm{ev}: \mathbf{A}_p(X, Y) \times X \to Y$ with $Y \in \mathbf{D}$ are morphisms [22, 3.1, 3.2].

Since the T_0-objects of a topological category **B** form a quotient-reflective subcategory of **B**, powers of T_0-objects are T_0-objects [23, 2.3.12], i.e., $\mathrm{T}_0\mathbf{B}$ is closed under formation of powers in **B**. In particular, every T_0-object that is cartesian in **B** is also cartesian in $\mathrm{T}_0\mathbf{B}$. We will show that the converse implication is true if **B** is saturated; for in that case, powers in $\mathrm{T}_0\mathbf{B}$ and **B** are formed in the same way.

In the following, T will denote the T_0-reflector of a given topological category, and τ the unit of the adjunction.

3.1. Lemma. *If* **B** *is saturated topological, then powers of* T_0*-objects in* **B** *and in* $\mathrm{T}_0\mathbf{B}$ *coincide.*

Proof. Denote $\mathbf{A} = \mathrm{T}_0\mathbf{B}$. Let $X, Y \in \mathbf{A}$. Since **A** is quotient-reflective in **B**, we have $\mathbf{B}_p(X, Y) \in \mathbf{A}$, hence $T\mathbf{B}_p(X, Y) = \mathbf{B}_p(X, Y)$. The sink

$$(h^*: W \to \mathbf{B}_p(X, Y) \mid W \in \mathbf{B},\ h: W \times X \to Y\ \mathbf{B}\text{-morphism})$$

is a final epi-sink in **B** by [22, 2.9(2)]. Consequently,

$$S = (Th^*: TW \to \mathbf{B}_p(X, Y) \mid W \in \mathbf{B},\ h: W \times X \to Y\ \mathbf{B}\text{-morphism})$$

is a final epi-sink in **A**. Since

$$R = (g^*: Z \to \mathbf{A}_p(X, Y) \mid Z \in \mathbf{A},\ g: Z \times X \to Y\ \mathbf{A}\text{-morphism})$$

is a final epi-sink in **A** and $\mathbf{A}(X, Y) = \mathbf{B}(X, Y)$, it is sufficient to show $S = R$ in order to prove $\mathbf{A}_p(X, Y) = \mathbf{B}_p(X, Y)$. $R \subset S$ is clear. Now let $W \in \mathbf{B}$ and $h: W \times X \to Y$ be a **B**-morphism. Since τ is initial and surjective, there is a morphism $m: TW \to W$ such that $\tau \circ m = 1_{TW}$. Then $g = h \circ (m \times 1_X): TW \times X \to Y$ is an **A**-morphism; we will show that $g^* = Th^*$. If $w, w' \in W$ are such that $\tau w = \tau w'$, then $h(w, x) = h^*(w)(x) = Th^*(\tau w)(x) = Th^*(\tau w')(x) = h^*(w')(x) = h(w', x)$ for any $x \in X$. With $w' = m(\tau w)$, we obtain $g^*(\tau w)(x) = g(\tau w, x) = h(m(\tau w), x) = h(w, x) = Th^*(\tau w)(x)$. □

Now Lemma 3.1 yields the following description of the cartesian objects of $\mathrm{T}_0\mathbf{B}$ by the cartesian objects of **B**:

3.2. Theorem. *A* T_0*-object of a saturated topological category* **B** *is cartesian in* T_0**B** *if and only if it is cartesian in* **B**.

Proof. If X is cartesian in T_0**B**, then for every $Y \in T_0\mathbf{B}$, the evaluation map $\mathrm{ev}: \mathbf{B}_p(X, Y) \times X \to Y$ is a **B**-morphism by Lemma 3.1. Since T_0**B** is initially dense in **B**, it follows that X is cartesian in **B**. □

In view of Theorem 2.1, the last theorem can be stated for MacNeille completions of monotopological categories:

3.3. Corollary. *An object of a monotopological category* **A** *is cartesian in* **A** *iff it is cartesian in the MacNeille completion of* **A**.

3.4. Remark. Let **B** be saturated topological. Denoting by Cart **B** the class of cartesian objects of **B**, Theorem 3.2 can be stated as: $\mathrm{Cart}\, T_0\mathbf{B} = \mathrm{Cart}\, \mathbf{B} \cap T_0\mathbf{B}$. Moreover, Theorem 3.2 implies that $\mathrm{Cart}\, T_0\mathbf{C} = \mathrm{Cart}\, \mathbf{C} \cap T_0\mathbf{C} = \mathrm{Cart}\, \mathbf{C} \cap T_0\mathbf{B}$ for every bireflective subcategory **C** of **B**, since every bireflective subcategory of the saturated topological category **B** is also saturated. (The last equality follows immediately from $T_0\mathbf{C} = \mathbf{C} \cap T_0\mathbf{B}$ [18, 3.3.4(1)].) For bireflective subcategories **C** of **B** = **Top**, this result was shown in [22, 6.3].

Now we turn to the question of how to determine the cartesian objects of a saturated topological category by means of the cartesian objects of the subcategory of T_0-objects.

The situation in **Top** gives an important clue: A topological space X is a cartesian object of **Top** iff the lattice of open sets of X is a continuous lattice [9, Theorem 3], [14, 4.2]. Since the lattices of open sets of X and of its T_0-reflection TX are isomorphic, it follows that X is cartesian in **Top** iff TX is, and the latter is, by Theorem 3.2, equivalent to TX being cartesian in T_0**Top**.

In case of topological spaces, open sets can be identified with continuous functions to the Sierpinski two-point space. Replacing the lattice structures by powers, an argumentation similar to that in **Top** can be utilized in any saturated topological category.

3.5. Lemma. *If* **B** *is saturated topological and* $X \in \mathbf{B}$, $Y \in T_0\mathbf{B}$, *then* $\mathbf{B}(\tau, 1): \mathbf{B}_p(TX, Y) \to \mathbf{B}_p(X, Y)$ *is an isomorphism.*

Proof. Obviously, $\mathbf{B}(\tau, 1)$ is a bijection. Let $m: TX \to X$ with $\tau \circ m = 1$. Then $\mathbf{B}(\tau, 1): \mathbf{B}_p(TX, Y) \to \mathbf{B}_p(X, Y)$ and $\mathbf{B}(m, 1): \mathbf{B}_p(X, Y) \to \mathbf{B}_p(TX, Y)$ are morphisms [25, 1.1], and $\mathbf{B}(m, 1) \circ \mathbf{B}(\tau, 1) = \mathbf{B}(\tau \circ m, 1) = \mathbf{B}(1, 1) = 1$. □

3.6. Theorem. *An object of a saturated topological category* **B** *is cartesian in* **B** *if and only if its* T_0*-reflection is cartesian in* $\mathrm{T}_0\mathbf{B}$.

Proof. Using the notations of Lemma 3.5, it is easily seen that for $X \in \mathbf{B}$, $Y \in \mathrm{T}_0\mathbf{B}$, the diagram

$$\begin{array}{ccc} \mathbf{B}_p(X, Y) \times X & \xrightarrow{\text{ev}} & Y \\ {\scriptstyle \mathbf{B}(\tau,1)\times m}\uparrow & \nearrow{\scriptstyle \text{ev}} & \\ \mathbf{B}_p(TX, Y) \times TX & & \end{array}$$

commutes in **Set**. Since $f(x) = f(x')$ whenever $f \in \mathbf{B}(X, Y)$ and $x, x' \in X$ with $\tau x = \tau x'$, we obtain, by choosing $x' = m(\tau x)$, that $\text{ev} \circ (\mathbf{B}(m, 1) \times \tau)(f, x) = \text{ev}(f \circ m, \tau x) = f(m(\tau x)) = f(x) = \text{ev}(f, x)$, i.e., the diagram

$$\begin{array}{ccc} \mathbf{B}_p(X, Y) \times X & \xrightarrow{\text{ev}} & Y \\ {\scriptstyle \mathbf{B}(m,1)\times \tau}\downarrow & \nearrow{\scriptstyle \text{ev}} & \\ \mathbf{B}_p(TX, Y) \times TX & & \end{array}$$

is also commutative in **Set**. Now let $X \in \mathbf{B}$. By Theorem 3.2, TX is cartesian in $\mathrm{T}_0\mathbf{B}$ iff TX is cartesian in **B**, and the latter is equivalent to X being cartesian in **B** by the commutativity of the above diagrams and the fact that $\mathrm{T}_0\mathbf{B}$ is initially dense in **B**. □

Application of Theorem 3.6 to Theorem 2.1 immediately yields:

3.7. Corollary ([10, 1.4(b)], [6, 3.4]). *A monotopological category is cartesian closed if and only if its MacNeille completion is cartesian closed.*

4. Relations between CCMT and CCT hulls

We are now ready to compare the cartesian closed topological and monotopological hull of a category, and to describe them in terms of one another. Let us start with an observation about the existence of these hulls, which is an easy consequence of Corollary 3.7 and the fact that a category has a CC(M)T hull whenever it can be finally dense embedded into a CC(M)T category.

4.1. Theorem (cf. [6, 4.3]). *A category has a CCMT hull if and only if it has a CCT hull.*

In particular, the internal characterization for the existence of a CCT hull of Adámek and Koubek [2, 3] applies to the existence of a CCMT hull as well.

4.2. Remark. In [2, 3], Adámek and Koubek give an example of a topological category which does not have a CCT hull. The category **A** of T_0-objects of that category is properly monotopological; virtually the same argumentation as in [3, Section 3] can be used to show that **A** fails to have a CCMT hull.

Moreover, Corollary 3.7 yields the "upwards description" of the CCT hull of a category in terms of its CCMT hull:

4.3. Theorem. *If a category* **A** *has a CCMT hull, then* CCTH **A** *is obtained as the MacNeille completion of* CCMTH **A**: CCTH **A** $= M($CCMTH **A**$)$.

Proof. By Corollary 3.7, $M($CCMTH **A**$)$ is a finally dense CCT extension of **A**. Consequently, CCTH **A** exists and is contained in $M($CCMTH **A**$)$. On the other hand, CCMTH **A** $\subset$ CCTH **A** implies $M($CCMTH **A**$) \subset$ CCTH **A**. □

Next we give the "downwards description":

4.4. Theorem. *If a category* **A** *has a CCT hull, then either* (a) CCMTH **A** $=$ $T_0($CCTH **A**$)$, *or* (b) CCMTH **A** $=$ CCTH **A**; (a) *applies iff* **A** *fulfils one of the following conditions*:

(1) CCMTH **A** *is properly monotopological.*

(2) CCMTH **A** *is an epireflective, but not bireflective subcategory of* CCTH **A**.

(3) **A** $\subset T_0($CCTH **A**$)$.

(4) *The epireflective hull of* **A** *in* CCTH **A** *does not contain the indiscrete two-point object of* CCTH **A**.

Proof. By Theorem 4.3, CCTH **A** $= M($CCMTH **A**$)$. If CCMTH **A** is properly monotopological, then CCMTH **A** $= T_0($CCTH **A**$)$ by Theorem 2.1. On the other hand, if CCMTH **A** is topological, then CCMTH **A** $=$ CCTH **A**. This proves also the equivalence of (1) and (a). Since CCMTH **A** is finally dense in CCTH **A**, we know that CCMTH **A** is an epireflective subcategory of CCTH **A**; hence (1)$\Rightarrow$(2) is immediate. (2)$\Rightarrow$(3) follows from [18, 3.3.6]. The equivalences (a)$\Leftrightarrow$(3)$\Leftrightarrow$(4) are clear. □

If a monotopological category **A** has a CCT hull, then **A** is epireflective in CCTH **A**, by final density. Since **A** is topological iff **A** contains an indiscrete two-point object [23, 1.2.17], and finally dense embeddings preserve indiscrete objects, Theorem 4.4(4) implies:

4.5. Corollary. *If a monotopological category* **A** *has a CC(M)T hull, then* **A** *is properly monotopological if and only if* CCMTH **A** *is properly monotopological.*

This result is remarkable in two ways: Most important, it shows that the concept of a CCMT hull fulfils our original requirement to "add cartesian closedness," while retaining separation" (and that the undesirable case of a topological category with a properly monotopological CCMT hull cannot occur). Moreover, it provides a way to characterize when a monotopological category is properly monotopological, by properties of its CCMT hull (existence of that hull assumed).

An immediate consequence of Corollary 4.5 and Theorems 4.4 and 4.3 is:

4.6. Corollary. *The CCT hull of a properly monotopological category, if its exists, is saturated.*

A similar statement about topological categories is not generally true; however, the CCT hull of a saturated topological category is always saturated [19, Corollary 3].

The preceding results give connections between CCT and CCMT hulls of a category **A** by reference to **A**. Now in case **A** is properly monotopological, $\mathbf{A} = \mathrm{T}_0\mathbf{B}$ for a saturated topological category **B**, and one might ask for the relationships between the CCT and CCMT hulls of $\mathrm{T}_0\mathbf{B}$ and **B**. In order to investigate these, we need the following lemma:

4.7. Lemma. *If* **C** *is a topological category and* **B** *a saturated, bireflective subcategory of* **C** *which is finally dense in* **C**, *then* $\mathrm{T}_0\mathbf{B}$ *is finally dense in* $\mathrm{T}_0\mathbf{C}$.

Proof. Let $Y \in \mathrm{T}_0\mathbf{C}$ and $(f_i : X_i \to Y \mid i \in I)$ a final sink in **C** with all X_i in **B**. Application of the T_0-reflector T of **C** yields a final sink $(Tf_i : TX_i \to Y \mid i \in I)$ in $\mathrm{T}_0\mathbf{C}$. Hence it suffices to show that $TX \in \mathrm{T}_0\mathbf{B}$ whenever $X \in \mathbf{B}$ (cf. [18. 3.4.7]). Since $\mathrm{T}_0\mathbf{B} \subset \mathrm{T}_0\mathbf{C}$, the $\mathrm{T}_0\mathbf{B}$-reflection $\sigma : X \to SX$ can be factorized over the $\mathrm{T}_0\mathbf{C}$-reflection $\tau : X \to TX$, i.e. $\sigma = \bar{\sigma} \circ \tau$ for some **C**-morphism $\bar{\sigma}$. Hence the initiality of σ (in **B** and thus, by final density, also in **C**) implies that τ is initial. Consequently, TX is an initial subobject of $X \in \mathbf{B}$, and $TX \in \mathbf{B} \cap \mathrm{T}_0\mathbf{C} = \mathrm{T}_0\mathbf{B}$. □

4.8. Theorem. *If a saturated topological category* **B** *has a CCT hull, then* $\mathrm{CCMTH}(\mathrm{T}_0\mathbf{B}) = \mathrm{T}_0(\mathrm{CCTH}\,\mathbf{B})$.

Proof. Denote $\mathbf{C} = \mathrm{CCTH}\,\mathbf{B}$. Then $\mathrm{T}_0\mathbf{C}$ is a finally dense CCMT extension of $\mathrm{T}_0\mathbf{B}$. Since $\mathrm{T}_0\mathbf{C}$ is closed under formation of function spaces in **C**, it remains to be shown that $\{\mathbf{C}[X, Y] \mid X, Y \in \mathrm{T}_0\mathbf{B}\}$ is initially mono-dense in $\mathrm{T}_0\mathbf{C}$. Let $Z \in \mathrm{T}_0\mathbf{C}$. There is an initial source $(f_i : Z \to \mathbf{C}[X_i, Y_i] \mid i \in I)$ in **C** with all X_i, Y_i in **B**. Since **C** is saturated, $(1, \tau) : \mathbf{C}[X_i, Y_i] \to \mathbf{C}[X_i, TY_i]$ is initial for each $i \in I$. By Lemma 3.5, $\mathbf{C}[X_i, TY_i] \cong \mathbf{C}[TX_i, TY_i]$. Composition yields an initial source $(g_i : Z \to \mathbf{C}[TX_i, TY_i] \mid i \in I)$ in **C**, which is a monosource because of $Z \in \mathrm{T}_0\mathbf{C}$. As in the proof of Lemma 4.7, all TX_i, TY_i are in $\mathrm{T}_0\mathbf{B}$. □

4.9. Proposition. *If* **B** *is a saturated topological category which has a CCT hull, and if* T_0**B** *is properly monotopological, then* CCTH(T_0**B**) = CCTH **B**.

Proof. By Remark 2.2(2), **B** = M(T_0**B**). Hence CCTH(T_0**B**) = CCTH(MT_0**B**) = CCTH **B**. □

The assumption that T_0**B** is properly monotopological cannot be omitted in Proposition 4.9, as is shown by the example of sets with equivalence relation (and relation-preserving maps).

5. Examples

Finally, we want to determine the CCMT hulls of some well-known categories.

5.1. Some well-known categories from general topology

The categories **Top** of topological spaces and **Unif** of uniform spaces are saturated; consequently, all bireflective subcategories **B** of **Top** or **Unif** are saturated. By Theorem 4.8, we obtain CCMTH(T_0**B**) = T_0(CCTH **B**) in all these cases. The CCT hulls of **Top, CReg** (completely regular spaces), **Unif** and **Prox** (proximity spaces) are well known (see [7, 3.3, 4.6], [4, 2.4], [21, Ex. 10c]): CCTH **Top** is the category of Antoine (or epitopological) spaces, CCTH **CReg** the category of c-spaces, CCTH **Unif** the category of bornological uniform spaces, and CCTH **Prox** the coreflective hull of **Prox** in **Unif**. Hence application of Theorem 4.8 yields the CCMT hulls of T_0**Top**, **Tych** (Tychonoff spaces), T_2**Unif** (Hausdorff uniform spaces), and T_0**Prox**.

5.2. Finite and countable posets

The MacNeille completion of the categories **FPos** of finite and **CPos** of countable partially ordered sets (and order-preserving maps) is the category **Qos** of quasi-ordered sets (or equivalently, Alexandroff-discrete spaces, i.e., topological spaces where arbitrary intersections of open sets are open); in fact, the two-point chain is initially and finally dense in **Qos** (e.g. [11, 3.1a]). Since **Qos** is cartesian closed [20, 3.1], we have CCTH **FPos** = CCTH **CPos** = **Qos**. Now T_0**Qos** is the category **Pos** of partially ordered sets. Hence Theorem 4.4 implies CCMTH **FPos** = CCMTH **CPos** = **Pos**. This result can also be found—with a much more laborious proof—in [6, 4.5].

5.3. Compact Hausdorff spaces

The category T_2**Comp** of compact Hausdorff spaces is closed under finite products in **Top**, and contained in Cart **Tych** and Cart **CReg**. Consequently, T_2**Comp** is an exponential subcategory [21, pp. 272, 280] of **Tych** and **CReg**, and the coreflective hulls of T_2**Comp** in **Tych** and **CReg** are CCMT and CCT categories, respectively

[21, p. 280, Theorem 5]. Since T_2**Comp** is initially dense in **CReg**, these coreflective hulls coincide with the CCMT and CCT hulls of T_2**Comp**. More illustrative descriptions can be given as follows: Call a topological space X *functionally compactly generated* iff it is completely regular and any real-valued function with domain X that is continuous on all compact subspaces of X is already continuous; X is called a *functionally compactly generated Tychonoff* space iff X fulfils the latter requirement and $X \in$ **Tych**. The corresponding subcategories of **CReg** are denoted by **FCG** and **FCGTych**, respectively. Then **FCG** is coreflective in **CReg**; **FCGTych** is coreflective in **Tych**; and both categories contain the compact Hausdorff spaces as a finally dense subclass. Hence they constitute the coreflective hulls of T_2**Comp** in **CReg** and **Tych**, respectively. In total, we have: CCTH T_2**Comp** = $M$$T_2$**Comp** = **FCG** and CCMTH T_2**Comp** = **FCGTych** = T_2**Top** ∩ **FCG** = T_0(CCTH T_2**Comp**). For descriptions of CCTH T_2**Comp**, see also [5, III.6].

5.4. Metrizable spaces

Basically the same proof as in Section 5.3 can be employed to determine the CCMT hull of the category **Met** of metrizable spaces (and continuous maps), with the minor addition that, to show that **Met** is exponential in **Tych** and **CReg**, one observes that the coreflective hull of **Met** (both in **Tych** and in **CReg**) coincides with the coreflective hull of the one-point compactification of the discrete countably infinite topological space. Replacing functions that are continuous on compact subspaces, by sequentially continuous functions (i.e. functions that preserve limits of sequences), the categories **FCG** and **FCGTych** are replaced by the categories **FS** of *functionally sequential* and **FSTych** of *functionally sequential Tychonoff* spaces. Then CCTH **Met** = M**Met** = **FS** and CCMTH **Met** = **FSTych** = T_2**Top** ∩ **FS** = T_0(CCTH **Met**). The description of CCTH **Met** is also given in [5, III.7].

5.5. L-embedded spaces

Another type of example can be obtained by modifying and refining Bourdaud's methods [7] (cf. also [17] and [25]); for more details and extensive references, see [24, Section 4].

Denote by **Lim**, **PsT** and **Reg** the categories of limit spaces (in the sense of [16, Definition 4], pseudotopological spaces and regular topological spaces. If $L \in T_0$**Lim** and **A** is the epireflective hull of L in **Lim**, then the CCMT hull of **A** is given by the category of *L-embedded* limit spaces, i.e. those limit spaces X for which the map $i_X : X \to$ **Lim**[**Lim**[X, L], L] defined by $i_X(x)(f) = f(x)$ for $x \in X$, $f \in$ **Lim**[X, L], is an embedding. In particular, for $L = \mathbb{R}$, this shows that the important category of c-embedded spaces constitutes the CCMT hull of the Tychonoff spaces. Analogously, the CCT hull of the bireflective hull of L in **Lim** consists of those limit spaces X for which i_X is initial.

More can be said in the following situation: Let **B** be a bireflective subcategory of **Reg**, and $L \in T_0$**B** be separating for **B**, i.e.: L contains two points y, z possessing disjoint neighborhoods, and points and closed sets of every **B**-object can be separated by continuous maps to L such that closed sets are mapped to y and points of the complement to z. Then CCMTH(T_0**B**) cannot only be described as the category of L-embedded pseudotopological spaces, but also as CCMTH(T_0**B**) = {$X \in$ **PsT** | X Hausdorff and **B**-regular} (where a pseudotopological space X is called **B**-regular if for every filter $\mathscr{F}$ on X that converges to a point $x \in X$, the filter generated by the closures—in the **B**-modification of X—of the sets $F \in \mathscr{F}$ also converges to x). This situation occurs if **B** is the category of zero-dimensional spaces and L the discrete two-point space or if **B** = **CReg** and L is the set of reals (or the closed unit interval) in their usual topology. The latter example is just the well-known description of the c-embedded spaces from [15, 2.4].

The CCT hull of a category **B** which fulfils the above conditions can be described similarly: CCTH **B** = {$X \in$ **PsT** | X **B**-closed-domained and **B**-regular} (where X is called **B**-closed-domained if for each filter $\mathscr{F}$ on X, the set of limit points of $\mathscr{F}$ is closed in the **B**-modification of X). This constitutes a partial answer to the problem in [5, p. 22]. (The assumption that $L \in T_0$**B** is not essential for this description.)

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