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RECURSIVELY RIGID BOOLEAN ALGEBRAS

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Introduction

A recursive Boolean algebra $B = (F_B, +_B, \cdot_B, -_B, 0_B, 1_B)$ consists of a recursive subset F_B of the natural numbers $\mathbb{N}$ called the field of B, partial recursive operations $+_B$ (join), $\cdot_B$ (meet), $-_B$ (complement), plus distinguished elements 0_B and 1_B of F_B which are the zero and one of B respectively. We let $\mathrm{Aut}(B)$ denote the group of automorphisms of B and $\mathrm{Aut}_r(B)$ denote the group of recursive automorphisms of B, i.e., those automorphisms of B which are partial recursive functions. For any $x \in B$, we let $B\restriction x$ denote the recursive B.A. determined by x. That is, $F_{B\restriction x} = \{y \in F_B \mid y \leqslant_B x\}$ where $y \leqslant_B x$ iff $x \cdot_B y = y$, the operations $+_{B\restriction x}$ and $\cdot_{B\restriction x}$ are the restrictions of $+_B$ and $\cdot_B$ respectively, $-_{B\restriction x} y$ is defined to be $x -_B y$, $0_{B\restriction x} = 0_B$, and $1_{B\restriction x} = x$. Given two recursive B.A.'s B_1 and B_2 we write $B_1 \approx B_2$ if B_1 is isomorphic to B_2 and $B_1 \approx_r B_2$ if B_1 is *recursively isomorphic* to B_2, i.e., if there is a partial recursive isomorphism from B_1 onto B_2. Moreover we let $B_1 \times B_2$ denote the recursive B.A. such that $F_{B_1 \times B_2} = \{\langle x, y\rangle \mid x \in B_1, y \in B_2\}$ where $\langle\,,\,\rangle$ is some fixed recursive pairing function and the operations are taken componentwise. Clearly if B is a recursive B.A. and $z \in B$, then $B \approx_r (B\restriction z) \times (B\restriction -z)$.

A B.A. D is said to be *rigid* if $\mathrm{Aut}(D)$ consists only of the identity. Now it is easy to see that the only rigid countable B.A. is the two element B.A. That is, let F_n denote the finite B.A. with exactly n atoms and $\tilde{Q}$ denote some fixed recursive presentation of the countable atomless B.A. Note since Cantor's back and forth argument is clearly effective, any two recursive countable atomless B.A.'s are recursively isomorphic and $\mathrm{Aut}_r(\tilde{Q})$ is countably infinite (see [4]). Now if B is any countable B.A. and $B \not\approx F_1$, then there exists $z \in B$ such that either $B\restriction z \approx F_2$ or $B\restriction z \approx \tilde{Q}$. Moreover for any $z \in B$, $\phi \in \mathrm{Aut}(B\restriction z)$, and $\psi \in \mathrm{Aut}(B\restriction -z)$, there is a unique $\theta \in \mathrm{Aut}(B)$ such that $\theta \restriction (B\restriction z) = \phi$ and $\theta \restriction (B\restriction -z) = \psi$. Thus we can construct a nontrivial $\theta \in \mathrm{Aut}(B)$ by letting ψ be the identity on $B\restriction -z$ and letting ϕ be the automorphism which interchanges the two atoms if $B\restriction z \approx F_2$ or letting ϕ be one of the nontrivial recursive automorphisms of $\tilde{Q}$ if $B\restriction z \approx \tilde{Q}$.

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Clearly, if B started out to be a recursive B.A., θ will be a recursive automorphism of B. Thus there are no interesting recursively rigid B.A.'s if we take the obvious definition of saying a recursive B.A. B is recursively rigid iff $\mathrm{Aut}_r(B)$ consists only of the identity. Instead, we shall say that B is recursively rigid if the only automorphisms in $\mathrm{Aut}_r(B)$ are induced by essentially trivial factorizations of B.

Definition. A recursive B.A. B is *recursively rigid* if for every recursive automorphism $\theta \in \mathrm{Aut}_r(B)$, there exists a $z \in B$ such that $B \restriction z$ is a finite B.A. and there exists a $\phi \in \mathrm{Aut}_r(B \restriction z)$ such that θ is induced by the pair $\langle \phi, \mathrm{I} \rangle$ where $\mathrm{I} \in \mathrm{Aut}_r(B \restriction -z)$ is the identity.

It follows from our analysis above that any recursive B.A. with an atomless element cannot be recursively rigid. In fact, the following easily follows by essentially the same argument.

Proposition 1. *Suppose B is a recursively rigid recursive B.A. Then either*

(i) *$B \approx F_n$ for some n, or*

(ii) *B is an atomic B.A. where the set of atoms of B,* $\mathrm{At}(B)$, *is infinite and each recursive auromorphism of B moves only finitely many atoms. Hence* $\mathrm{Aut}_r(B)$ *is isomorphic to* $\mathrm{FP}(\omega)$ *the group of permutations of ω which move only finitely many elements.*

Now the existence of recursively rigid BA's was first proved by Morozov [3] although he did not use the term recursively rigid. He proved the following.

Theorem 2 (Morozov [3]). *Suppose B is an atomic B.A. and* $\mathrm{At}(B)$ *is recursive. Then there exists a recursive B.A. D isomorphic to B such that every recursive automorphism of D moves only finitely many atoms.*

Theorem 2 is not the best possible result due to the restriction that the set of atoms of B must be recursive. That is, it is a result of Goncharov [1] that there exist atomic recursive B.A.'s B_n for each $n \geqslant 0$ such that $B_n \not\approx B_m$ if $m \neq n$ and B_n is not isomorphic to any recursive B.A. B such that $\mathrm{At}(B)$ is recursive. Thus, Theorem 2 does not cover all atomic recursive B.A.'s. The main result of this paper will show that we can drop the hypothesis that $\mathrm{At}(B)$ be recursive in Theorem 2. In fact, we can even drop the hypothesis that B must be atomic. That is, ignoring the obviously trivial cases in Theorem 2 where $\mathrm{At}(B)$ is finite, we shall prove the following.

Theorem 3. *Let D be any recursive B.A. such that* $\mathrm{At}(D)$ *is infinite. Then for each $n \geqslant 0$, there exists a recursive B.A. D_n isomorphic to D such that*

(i) $\mathrm{At}(D_n)$ *is immune, i.e.,* $\mathrm{At}(D_n)$ *contains no infinite r.e. set.,*
(ii) *every recursive automorphism of* D_n *moves only finitely many atoms, and*
(iii) $D_n \not\approx_r D_m$ *if* $n \neq m$.

We shall give the proof of Theorem 3 in Section 1. The proof of Theorem 3 requires some of the machinery developed in [4, 5] which was used to prove the same result with condition (ii) removed. However, our proof here requires the use of an infinite injury priority argument as opposed to the finite injury priority arguments used to prove the corresponding result in [4] and Theorem 2. Indeed, this is the first example of the use of infinite injury priority argument in recursive algebra for other than degree-theoretic results.

Clearly as corollaries of Theorem 3, we get the following results.

Corollary 4. *Every infinite atomic recursive B.A. is isomorphic to a recursively rigid recursive B.A.*

Corollary 5. *Every infinite atomic recursive B.A. D is isomorphic to a recursive B.A. B such that*

$$\mathrm{Aut}_r(B) \approx \mathrm{FP}(\omega) \quad \textit{and} \quad \mathrm{Aut}_r(B \times \tilde{Q}) \approx \mathrm{FP}(\omega) \times \mathrm{Aut}_r(\tilde{Q}).$$

We note that Corollary 5 is in great contrast with the following result of McKenzie.

Theorem 6 (McKenzie [2]). *Suppose* B_1 *is any countable B.A. of the form D or* $D \times \tilde{Q}$ *where D is an atomic B.A. with at least two atoms and* B_2 *is any countable B.A. Then* $B_1 \approx B_2$ *if and only if* $\mathrm{Aut}(B_1) \approx \mathrm{Aut}(B_2)$.

Of course as was pointed out by Morozov [3], it already follows from Theorem 2 that the obvious effective version of McKenzie's result is false but we see from Corollary 5 that the effective version of McKenzie's theorem fails in the strongest possible terms. Thus, in general the group of recursive automorphisms of a recursive B.A. may not tell us much about the B.A.

We make one final remark to show that yet another interesting result follows from Theorem 3. That is, we cannot determine whether or not a recursive B.A. B has a decidable presentation from its group of recursive automorphisms even for atomic B.A.'s. Here we say that a recursive B.A. D is *decidable*[1] if $\mathrm{Th}(D, d)_{d \in D}$ is a decidable theory. Now it follows from Tarski's elimination of quantifiers for B.A.'s that an atomic recursive B.A. D is decidable if and only if $\mathrm{At}(B)$ is recursive (see [4]). Thus the atomic B.A.'s constructed by Goncharov which were mentioned previously are all examples of atomic recursive B.A.'s which are not isomorphic to any decidable B.A. Thus by Corollary 5, we have the following.

[1] We note that in the Russian literature, the term constructive B.A. and strongly constructive B.A. are used for our recursive B.A. and decidable B.A. respectively.

Corollary 7. *There exist atomic recursive B.A.'s B_1 and B_2 such that* $\mathrm{Aut}_r(B_1) \approx \mathrm{Aut}_r(B_2)$ *where B_1 is isomorphic to a decidable B.A. but B_2 is not isomorphic to any decidable B.A.*

We note that one cannot replace the hypothesis that B_1 and B_2 are recursive by the hypothesis that B_1 and B_2 are decidable since Morozov has proved the following which may be regarded as an effective version for Theorem 6.

Theorem 8 (Morozov [3]). *If B_1 is an atomic decidable B.A. and B_2 is an atomic recursive B.A. such that* $\mathrm{Aut}_r(B_1) \approx \mathrm{Aut}_r(B_2)$, *then* $B_1 \approx_r B_2$.

We should also note that if one is willing to drop the hypothesis that the B.A.'s are atomic in Corollary 7, then even the full group of automorphisms cannot determine if a B.A. is isomorphic to a recursive B.A. That is, Morozov [3] has exhibited a decidable B.A. D_1 and a countable B.A. D_2 which is not even isomorphic to a recursive B.A. such that $\mathrm{Aut}(D_1) \approx \mathrm{Aut}(D_2)$. Note that by McKenzie's theorem, D_1 and D_2 are necessarily non-atomic.

1. The proof of our main result

In this section, we shall give the proof of Theorem 3. Before we give the proof, we need a bit of notation and to quote some results from [4]. First if S is any subset of a B.A. B, we let $\langle S \rangle$ denote the subalgebra generated by B. The set of atoms of B will be denoted by $\mathrm{At}(B)$. Note that if C is a subalgebra of B, $\mathrm{At}(C)$ denotes the atoms of C and hence an $x \in \mathrm{At}(C)$ is not necessarily an atom in B. Now we shall think of $\tilde{Q}$ as the B.A. generated by the left-closed right-open intervals of the rationals $\mathbb{Q}$ under an appropriate Gödel numbering. In fact we shall assume the field of $\tilde{Q}$ is equal to $\mathbb{N}$, the set of natural numbers. Now in general, we shall construct our recursive B.A.'s to be recursive subalgebras of $\tilde{Q}$. This causes no restriction due to the following well known result which is proved in [4].

Theorem 9. *Every recursive B.A. is recursively isomorphic to a recursive subalgebra of $\tilde{Q}$.*

Next suppose D is a recursive B.A. When we take the least element x in D that satisfies a certain property, we mean the least x with respect to the usual ordering of the natural numbers as $F_D \subseteq \mathbb{N}$. We shall always use $\leqslant$ to refer to the order of $\mathbb{N}$ and hence we will use subscripts $\leqslant_D$ to refer to the ordering induced by the B.A. D. We say that an r.e. sequence $0_D = d_0, d_1, \ldots$ is an *r.e. generating sequence* for D if letting D^s denote $\langle \{d_0, \ldots, d_s\} \rangle$, we have (i) $D = \bigcup_s D^s$ and

(ii) for all s, $D^{s+1} \supset D^s$ and there is an atom $a \in \mathrm{At}(D^s)$ such that $0_D <_D d_{s+1} <_D a$. It is easy to see that every recursive B.A. D has an r.e. generating sequence; see [4] for details.

The purely Boolean-algebraic key result we shall need is the following which states that if we start with any countable B.A. B such that $\mathrm{At}(B)$ is infinite and split each of the atoms of B into finitely many pieces, we do not change the isomorphism type.

Theorem 10 (Remmel [4]). *Let B be a subalgebra of $\tilde{Q}$ such that $\mathrm{At}(B) = \{d_0, d_1, \ldots\}$ is infinite. Assume that for each i, $e_1^i, \ldots, e_{k_i}^i$ are nonzero pairwise disjoint elements of $\tilde{Q}$ such that*

$$d_i = \sum_{j=1}^{k_i} e_j^i.$$

Let $C = \langle B \cup \{e_j^i \mid i \geqslant 0 \;\&\; 1 \leqslant j \leqslant k_i\}\rangle$. Then $B \approx C$.

We claim that Theorem 3 will follow once we can prove the following.

Theorem 11. *Let D be a recursive B.A. such that $\mathrm{At}(D)$ is infinite. Then there exists a recursive B.A. C isomorphic to D such that $\mathrm{At}(C)$ is immune and any recursive automorphism of C moves only finitely atoms.*

To see that Theorem 3 follows from Theorem 10, we need one other purely Boolean-algebraic result from [4].

Theorem 12 (Remmel [4]). *Suppose B is a countable B.A. such that $\mathrm{At}(B)$ is infinite. Then for any $z \in B$ such that $B \restriction z$ is a finite B.A., $B \approx B \restriction (-_B z)$.*

Proof of Theorem 3. Let D be any recursive B.A. such that $\mathrm{At}(D)$ is infinite and let C be the recursive B.A. whose existence is guaranteed by Theorem 11. Next let $\mathrm{At}(C) = \{a_0, a_1, \ldots\}$ and let $z_n = \sum_{i=0}^{n} a_i$ for each n. Then we claim that we can let $D_n = C \restriction (-_C z_n)$ for each n. That is, since the set of atoms of C is immune and each recursive automorphism of C moves only finitely many atoms, it is clear each D_n inherits those same two properties. Thus we need only show that $D_n \not\approx_r D_m$ if $m \neq n$. So suppose for example that $m < n$ and there is a recursive isomorphism $\phi : D_m \to D_n$. Note that $\phi \restriction \mathrm{At}(D_m)$ must map $\mathrm{At}(D_m)$ onto $\mathrm{At}(D_n)$. But then consider the r.e. set $S = \{\phi(d_{m+1}), \phi^2(d_{m+1}), \phi^3(d_{m+1}), \ldots\}$. As $d_{m+1} \notin \mathrm{At}(D_n)$ and ϕ is 1:1, it would follow that S is an infinite r.e. set contained in $\mathrm{At}(D_n)$ violating the immunity of $\mathrm{At}(D_n)$. Thus there can be no such ϕ and hence $D_0, D_1, \ldots$ have the properties required by Theorem 3.

Proof of Theorem 11. Let $d_0, d_1, \ldots$ be an r.e. generating sequence for D and let $D^s = \langle\{d_0, \ldots, d_s\}\rangle$. We shall build the desired recursive B.A. C in stages so that C is a recursive subalgebra of $\tilde{Q}$. At each stage of our construction, we will specify two finite subalgebras B^s and C^s of $\tilde{Q}$ and an isomorphism $f^s: D^s \to B^s$. We will ensure that for all s, $B^s \subseteq C^s$, $B^s \subseteq B^{s+1}$, $C^s \subseteq C^{s+1}$, and $f^s \subseteq f^{s+1}$. At the end of our construction $B = \bigcup B^s$ will be an r.e. subalgebra of $\tilde{Q}$, $C = \bigcup C^s$ will be a recursive subalgebra of $\tilde{Q}$ and $f = \bigcup f^s$ will be a partial recursive isomorphism from D onto B. C will be in relation to B as in Theorem 10. That is, if $\mathrm{At}(B) = \{a_0, a_1, \ldots\}$, then for each i, there will be finitely many pairwise disjoint nonzero elements of $\tilde{Q}$, $e_1^i, \ldots, e_{k_i}^i$, such that

$$a_i = \sum_{i=1}^{k_i} e_j^i \quad \text{and} \quad C = \langle B \cup \{e_j^i \mid i \geq 0 \;\&\; 1 \leq j \leq k_i\}\rangle.$$

Thus f will ensure that $D \approx B$ and Theorem 10 will ensure that $B \approx C$ so that $D \approx C$.

To ensure that $\mathrm{At}(C)$ is immune, we shall meet the following set of requirements for $e = 0, 1, \ldots$.

R_{2e}: If $W_e \cap C$ is infinite, then $W_e \cap (C - \mathrm{At}(C)) \neq \emptyset$.

We say requirement R_e is *satisfied* at stage s if $W^s \cap (C^s - \mathrm{At}(C^s)) \neq \emptyset$.

To ensure that there is no recursive automorphism ϕ of C such that ϕ moves infinitely many atoms of C, we shall meet the following set of requirements.

R_{2e+1}: If ϕ_e is an order preserving 1:1 map on some subset of C and there are infinitely many $x \in \mathrm{At}(C)$ such that $\phi_e(x)\downarrow$ and $\phi_e(x) \neq x$, then there is an $x_e \in \mathrm{At}(C)$ such that $\phi_e(x_e)\downarrow$ and $\phi_e(x_e) \notin \mathrm{At}(C)$.

Our basic strategy to meet the requirements R_{2e} is as follows. Suppose at stage s, requirement R_{2e} is not satisfied and there is an $x \in \mathrm{At}(C^s)$ such that $x \in W_e^{s+1}$. Then at state $s+1$, we let x_1 and x_2 be two nonzero disjoint element of $\tilde{Q}$ such that $x = x_1 + x_2$ and let $C^{s+1} = \langle C^s \cup \{x_1, x_2\}\rangle$. Thus $x \notin \mathrm{At}(C^{s+1})$ so that $x \notin \mathrm{At}(C)$ and x will witness that R_{2e} is satisfied.

Our strategy to meet the requirements R_{2e+1} requires a similar action. The idea is to find some element y such that y currently looks like an atom of C in the sense that $y \in \mathrm{At}(C^s)$, $\phi_{e,s}(y)\downarrow$ and $\phi_{e,s}(y) = x \neq y$. Then there are two possibilities. One is that $x \notin C^s$ in which case we will place x into a set F^{s+1} of forbidden elements and simply ensure that no element in F^{s+1} ever gets into C so that ϕ_e cannot possibly be a recursive automorphism of C. The other possibility is that $x \in C^s$ in which case we will ensure that at stage $s+1$, x is split into two nonzero elements x_1 and x_2 such that $x_1 + x_2 = x$ and $C^{s+1} = \langle C^s \cup \{x_1, x_2\}\rangle$. We shall then place a Δ_{2e+1} marker on y. Our hope is that $y \in \mathrm{At}(C)$ so that once again ϕ_e cannot be an automorphism of C since ϕ_e takes an atom y to non-atom x in C. However, there are two ways that this attempt can fail. One is that $y \in B^s$ and at some later stage t, we are forced to split y into two nonzero pieces in B^t in order to ensure that B is isomorphic to C. The other way is that y could be split in C for the sake of meeting some higher priority requirement. In either case, we will simply remove the Δ_{2e+1} marker from y, search for a new element that currently looks like an atom, and apply the same strategy. We shall see that if ϕ_e really does move infinitely many atoms, then we will require only finitely many applications of this strategy before we find a y which is actually in $\mathrm{At}(C)$. However, because our original B.A. may have atomless elements, there may exist recursive automorphisms ϕ_e which move infinitely many elements but leave all atoms of C fixed. For such ϕ_e, we may not be able to avoid applying our basic strategy infinitely many times due to the fact that we are not assuming we can effectively tell atoms from atomless elements in our original B.A. D. This fact will necessarily complicate our construction. The key point is that such ϕ_e will impose only finitely much permanent restraints so that we will still have enough freedom to met the other requirements.

The need for restraint arises in our construction due to the fact that our strategies for meeting the requirements R_e may conflict with ensuring that C is isomorphic to D. That is, both types of strategy can cause some $x \in B$ to be split into two nonzero pieces x_1 and x_2 in C. Now if we blindly follow such a procedure to satisfy the requirements, then at some later stages we may split x_1 and x_2 into nonzero disjoint elements so that $x_1 = y_1 + y_2$ and $x_2 = z_1 + z_2$ for the sake of other requirements R_i and R_j, and then at even later stages split each of y_1, y_2, z_1, and z_2, etc. In this way, x may end up to be an atomless element in C even though D is atomic. Our idea to control the isomorphism type of C is to build an isomorphic copy of D, B, in $\bar{Q}$ and to ensure that C only differs from B as described above. In particular, we must ensure that if $x \in \mathrm{At}(B)$, then x is a union of finitely many atoms in C. We shall priority rank our requirements as $R_0, R_1, \ldots$, that is, R_0 has highest priority, R_1 has next highest priority, etc. We shall use a set of movable markers in the construction. We imagine we have a potentially infinite set of markers Γ_e for each requirement R_e. When we split an $x \in \mathrm{At}(C^s)$ at state $s+1$ into two nonzero disjoint elements x_1 and x_2 for the sake of requirement R_e as described above, then we will place a Γ_e marker on each of x_1 and x_2. As long as a Γ_e marker remains on x_1 and x_2, we will allow x_1 and x_2 to be split at some later stage only for the sake of some requirement R_j which has a higher priority than R_e, i.e., only if $j < e$. In such a case those elements into which x_1 or x_2 are split will also have Γ_j markers for some $j < e$. From this, it is easy to see that if an element x_i has a permanent Γ_e marker in it, then x_1 will be a union of at most 2^e atoms in C. However, there is one situation where we could remove the Γ_e markers from x_1 and x_2. Namely it may be the case that at some stage u, $x = f^u(a)$ where $a \in \mathrm{At}(D^u)$ and then at stage $u+1$, $d_{u+1} \cdot a$ and $a - d_{u+1}$ are nonzero so that we must split x for the sake of building B isomorphic to D. In such a situation, we will let $B^{u+1} = \langle B^u \cup \{x_1, x_2\} \rangle$ and let $f^{u+1}(a \cdot d_{u+1}) = x_1$ and $f^{u+1}(a - d_{u+1}) = x_2$ and remove the Γ_e markers from x_1 and x_2. In this way, if a

turned out to be an atomless element in D, we will be free to make x an atomless element in C.

Now that we know the nature of the restraints imposed by our requirements, we can better describe our remark concerning those odd requirements R_{2e+1} which may act infinitely often. The idea is that we shall assume that only finitely many atoms of C have permanent Γ_j markers on them for $j \leq 2e$. Now if at stage s, we find $x, y \in \mathrm{At}(C^s)$ such that $\phi^{s+1}(y) = x$ and $x \cdot_{\tilde{Q}} y = 0_{\tilde{Q}}$, then as described above we shall place a Δ_{2e+1} marker on y and split x into x_1 and x_2 and place Γ_{2e+1} markers on x_1 and x_2. Now if later y is split, say due to ensuring that B is isomorphic to D, we shall remove the Δ_{2e+1} marker from y. However, we are not free to remove the Γ_{2e+1} markers from x_1 and x_2 unless once again the process of ensuring that D is isomorphic to B forces us to split x. Thus it is possible for an abandonded attempt to meet R_{2e+1} to leave permanent restraint on x_1 and x_2. The key point which saves us however is that if x_1 and x_2 are permanently restrained, then $x = x_1 + x_2$ will be a finite union of atoms in C. But as $y \cdot x = 0_{\tilde{Q}}$ and $\phi_e(y) = x$, it necessarily follows that if ϕ_e is to be an isomorphism, then at least one atom, say z, of C under x must be mapped outside of x under ϕ_e. In this way, we will be able to argue that if infinitely many elements of C were to end up with permanent Γ_{2e+1} markers on them, then ϕ_e would have to move infinitely many atoms. However, as mentioned above, if ϕ_e moves infinitely many atoms, then we will eventually find a $y \in \mathrm{At}(C)$ on which we can successfully apply our strategy and at which point we will cease to act for R_{2e+1}. But then R_{2e+1} would impose only finitely much restraint contradicting our original assumption. Of course, the even requirements act at most once since once an even requirement is satisfied at stage s, it remains satisfied at all later stages. Thus we will be able to show that each requirement can impose at most finitely much permanent restraint.

We shall say an odd requirement R_{2e+1} is *satisfied* at stage s if either there is a Δ_{2e+1} marker on some element of C^s at stage s or if we have witnesses to the fact that ϕ_e is not an isomorphism at stage s, for example there exist x and $y \in C^s$ such that $\phi_e(x)\downarrow$ and $\phi_e(y)\downarrow$ and either $x \in C^s$ and $\phi_e(x) \in F^s$ or $x \neq y$ and $\phi_e(x) = \phi_e(y)$ or $x <_{\tilde{Q}} y$ but $\phi_e(x) \not<_{\tilde{Q}} \phi_e(y)$, etc.

Finally, we need to state and prove one simple lemma before actually giving the construction. This lemma will be used to justify our ability to choose new elements to add to C^s so that we can ensure C is recursive and keep any forbidden elements in F^s out of C.

Lemma 11.1. *Suppose B is a finite subalgebra of $\tilde{Q}$, F is a finite subset of $\mathbb{N}$ such that $F \cap B = \emptyset$, and a is any atom of B. Then there exists a nonzero $x \in \tilde{Q}$ such that $x <_{\tilde{Q}} a$ and $\langle B \cup \{x\} \rangle \cap F = \emptyset$.*

Proof. It is easy to see that if $x, y <_{\tilde{Q}} a$ and $x \notin \{y, a - y\}$, then $\langle B \cup \{x\} \rangle \cap \langle B \cup \{y\} \rangle = B$. Next observe that since $\tilde{Q}$ is atomless, there are infinitely many nonzero x such that $x <_{\bar{Q}} a$. Thus since F is only finite, there must be a nonzero $x <_{\bar{Q}} a$ such that $\langle B \cup \{x\}\rangle \cap F = \emptyset$. $\square$

We now proceed to give the formal description of our construction. Each stage $s > 0$ is divided into two substages. The first substage will be devoted to extending the isomorphism $f^s : D^s \to B^s$ to an isomorphism $f^{s+1} : D^{s+1} \to B^{s+1}$. The second substage will be devoted to meeting the requirements R_e. Moreover, we shall use even stages to meet the even requirements and odd stages to meet the odd requirements.

Construction

Stage 0. Let $B^0 = C^0 = \{1_{\bar{Q}}, 0_{\bar{Q}}\}$, $f^0(1_D) = 1_{\bar{Q}}$, $f^0(0_D) = 0_{\bar{Q}}$, and $F^0 = \emptyset$.

Stage $s+1$. Assume B^s, C^s, F^s, and f^s have been defined so that $f^s : D^s \to B^s$ is an isomorphism, $B^s \subseteq C^s$, and $C^s \cap F^s = \emptyset$.

Substage i. Let a be the atom of D^s such that $d_{s+1} <_D a$ and let $x = f^s(a)$. If $x \in \mathrm{At}(C^s)$, then let x_1 be the least nonzero element of $\bar{Q}$ such that $x_1 <_{\bar{Q}} x$ and $\langle C^s \cup \{x_1\}\rangle \cap \{0, \ldots, s\} = C^s \cap \{0, \ldots, s\}$ and $\langle C^s \cup \{x_1\}\rangle \cap F^s = \emptyset$. If $x \notin \mathrm{At}(C^s)$, let t be the least stage such that $x \in \mathrm{At}(C^{t-1})$ but $x \notin \mathrm{At}(C^t)$. Then at stage t, there will be a first time either in substage i or substage ii where x is split into two elements $x_1 < x_2$. In fact it will turn out that our construction will ensure that x_1 is the least element in C^s with $0 <_{\bar{Q}} x_1 <_{\bar{Q}} x$. In either case we define $B^{s+1} = \langle B^s \cup \{x_1\}\rangle$ and define f^{s+1} by defining it on the atoms of D^{s+1} and extending it to be a homomorphism. That is, if $d \in \mathrm{At}(D^{s+1})$ and $d \notin \{d_{s+1}, a - d_{s+1}\}$, then let $f^{s+1}(d) = f^s(d)$. Otherwise let $f^{s+1}(d_{s+1}) = x_1$ and $f^{s+1}(a - d_{s+1}) = x - x_1$. It is clear that $f^s \subseteq f^{s+1}$ and f^{s+1} is an isomorphism from D^{s+1} onto B^{s+1}. Also if we are in the case where $x \notin \mathrm{At}(C^s)$, remove any markers on x_1 and $x - x_1$. Remove any Δ markers from elements not in $\mathrm{At}(\langle C^s \cup \{x_1\}\rangle)$.

Substage ii.

Case 1: $s+1$ *is even.*

Look for an $e \leqslant s+1$ such that $e = 2i$, R_e is not satisfied at stage s, and there is an $x \in \mathrm{At}(C^s)$ such that $x \in W_i^s$ and x has no Γ_j or Δ_j markers on it with $j < e$. If there is no such e, then let $C^{s+1} = \langle C^s \cup \{x_1\}\rangle$ and to onto the next stage. Otherwise, let $e(s+1)$ be the least such e and $x(s+1)$ be the least x corresponding to $e(s+1)$. If $x(s+1)$ equals the x chosen at substage i, let $C^{s+1} = \langle C^s \cup \{x_1\}\rangle$. Otherwise let z_1 be the least nonzero $z <_{\bar{Q}} x(s+1)$ such that $\langle C^s \cup \{x_1\} \cup \{z\}\rangle \cap \{0, \ldots, s\} = \langle C^s \cup \{x_1\}\rangle \cap \{0, \ldots, s\}$ and $\langle C^s \cup \{x_1\} \cup \{z\}\rangle \cap F^s = \emptyset$. Let $C^{s+1} = \langle C^s \cup \{x_1, z_1\}\rangle$. Place Γ_e markers on z_1 and $x(s+1) - z_1$. Remove any Δ markers from elements not in $\mathrm{At}(C^{s+1})$.

Case 2: $s+1 = 2r+1$ *is odd.*

We build a sequence of sets $G_{-1} \subseteq G_0 \subseteq \cdots \subseteq G_r$ and a sequence of subalgebras

$E_{-1} \subseteq E_0 \subseteq \cdots \subseteq E_r$ by induction. We let $G_{-1} = F_s$ and $E_{-1} = \langle C^s \cup \{x_1\} \rangle$. Then assuming we have constructed G_{j-1} and E_{j-1}, we construct G_j and E_j as follows. If either there is an element $y \in E_{j-1}$ with a Δ_{2j+1} marker on it, there are witnesses in E_{j-1} which show that ϕ_j is not consistent with being an automorphism of E_{j-1}, or there are $w, z \in E_{j-1}$ such that $w \leqslant_{\tilde{Q}} z$, $\phi_j^{s+1}(w)\uparrow$, and z was split into two elements u_j and w_j with Γ_{2j+1} markers on them, i.e., we split z for R_{2j+1}, at some previous stage, then let $G_j = G_{j-1}$ and $E_j = E_{j-1}$. Otherwise, look for the least $y \in \mathrm{At}(E_{j-1})$ such that

(a) $\phi_j^{s+1}(y)\downarrow$ and $x \cdot y = 0_{\tilde{Q}}$ where $x = \phi_j(y)$, and

(b) x has no Γ_i or Δ_i marker on it for $i \leqslant 2j+1$.

If there is no such y, then again let $G_j = G_{j-1}$ and $E_j = E_{j-1}$. Otherwise, let $y(j, s+1)$ be the least such y and $x(j, s+1) = \phi_j(y(j, s+1))$. If $x(j, s+1) \notin E_{j-1}$ let $G_j = G_{j-1} \cup \{x(j, s+1)\}$ and let $G_j = G_{j-1}$ otherwise. Now if $x(j, s+1) \notin \mathrm{At}(E_{j-1})$, then let $E_j = E_{j-1}$ and place a Δ_{2j+1} marker on $y(j, s+1)$. Finally if $x(j, s+1) \in \mathrm{At}(E_{j-1})$, then let z_j be the least nonzero $z <_{\tilde{Q}} x(j, s+1)$ such that $\langle E_{j-1} \cup \{z\} \rangle \cap \{0, \ldots, s\} = E_{j-1} \cap \{0, \ldots, s\}$ and $\langle E_{j-1} \cup \{z\} \rangle \cap G_j = \emptyset$. Then let $E_j = \langle E_{j-1} \cup \{z_j\} \rangle$, place a Δ_{2j+1} marker on $y(j, s+1)$, and place a Γ_{2j+1} marker on z_j and $x(j, s+1) - z_j$. Remove any Δ marker from elements not in $\mathrm{At}(E_j)$. Finally, we let $C^{s+1} = E_r$ and $F^{s+1} = G_r$.

This completes the description of the construction. It is clear that each stage is completely effective. We let $B = \bigcup_s B^s$, $C = \bigcup_s C^s$, and $f = \bigcup_s f^s$. Our construction clearly ensures that $f: D \to B$ is an isomorphism and that C is a recursive subalgebra of $\tilde{Q}$ since we have ensured that for any fixed s, $s \in C$ iff $s \in C^s$.

We are now left with two lemmas to prove to see that C has the desired properties.

Lemma 11.2. *All the requirements are met.*

Proof. Fix e and assume by induction that for all $i < e$, requirement R_j is met and that there are only finitely many elements of C which have permanent Γ_i markers on them. First suppose $e = 2i$. Note that once an even requirement is satisfied at a stage s, it remains satisfied at all later stages. Hence there is at most one state s where $e(s+1) = e$ and there can be at most two permanent Γ_e markers that are ever introduced. Thus we need only see that requirement R_e is met. So suppose to the contrary that W_i is infinite and $W_i \subseteq \mathrm{At}(C)$. It then follows that there must exist a least $x \in W_i$ such that x has no permanent Γ_j or Δ_j markers on it for $j \leqslant e$. Thus there is a stage t large enough so that t is odd, $x \in W^t \cap \mathrm{At}(C^t)$, x has no Γ_j or Δ_j markers on it with $j \leqslant e$, and $e(2s) \geqslant e$ for all $2s \geqslant t+1$. But then at stage $t+1$, either requirement R_e is already satisfied or $e = e(t+1)$ and we take action at stage $t+1$ to satisfy R_e. In either case, we are assured that $W_i^{t+1} \cap (C^{t+1} - \mathrm{At}(C^{t+1})) \neq \emptyset$ contrary to our assumption.

Suppose $e = 2j + 1$. Note there are essentially 4 ways in which we can be forced into taking no further action for R_e past a certain stage. First we can find witnesses at some stage which show that ϕ_e cannot be an automorphism in which case R_e is met. A second is that there is a permanent Δ_e marker in which case our basic strategy guarantees R_e is met. A third way is that at some stage u, $y(j, u)$ and $x(j, u)$ are defined, the Γ_e markers on the elements which split $x(j, u)$ are permanent, but there never a stage $t > u$ such that $\phi^t(w)\downarrow$ for all $w \in C^t$ with $w \leqslant_{\bar{Q}} x(j, u)$. In this case, it is easy to see that our construction ensures that the only way any predecessors of $x(j, u)$ can be split after stage u is due to an action for some requirement R_i with $i < e$. Moreover, it follows that any Γ_i markers placed on predecessors of $x(j, u)$ are also permanent since our construction ensures that before we can remove such markers we must first remove the Γ_e markers from our original splitting of $x(j, u)$. Since there are only finitely many permanent Γ_i markers with $i < e$, we know that $x(j, u)$ must be a finite union of atoms of C. Thus, it follows that there must be at least one $w \in C$ where $w \leqslant_{\bar{Q}} x(j, u)$ and $\phi_j(w)\uparrow$, so once again R_e is met. Finally, it may be the case that for all sufficiently large t, we have no candidates meeting conditions (a) and (b). But then it is easy to see that since there are only finitely many permanent Γ_i and Δ_i markers for $i < e$, it cannot possibly be the case that ϕ_1 is an automorphism of C which moves infinitely many atoms. Thus in all cases, we can clearly continue the induction.

We have thus reduced ourselves to the case where we can assume (i) ϕ_j^s is always consistent with being an automorphism of C, (ii) there are no permanent Δ_e markers, (iii) if z is split into two elements with permanent Γ_e markers on them, then $\phi_j(w)\downarrow$ for all $w \leqslant_{\bar{Q}} z$ with $w \in C$, and (iv) we take action for R_e at infinitely many stages. Next let us explore how we can get permanent Γ_e markers without having permanent Δ_e markers. Consider a stage s where a permanent Γ_e marker is introduced. At such a stage s, we have $y(j, s) \cdot x(j, s) = 0_{\bar{Q}}$, $\phi_j^s(y(j, s)) = x(j, s)$, and we split $x(j, s)$ into two elements z_j and $w_j = x(j, s) - z_j$. Now if the Γ_e markers we place on z_j and w_j are permanent, then by our previous arguments we know both z_j and w_j are finite unions of atoms in C. It then follows, due to the fact that by assumption ϕ_j^t is always consistent with being an automorphism of C, that there must be some $a \leqslant z_j + w_j = x(j, s)$ such that $a \in \mathrm{At}(C)$ and $\phi_j(a) \cdot x(j, s) = 0_{\bar{Q}}$. That is, by our assumption, $\phi_j(x(j, s))$ must be defined and since $\phi_j(y(j, s)) = x(j, s)$ and $y(j, s) \cdot x(j, s) = 0_{\bar{Q}}$ we must have $\phi_j(y(j, s)) \cdot \phi_j(x(j, s)) = x(j, s) \cdot \phi_j(x(j, s)) = 0_{\bar{Q}}$. Now consider $\phi_j(a)$ which again must be defined by our assumptions. We claim that $\phi_j(a)$ has a permanent Γ_i or Δ_i marker on it for some $i \leqslant e$. Otherwise for all sufficiently large t, a is a candidate to have a Δ_e marker on it at stage $2t + 1$. By our assumptions, we take action for R_e at infinitely many stages. It then follows that since once we remove a Δ_e marker from an element y we can never reconsider y to have another Δ_e marker placed on it, eventually a would become the least candidate to have a Δ_e marker placed on it. Thus eventually we would take an action for R_e at a stage t

where $y(j, t) = a$. But then since $a \in \mathrm{At}(C)$, the Δ_e marker we placed on a would be permanent contradicting our assumptions.

We now claim that under the above circumstances either

(a) there is some $d_1 \in \mathrm{At}(C)$ such that $d_1 \leqslant_{\bar{Q}} x(j, s)$ and $\phi_j(d_1)$ has a permanent Γ_i or Δ_i marker on it for $i < e$, or

(b) there is some $d_2 \in \mathrm{At}(C)$ such that $d_2 \leqslant_{\bar{Q}} x(j, s)$ and d_2 has a permanent Γ_i marker on it for $i < e$.

Note that once we prove our claim, it will follow that there are only finitely many permanent Γ_e markers since by induction there are only finitely many permanent Γ_i or Δ_i markers with $i < e$ and we are assuming ϕ_j is always consistent with being an automorphism of C. To prove our claim we need to examine the case where $\phi_j(a)$ has a permanent Γ_e marker on it. Suppose $\phi_j(a)$ got its permanent Γ_e marker at stage t. Note that $t \neq s$. So first assume $t > s$ and consider $y(j, t)$ and $x(j, t)$. Note that $\phi_j(a)$ must be one half of the splitting of $x(j, t)$. Let us assume that $a \leqslant_{\bar{Q}} z_j$. Note that at stage t, $y(j, t)$ is an atom in the E_{j-1} of stage t and $z_j \in E_{j-1}$ so that we must have that either $y(j, t) \cdot z_j = 0_{\bar{Q}}$ or $y(j, t) \leqslant_{\bar{Q}} z_j$. It cannot be that $y(j, t) \cdot z_j = 0_{\bar{Q}}$ since otherwise $a \cdot y(j, t) = 0_{\bar{Q}}$ but $\phi_j(a) \leqslant_{\bar{Q}} \phi_j(y(j, t)) = x(j, t)$ so that a and $y(j, t)$ would eventually witness that ϕ_i is not consistent with being an automorphism of C. Now if $y(j, t) \leqslant_{\bar{Q}} z_j$, then since the Δ_e markers on $y(j, t)$ must eventually be abandoned, we must conclude that $y(j, t)$ and hence z_j must have been split in C due to the action of some requirement R_i with $i < e$. Hence there is an atom of C strictly below z_j with a permanent Γ_i marker on it for some $i < e$. Thus if $s < t$, case (b) of our claim holds. Lastly consider the case when $t < s$. Let u_j and v_j be the elements into which $x(j, t)$ was split at stage t and say $\phi_j(a) = u_j$. Since we are assuming that the Γ_e markers on u_j and v_j are permanent, it follows from our construction that every $b \leqslant_{\bar{Q}} u_j + v_j$ which is an atom either in C^r or the sequence of subalgebras constructed at stage r for any $r \geqslant s$ must have a Γ_i marker on it for some $i \leqslant e$. But note that to split $x(j, s)$ at stage s, it must be that $x(j, s)$ is an atom in the E_{j-1} constructed at stage s. Thus we must conclude that since $x(j, s)$ has no Γ_i or Δ_i markers on it with $i < e$ when it was split, $x(j, s) \cdot (u_j + v_j) = x(j, s) \cdot x(j, t) = 0_{\bar{Q}}$. Moreover note that since $a \leqslant_{\bar{Q}} x(j, s)$ and $\phi_j(a) = u_j(a) = u_j \leqslant_{\bar{Q}} x(j, t)$, then $\phi_j(x(j, s)) \cdot x(j, t) \neq 0_{\bar{Q}}$. But then $x(j, s) \cdot y(j, t) \neq 0_{\bar{Q}}$ since again we can assume $\phi_j(x(j, s))\downarrow$ and ϕ_j preserves intersections. It then follows that since $x(j, s) \in \mathrm{At}(E_{j-1})$ and $y(j, t) \in E_{j-1}$ that $x(j, s) \leqslant_{\bar{Q}} y(j, t)$. In fact $x(j, s) <_{\bar{Q}} y(j, t)$. That is, $y(j, t) \notin \mathrm{At}(E_{j-1})$ because otherwise $y(j, t)$ would retain its Δ_e marker and then we could not act for R_e at stage s. Now due to the fact that ϕ_j is always consistent with being an automorphism, we know that $x(j, s) <_{\bar{Q}} y(j, t)$ implies $\phi_j(x(j, s)) <_{\bar{Q}} x(j, t)$. But as $\phi_j(a) = u_j$, we must then have that $\phi_j(x(j, s) - a) <_{\bar{Q}} v_j$. Thus we can now conclude that some atom $a' \in \mathrm{At}(C)$ where $a' \leqslant_{\bar{Q}} x(j, s) - a$ is such that $\phi_j(a') <_{\bar{Q}} v_j$. But then since $x(j, s) \cdot x(j, t) = 0_{\bar{Q}}$, we have $\phi_j(a') \cdot a' = 0_{\bar{Q}}$. Moreover we must conclude that v_j must have been split and that $\phi_j(a')$ must be an element with a Γ_i marker on it for $i < e$. That is, $\phi_j(a')$ must also have permanent Γ_R marker on it for some $R \leqslant e$ or by our previous argument, we can conclude that a' would eventually get a permanent Δ_e marker on it. Moreover the case $R = e$ is ruled out since $\phi_j(a') <_{\bar{Q}} v_j$ and v_j has a permanent Γ_e marker on it. Thus if $t < s$, we are guaranteed that case (a) of our claim holds.

Having established the fact that there are only finitely many Γ_e markers given assumptions (i)–(iv), we are left only with proving that requirement R_e is met. So suppose ϕ_j is an automorphism of C which moves infinitely many atoms of C. It then follows that there must be some $a \in \mathrm{At}(C)$ such that $\phi_j(a)$ has no permanent Γ_i or Δ_i markers on it for $i \leqslant e$. But then by our previous arguments, our current assumptions ensure that eventually we would place a Δ_e marker on a and ensure that $\phi_j(a) \notin \mathrm{At}(C)$. But then this Δ_e marker would be permanent contrary to our assumptions. Thus ϕ_j cannot be an automorphism of C which moves infinitely many atoms, so R_e is met. □

To complete our proof, we need only prove the following.

Lemma 11.3. $D \approx C$.

Proof. Our construction ensures that $D \approx B$. Thus we show $B \approx C$. Let $\mathrm{At}(B) = \{a_0, a_1, \ldots\}$. First we shall show that for each a_i there exist $e_1^i, \ldots, e_{k_i}^i$, elements of $\mathrm{At}(C)$, such that $a_i = \sum_{j=1}^{k_i} e_j^i$ and then we shall show that $C = \langle B \cup \{e_j^i \mid i \geqslant 0 \mathbin{\&} 1 \leqslant j \leqslant k_i\}\rangle$. Thus it will follow from Theorem 10 that $B \approx C$.

Given i, let t be the first stage such that $a_i \in B^t$. Since $a_i \in \mathrm{At}(B)$ there is an atom d of D such that $f^t(d) = a_i$. It follows that there is no stage $s > t$ such that at substage i of stage s we introduce a nonzero $x \in B^s$ such that $x <_{\bar{Q}} a_i$. Now if a_i is not an atom of C, then there is a stage s such that the $x(s)$ or $x(j, s)$ chosen at substage ii of stage s is a_i. Thus there exists an $x \in C^s$ such that x is nonzero and $x <_{\bar{Q}} a_i$ and both x and $a_i - x$ have Γ_e markers on them for some e. Since a_i is never split in substage i at any stage u, it follows that the Γ_e markers are never removed from x and $x - a_i$. Our construction now ensures that if there is a stage u and a nonzero y such that $y <_{\bar{Q}} x$ or $y <_{\bar{Q}} x - a_i$ and $y \in \mathrm{At}(C^u) - \mathrm{At}(C^{u-1})$, then y must have a Γ_j marker on it for some $j < e$ and this Γ_j marker will never be removed from y. Since we have proved in Lemma 11.2 that there are only finitely many Γ_i markers for any i, it easily follows that both x and $a_i - x$ is a union of finitely many atoms of C.

Next suppose z is an arbitrary nonzero element of C. Let s be the stage where $z \in C^s - C^{s-1}$. Thus we can express z as a finite union of atoms of C^s, $z = \sum_{i=1}^{k} z_i$. It is an easy finite induction to show that if $z' \in \mathrm{At}(C^s)$, then either $z' \in B^s$ or z' has a Γ_j marker on it for some j. Moreover, our construction ensures that if a Γ_j marker is ever removed from any x at some stage t, then $x \in B^t$. Now let u be a stage large enough so that if $x \in C^s$ and x has a Γ_j marker on it at stage s, then either x no longer has a Γ_j marker on it at stage u, in which case $x \in B^u$, or the Γ_j marker remains on x at all stages $t \geqslant s$. Consider the subalgebra $E = B^u \cap C^s$. For each z_i as above, either $z_i \in B^u$ or z_i has a Γ_j marker on it at stage u. In the latter case, let a be the atom of E such that $z_i <_{\bar{Q}} a$. Consider the first stage w where $a \in \mathrm{At}(C^w) - \mathrm{At}(C^{w+1})$. Then at stage $w+1$, there are x_1 and x_2 in $\mathrm{At}(C^{w+1})$ or $\mathrm{At}(E_j)$ for some j such that $a = x_1 + x_2$. Since $a \in C^s$, $z_i \in C^s$, and $z_i <_{\bar{Q}} a$, it follows that $w+1 \leqslant s$ and x_1 and x_2 are not in $B^u \cap C^s$. Moreover, we must have split a at substage ii of stage $w+1$ and hence at stage $w+1$, x_1 and x_2 have Γ_n markers on them for some n. Since x_1 and x_2 are not in B^u, it follows that the Γ_n markers on x_1 and x_2 were not removed by stage u. Thus x_1 and x_2 have Γ_n markers on them at stage u and hence by our choice of u, x_1 and x_2 have Γ_n markers on them at all stages $t \geqslant s$. It now follows that a must be an atom of B for if there is a stage t such that $a \in \mathrm{At}(B^{t+1}) - \mathrm{At}(B^t)$, then we split a at stage i of stage $t+1$ and our construction would force us to remove the Γ_n markers on x_1 and x_2. Thus each z_i is either in B or $z_i <_{\bar{Q}} a_j$ where a_j is some atom of B in which case z_i is a finite union of some of $e_1^j, \ldots, e_{k_j}^j$ where $e_i^j, \ldots, e_{h_j}^j$ are the atoms of C under a_j. Thus we can conclude that

$$C = \langle B \cup \{e_j^i \mid i \geqslant 0 \;\&\; 1 \leqslant j \leqslant k_i\} \rangle. \quad \square$$

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