A Generalization of Walsh's Two-Circle Theorem to Vector Spaces

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In this article we investigate the location of the null sets of generalized pseudo-derivatives of the product (or quotient) of abstract polynomials. Some special cases of this general problem were studied by Walsh as geometry of polynomials in the complex plane. One of our two results deduces an improved version of the first part of the two-circle theorem due to Walsh on the quotient of two polynomials and the other furnishes an improved version of the two-circle theorem on the product of two polynomials.

1. INTRODUCTION

Throughout, unless mentioned otherwise, $E$ denotes a vector space over an algebraically closed field $K$ of characteristic zero. It is known (see [6, pp. 248–255]) that $K = K_0(i)$, where $K_0$ is a maximal ordered subfield of $K$ and $-i^2$ is the unit element of $K$. By [10, Remark 1.1], every vector space $E$ can be made into a $K$-resp. $K_0$ inner product space (written briefly $K$-i.p.s.).

Let $K_\infty = K \cup \{\infty\}$ denote the projective field (see [12, p. 352] or [9, p. 116]) obtained by adjoining to $K$ an element $\infty$ (called the scalar infinity). Also, let $E \cup \{\omega\} = E_\omega$ as in [12, p. 372]. For $E = K$, we can use $\omega$ and $\infty$ interchangeably.

The details regarding the rest of the material in this section can be seen in [10, pp. 833–835, 839–843; 11, pp. 268–272]. It is quite interesting to note how certain geometrical configurations in $\mathbb{C}$ have analytical analogues...

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in abstract spaces that are associated with abstract polynomials. The first such instance is in the work of Zervos [12, pp. 352–353] (or [9, p. 116]) in certain fields, as described below.

A subset $A$ of $K_n$ is called a generalized circular region (g.c.r.) of $K_n$ if either $A$ is one of $\phi, K, K', \Psi$ or if $A$ satisfies the following two properties: (i) $\theta_t(A)$ is $K_t$-convex for all $\zeta \in K - A$, where $\theta_t(z) = (z - \zeta)^{-1}$ for all $z \in K_n$, and (ii) $\infty \in A$ if $A$ is not $K_t$-convex. The family of all g.c.r.’s will be denoted by $D(K_n)$. For $K = C$, we have the following result (see [12, p. 352] or [9, p. 116]): The nontrivial members of $D(C_n)$ are the open interior (or exterior) of circles or the open half-planes, adjoined with a connected subset possibly empty of their boundary. The concept of $D(K_n)$ was further extended to the family $D(E_w)$ of g.c.r.’s of $E_w$ when $E$ is a K-i.p.s., with an abundance of nontrivial g.c.r.’s [10, Proposition 1.5]. Another instance is that of hermitian cones, due to Hormander. Now we describe a still more general concept due to Zaheer [11], of the family $D*(E_w)$ in vector spaces, which is even richer than that of $D(E_w)$.

Given $S \subseteq E_w$, we write

$$G_t(x, y) = \{ \rho \in K_n \mid x + \rho y \in S \} \forall x, y \in E. \quad (1.1)$$

We say that $S$ is a supergeneralized circular region (briefly, sg.c.r.) if $G_t(x, y) \in D(K_n)$ for every $x, y \in E$. The family of all sg.c.r.’s will be denoted by $D*(E_w)$. Clearly $\phi, E,$ and $E_w$ are trivial members of $D*(E_w)$. Properties of $\omega$ and $\infty$ [10, p. 834] imply that (since $G_t(x, 0) = K$ or $G_t(x, 0) = \phi$ depending on whether $x \in S$ or $x \notin S$)

$$\infty \notin G_t(x, 0) \in D(K_n) \forall x \in S \quad (1.2)$$

and that

$$\infty \in G_t(x, y) \forall x \in E, y \in E - \{0\} \text{ if and only if } \omega \in S. \quad (1.3)$$

Therefore

$$S \in D*(E_w) \quad \text{if and only if } G_t(x, y) \in D(K_n) \forall x, y \in E \ (y \neq 0). \quad (1.4)$$

The relations between $D(E_w)$ and $D*(E_w)$ are given in the following:

**Proposition (1.1) [11, pp. 269–272].** Let $E$ be a K-i.p.s. Then

(a) $D(E_w) \subseteq D*(E_w)$ if dim $E \geq 2$

(b) $D(E_w) = D*(E_w)$ if dim $E = 1$.

(c) $D(K_n) = D*(K_n)$ if $E = K$ is taken as a 1-dimensional natural K-i.p.s., where $D(K_n) = D(K_n)$ as in [11, Remark 3.1]. Hence $D*(C_n)$ coincides with the family $D(C_n)$ of all g.c.r.’s of $C_n$. Note that $C$ is convex if $\infty \notin C \subseteq D(C_n)$. 
Remark (1.2). (i) The above proposition shows that the family $D^*(E_w)$ is a natural generalization to vector spaces of the concept of g.c.r.'s [10, Remark 1.6] in the complex plane and that it offers a richer class when $E$ is a $K$-i.p.s.

(ii) All maximal subspaces and their translations are members of $D^*(E_w)$ in a vector space, but are not members of $D(E_w)$ when $E$ is a $K$-i.p.s.

(iii) If $S \in D^*(E_w)$, then $E_w - S$ may not belong to $D^*(E_w)$ [11, Remark 3.7(iii)].

(iv) There are sets in $D^*(E_w)$ whose complements in $E_w$ are also in $D^*(E_w)$, for example, maximal subspaces and their translations [11, Remark 3.7(iv)].

A mapping $P: E \to K$ is called an abstract polynomial (written briefly a.p.) of degree $n$ if for every $x, y \in E$,

$$P(x + \rho y) = \sum_{k=0}^{n} A_k(x, y) \rho^k \quad \forall \rho \in K,$$

where the coefficients $A_k(x, y) \in K$ are independent of $\rho$ and $A_n(x, y) \neq 0$. We shall denote by $P_n(E, K)$ the class of all a.p.'s of degree $n$ from $E$ to $K$. In particular, we shall write $P_n(K) = P_n(K, K)$ and $P_n = P_n(\mathbb{C})$. It is known [5, Theorem 2.2] that $A_0(x, y)$ is independent of $y$, $A_n(x, y)$ is independent of $x$, and $A_k(x, y)$ is an abstract polynomial of degree $n - k$ in $x$ (for each fixed $y$) and also an abstract homogeneous polynomial of degree $k$ in $y$ (for each fixed $x$). Note that $A_n(x, y) \neq 0$ implies the existence of at least one nonzero element $h \in E$ for which $A_n(x, h) = A_k(0, h) \neq 0$ for every $x \in E$. Such elements $h$ are termed faithful to $P$. The set of all elements faithful to $P$ will be denoted by $F(P)$. The null-set of $P$ is the set $Z(P) = \{x \in E \mid P(x) = 0\}$.

Given $P \in P_n(E, K)$ and $h \in F(P)$, since $K$ is algebraically closed, we may write

$$P(x + \rho h) = A_n(x, h) \prod_{j=1}^{n} [\rho - \rho_j(x, h)] \quad \forall \rho \in K,$$

where the coefficients $\rho_j(x, h)$ belong to $K$ and are independent of $\rho$ such that $A_n(x, h) = A_0(0, h)$ for all $x \in E$. Let $\Delta(m, n)$ denote the sum of all possible products formed out of the scalars $\rho_j(x, h), j = 1, 2, \ldots, n$, taken $m$ at a time, so that Eq. (1.6) yields

$$P(x) = A_0(x, h) = (-1)^n A_n(x, h) \Delta(n, n)$$

and

$$P_h(x) = A_1(x, h) = (-1)^{n-1} A_n(x, h) \Delta(n - 1, n).$$
Definition 1.3. Given \( P \in P_n(E, K) \) (via (1.6)) and \( h \in F(P) \), we define for each \( k = 1, 2, \ldots, n \), the \( k \)th pseudo-derivative (or \( p \)-derivative) \( P_h^{(k)} \) of \( P \) (relative to \( h \)) to be the mapping from \( E \) to \( K \) given by

\[
P_h^{(k)}(x) = k! A_k(x, h) \quad \forall x \in E.
\]

Indeed, \( P_h^{(k)} \in P_{n-k}(E, K) \). The first few members in (1.9) will be written as \( P'_h, P''_h \), etc. Some basic properties \( 10, \) pp. 841–843 of the family \( P_h(E, K) \) are collected in the following:

Proposition 1.4. If \( P \in P_n(E, K) \) and \( h \in F(P) \), then \( h \in F(P_h^{(k)}) \) for \( k = 1, 2, \ldots, n - 1 \), and we have

\[
P_h^{(k+1)}(x) = (P_h^{(k)})'(x) \quad \forall x \in E, k = 1, 2, \ldots, n - 1.
\]

Remark 1.5. (i) If \( P \in P_n(E, K), h \in F(P) \), then \( P_h^{(k)} \in P_{n-k}(E, K) \) and the a.p.'s \( P_h^{(k)} \) \((k = 1, 2, \ldots, n)\) deserve to be called successive pseudo-derivatives of \( P \).

(ii) Let \( P_n(K) = P_n(E, K) \) denote the collection of all \( n \)th degree (ordinary) polynomials from \( K \) to \( K \). If \( f^{(k)} \) stands for the \( k \)th formal derivative of \( f \in P_n(K) \), then

(i) \( f \) is necessarily an a.p. of degree \( n \) from \( K \) to \( K \) (i.e., \( f \in P_n(K) \));

(ii) Every nonzero element of \( K \) is faithful to \( f \) when \( f \) is regarded as a member of \( P_n(K) \), that is, \( F(f) = K - \{0\} \);

(iii) For each value of \( k = 1, 2, \ldots, n \), the \( k \)th pseudo-derivative \( f_h^{(k)} \) of \( f \) (treated as a member of \( P_n(K) \)) and the \( k \)th formal derivatives \( f^{(k)} \) of \( f \) (treated as a member of \( P_n(K) \)) satisfy the relation

\[
f_h^{(k)}(x) = h^k f^{(k)}(x) \quad \forall x \in E, h \in K - \{0\}.
\]

In particular (for \( h = 1 \)), \( f_1^{(k)} = f^{(k)} \), and the two notations coincide. The same is true in particular when \( K = \mathbb{C} \), but then the formal derivative \( f^{(k)} \) becomes the usual \( k \)th derivative of \( f \) as defined via calculus.

Definition 1.6. Given a.p.'s \( P_k \in P_{n_k} \) and scalars \( m_k \in K, k = 1, 2, \ldots, q \), set

\[
Q(x) = P_1(x) \cdots P_q(x),
\]

\[
Q_k(x) = P_1 \cdots P_{k-1}(x) \cdot P_{k+1} \cdots P_q(x),
\]

and define, for each \( h \in \bigcap_{k=1}^q F(P_k) \),

\[
R_h(x) = \sum_{k=1}^q m_k Q_k(x) \cdot (P_k)'(x) \quad \forall x \in E.
\]
We call \( R_h(x) \) a generalized pseudo-derivative (or \( p \)-derivative) of the product \( Q(x) \).

Note that if \( n = n_1 + n_2 + \cdots + n_q \), then \( Q \in \mathcal{P}_n(E, K) \), \( Q_k \in \mathcal{P}_{n_k-n}(E, K) \), and \( (P_k)_h \) is an a.p. of degree \( n_k - 1 \) in \( x \), \( 1 \leq k \leq q \). Therefore, \( R_h(x) \) is an a.p. of degree at most \( n - 1 \) in \( x \). Also note that, if in the notations of Definition (1.6), \( m_k = 1 \) for \( k = 1, 2, \ldots, q \), then \( R_h(x) = Q'_h(x), \forall x \in E, \forall h \in \bigcap_{k=1}^q F(P_k) \). That is, \( R_h(x) \) of the product \( Q(x) \) is essentially the first \( p \)-derivative \( Q'_h(x) \) of \( Q(x) \) [1, Proposition 2.2]. This justifies the terminology for \( R_h(x) \) to be called a “generalized \( p \)-derivative of \( Q(x) \).”

2. THE MAIN RESULTS

In this section we establish two main results. The first one deduces as a corollary the first part of Walsh’s two-circle theorem (see [8] or [4, Theorem (20, 1)]) and the second generalizes Walsh’s two circle theorem (see [7] or [4, Theorem (19, 1)]) on the product of two polynomials.

Theorem (2.1). Let \( P_k \in \mathcal{P}_{n_k}(E, K) \) \( 1 \leq k \leq q \), \( S_i \in D^*(E_\omega) \) with \( \omega \notin S_i \) \( i = 1, 2 \), such that

\[
Z(P_k) \subseteq \begin{cases} S_1, & 1 \leq k \leq p (< q), \\ S_2, & p + 1 \leq k \leq q. \end{cases}
\]  

(2.1)

If \( R_h(x) \) is given by (1.10), with

\[
m_k \begin{cases} > 0, & 1 \leq k \leq p, \\ < 0, & p + 1 \leq k \leq q, \end{cases}
\]  

(2.2)

such that \( \sum_{k=1}^q m_k n_k \neq 0 \), then

\[
Z(R_h) \subseteq S_1 \cup S_2 \cup S_3 \quad \forall h \in \bigcap_{k=1}^q F(P_k),
\]

where

\[
S_3 = \frac{A_1 S_2 + A_2 S_1}{A_1 + A_2}, \quad A_1 = \sum_{k=1}^p m_k n_k, \quad A_2 = \sum_{k=p+1}^q m_k n_k. 
\]  

(2.3)

Proof. Suppose to the contrary of \( x \in Z(R_h) \) that \( x \notin S_1 \cup S_2 \cup S_3 \) for some \( h \in \bigcap_{k=1}^q F(P_k) \). By the pattern of (1.6), let \( P_k \) be given by

\[
P_k(x + \rho h) = A_{n_k}(x, h) \prod_{j=1}^{n_k} \left[ \rho - \rho_{k,j}(x, h) \right] \quad \forall \rho \in K.
\]
Since \( P_k(x) \neq 0 \) for all \( k (1 \leq k \leq q) \), (1.7) gives
\[
\rho_{k,j}(x,h) = \rho_{k,j} \neq 0 \quad \forall j = 1, 2, \ldots, n_k, 1 \leq k \leq q.
\]
Further, since \( P_k(x + \rho_{k,j} h) = 0 \) for all \( j \) and \( k, x + \rho_{k,j} h \in Z(P_k) \subseteq S_1 \) for all \( j \) and for \( 1 \leq k \leq p \) and \( x + \rho_{k,j} h \in Z(P_k) \subseteq S_2 \) for all \( j \) and for \( p + 1 \leq k \leq q \). Hence, \( \rho_{k,j} \in G_i(x,h) \equiv G_1 \) (say) for all \( j \) and for \( 1 \leq k \leq p \), and \( \rho_{k,j} \in G_i(x,h) \equiv G_2 \) (say) for all \( j \) and for \( p + 1 \leq k \leq q \).

Since \( S_i \in D^*(E'_\omega) \) and \( h \neq 0 \), we observe that \( G_i \in D(K_\omega) \) where \( 0, \infty \in G_i \) (because \( x, \omega \in S_i \)) so that \( G_i \) is \( K_\omega \)-convex and \( \rho_{k,j} \neq 0, \infty \) for all \( k \) and \( j \).

Now consider the mapping \( \theta_0(\rho) = 1/(\rho - \zeta) \forall \rho \in K_\omega \). Since \( \theta_0(G_i) \) is \( K_\omega \)-convex and \( 0, \infty \in \theta_0(G_i) \), we have
\[
0 \neq \frac{1}{n_k} \sum_{j=1}^{n_k} \frac{1}{\rho_{k,j}} = \begin{cases} \theta_0(G_1) & \text{for } 1 \leq k \leq p, \\ \theta_0(G_2) & \text{for } p + 1 \leq k \leq q. \end{cases}
\]

Let
\[
A_1 = \sum_{k=1}^{p} m_k n_k, \quad A_2 = \sum_{k=p+1}^{q} m_k n_k.
\]

In view of (2.2), the scalars \( m_k n_k/A_1 \) (resp. \( m_k n_k/A_2 \)) are positive elements of \( K_\omega \) for \( k = 1, 2, \ldots, p \) (resp. \( k = p + 1, \ldots, q \)) with sum 1. Now, (2.3) and the \( K_\omega \)-convexity of \( \theta_0(G_i) \) imply that \( \mu_i/A_i \in \theta_0(G_i) \) for \( i = 1, 2 \), where
\[
\mu_1 = \sum_{k=1}^{p} \sum_{j=1}^{n_k} \frac{m_k}{\rho_{k,j}}, \quad \mu_2 = \sum_{k=p+1}^{q} \sum_{j=1}^{n_k} \frac{m_k}{\rho_{k,j}}.
\]

Therefore
\[
0, \infty \neq \mu_i/A_i = \theta_0(\rho_i) = \frac{1}{\rho_i} \quad \text{for some } \rho_i \in G_i \ (i = 1, 2). \quad (2.6)
\]

That is, \( \rho_i = A_i/\mu_i \in G_i \) and so \( x + \rho_i h = x + (A_i/\mu_i)h \in S_i \) for \( i = 1, 2 \). Hence (cf. (2.3) and (2.6)),
\[
\frac{A_1(x + \rho_2 h) + A_2(x + \rho_1 h)}{A_1 + A_2} = \frac{A_1(x + (A_2/\mu_2)h) + A_2(x + (A_1/\mu_1)h)}{A_1 + A_2} \in S_3
\]
We claim that \( m_1 + m_2 \neq 0 \). For otherwise \( m_1 = -m_2 \). Consequently,

\[
\frac{A_1(x - (A_2/\mu_1)h) + A_2(x + (A_2/\mu_1)h)}{A_1 + A_2} = x \in S_3,
\]

contradicting the fact that \( x \not\in S_1 \cup S_2 \cup S_3 \). Hence

\[
\mu_1 + \mu_2 = \sum_{k=1}^{q} \sum_{j=1}^{n_k} \frac{m_k}{\rho_{k,j}} \neq 0,
\]
as claimed. Also by (1.7) and (1.8), we have

\[
(P_k)'(x) = \left( \sum_{j=1}^{n_k} \frac{1}{\rho_{k,j}} \right) \cdot P_k(x), \quad 1 \leq k \leq q. \tag{2.7}
\]

Finally, since \( Q(x) \cdot P_k(x) = Q(x) \neq 0 \), definition (1.10) and Eqs. (2.5) and (2.7) imply that

\[
R_k(x) = \left( \sum_{k=1}^{q} \sum_{j=1}^{n_k} \frac{m_k}{\rho_{k,j}} \right) \cdot Q(x) \neq 0.
\]

This contradicts the fact that \( x \in Z(R_k) \), and the proof is complete. \( \blacksquare \)

The above theorem deduces as a corollary the following result, which is an improved version of the first part of the two-circle theorem due to Walsh [4, Theorem (20, 1)].

**Corollary (2.2).** Let \( f_i \in P_{n_i}(K) \), \( n_1 \neq n_2 \), and \( C_i \in D(K_i) \) with \( \omega \not\in C_i \) (\( i = 1, 2 \)). If \( Z(f_i) \subset C_i \) for \( i = 1, 2 \), then all the finite zeros of the formal derivative of the quotient \( f = f_1/f_2 \) lie in \( C_1 \cup C_2 \cup C_3 \), where

\[
C_3 = \frac{n_2 C_1 - n_1 C_2}{n_2 - n_1}.
\]

**Proof.** By Remark (1.5)(II), \( f_i \in P_{n_i}(K) \), \( F(f_i) = K - \{0\} \), and the \( R_k(x) \) of Theorem (2.1) (with \( p = 1 \), \( q = 2 \), and \( m_1 = -m_2 = 1 \)) is given by

\[
R_k(x) = (f_1)'(x) f_2(x) - (f_2)'(x) f_1(x) = h \left[ f_1'(x) f_2(x) - f_2'(x) f_1(x) \right] = h f'(x) \left[ f_2(x) \right]^2, \tag{2.8}
\]
where \( f' \) denotes the formal derivative of the quotient \( f_1/f_2 \).
Since \( R_h(x) \), the a.p.'s \( f_i \) and the \( C_i \in D^s(K_w) \equiv D(K_w) \) satisfy the hypotheses of Theorem (2.1) for \( E = K \), we conclude that \( Z(R_h) \subseteq C_1 \cup C_2 \cup C_3 \) for all \( h \in \bigcap_{i=1}^{q} F(f_i) \). The equality (2.8) then demonstrates that the finite zeros of \( f' \) lie in \( C_1 \cup C_2 \cup C_3 \), as was to be shown.

For \( K = \mathbb{C} \) and the \( C_i \) taken as disks \( D_i \equiv D(c_i, r_i) \) (a special subclass of \( D(C_w) \equiv D^s(C_w) \)), the above corollary is the following result due to Walsh (cf. [4, the first part of Theorem (20, 1)]), on the finite zeros of the derivative of the quotient of two polynomials of different degrees.

**Corollary (2.3).** Let \( f_i \) be a polynomial of degree \( n_i \), \( i = 1, 2 \). If \( f_i \) has all its zeros in or on a circle \( C_i \) with center \( c_i \) and radius \( r_i \), and if \( n_1 \neq n_2 \) then all the finite zeros of the derivative of the quotient \( f = f_1/f_2 \) lie in \( C_1 \cup C_2 \cup C_3 \), where \( C_3 \) is the circle with center \( c_3 \) and radius \( r_3 \) given by

\[
c_3 = \frac{n_2 c_1 - n_1 c_2}{n_2 - n_1}, \quad r_3 = \frac{n_2 r_1 + n_1 r_2}{|n_2 - n_1|}.
\]

Our next result immediately follows from Theorem (2.1), on taking all the constants \( m_k \) to be positive.

**Theorem (2.4).** Let \( P_k \in \mathcal{P}_{n_k}(E, K) \) (1 \( \leq k \leq q \)), \( S_i \in D^s(E_w) \) with \( \omega \notin S_i \) for \( i = 1, 2 \), such that

\[
Z(P_k) \subseteq \begin{cases} S_1, & 1 \leq k \leq p(< q), \\ S_2, & p + 1 \leq k \leq q. \end{cases}
\]

If \( R_h(x) \) is given by (1.10) with \( m_k > 0 \), then \( Z(R_h) \subseteq S_1 \cup S_2 \cup S_3 \) \( \forall h \in \bigcap_{k=1}^{q} F(P_k) \), where \( S_3 \) is given by (2.3).

The above theorem furnishes the following result which is an improved version of the two-circle theorem due to Walsh (see [7] or [4, Theorem (19, 1)]) on the product of two polynomials.

**Corollary (2.5).** Let \( f_i \in \mathcal{P}_{n_i}(K), \ n_1 \neq n_2 \), and \( C_i \in D(K_w) \) with \( \omega \notin C_i \) for \( i = 1, 2 \) (not necessarily disjoint). If \( Z(f_i) \subseteq C_i \) for \( i = 1, 2 \), then all the finite zeros of the formal derivative of the product \( f = f_1 f_2 \) lie in \( C_1 \cup C_2 \cup C_3 \), where

\[
C_3 = \frac{n_2 C_1 + n_1 C_2}{n_2 + n_1}.
\]

**Proof.** Proceeding as in the proof of Corollary (2.2) (with \( p = 1, q = 2 \), and \( m_1 = m_2 = 1 \), \( R_h(x) \) in this case is given by

\[
R_h(x) = (f_1)'_h(x)f_2(x) + (f_2)'_h(x)f_1(x) = hf'(x).
\]
For $K = \mathbb{C}$ and the $C_i$ taken as disks $D_i = D(c_i, r_i)$, the above corollary is precisely the following two-circle theorem due to Walsh (cf. [4, the first part of Theorem (19, 1)]) on the finite zeros of the derivative of the product of two polynomials of different degrees.

**Corollary (2.6).** Let $f_i$ be a polynomial of degree $n_i$, $i = 1, 2$. If $f_i$ has all its zeros in or on a circle $C_i$ with center $c_i$ and radius $r_i$, and if $n_1 \neq n_2$ then all the finite zeros of the derivative of the product $f = f_1 f_2$ lie in $C_1 \cup C_2 \cup C_3$, where $C_3$ is the circle with center $c_3$ and radius $r_3$ given by

\[ c_3 = \frac{n_2 c_1 + n_3 c_2}{n_2 + n_1}, \quad r_3 = \frac{n_2 r_1 + n_1 r_2}{n_2 + n_1}. \]

**Remark (2.7).** (I) If Theorem (2.4) is specialized for the case when $S_1 = S_2 = S$, then $Z(R_h) \subseteq S_1 \cup S_0 \forall h \in \bigcap_{k=1}^{q} F(P_k)$, where $S_0$ is given by (2.3). Therefore, $Z(R_h) \subseteq H(s)$, where $H(s)$ is the convex hull of $S$.

(I) If $S$ is convex, then $Z(R_h) \subseteq S$.

(III) If $E$ is a $K$-i.p.s. and $S$ is a ball or a half-space then $Z(R_h) \subseteq S$. Furthermore, if $m_k = 1$ for all $k$, then $Z(Q_h) \subseteq S$.

(IV) If $P \in P_n(K)$ and $S \in D(K_n) = D^*(K_\omega)$ with $\omega \notin S$, then $Z(P_h) \subseteq S \forall h \in F(P)$. This is the generalization of Lucas' theorem due to Zervos [12]. If in addition $E = K = \mathbb{C}$, then (IV) reduces to Lucas' theorem in the complex plane.

### 3. SOME GENERAL EXAMPLES

In this section, we discuss some general examples to support the validity of the hypotheses and the degree of generality of our main theorems.

**Example (3.1).** Let $E$ be a vector space over $K$ of arbitrary dimension. Consider the sets $S_i = a_i + E_0$, where $E_0$ is a maximal subspace of $E$ and $a_i \in E$ ($i = 1, 2$) with $a_1 - a_2 \notin E_0$. Then $S_i \in D^*(E_0)$ by Remark (1.2), such that $\omega \notin S_i$ and $S_1 \cap S_2 = \emptyset$. Given $z \notin E_0$ (it is possible to choose one), every element $x \in E$ has the unique representation $x = y + tz$ for some $y \in E_0$ and $t \in K$. With this representation, let

\[ a_i = y_i + t_i z, \quad \text{where } y_i \in E_0, \text{ and } t_i \in K \quad (i = 1, 2), \]

so that

\[ x \in E_0 \Leftrightarrow t = 0 \quad \text{and} \quad x \in S_i \Leftrightarrow t = t_i \quad \text{for } i = 1, 2. \quad (3.1) \]
For each \( k = 1, 2, \ldots, q \), we now define
\[
P_k(x) = \begin{cases} \left( -t_1 \right)^{n_k} & \text{for } 1 \leq k \leq p < q, \\ \left( -t_2 \right)^{n_k} & \text{for } p + 1 \leq k \leq q, \end{cases}
\]
for all \( x = y + tz \), where \( y \in E_0 \) and \( t \in K \). Then (cf. (3.1))
\[
P_k(x) = 0 \iff t = t_1 \iff x \in S_1 \quad \text{for } 1 \leq k \leq p,
\]
\[
P_k(x) = 0 \iff t = t_2 \iff x \in S_2 \quad \text{for } p + 1 \leq k \leq q,
\]
and so
\[
Z(P_k) \subseteq \begin{cases} S_1 & \text{for } 1 \leq k \leq p, \\ S_2 & \text{for } p + 1 \leq k \leq q. \end{cases}
\]  \hspace{1cm} (3.2)

For elements \( h = y' + t'z \in E \ (y' \in E_0, t' \in K) \), we have
\[
P_h(x + \rho h) = \begin{cases} \left( t + \rho t' - t_1 \right)^{n_k} & \text{for } 1 \leq k \leq p < q, \\ \left( t + \rho t' - t_2 \right)^{n_k} & \text{for } 1 + p \leq k \leq q, \end{cases}
\]  \hspace{1cm} (3.3)
\[
= \sum_{j=0}^{n_k} A_{k,j}(x, h) \rho^j,
\]  \hspace{1cm} (3.4)
where the coefficients are given by
\[
A_{k,0}(x, h) = \begin{cases} \left( -t_1 \right)^{n_k} & \text{for } 1 \leq k \leq p < q, \\ \left( -t_2 \right)^{n_k} & \text{for } 1 + p \leq k \leq q, \end{cases}
\]  \hspace{1cm} (3.5)
and
\[
A_{k,j}(x, h) = \begin{cases} C(n_k, j)(t')^j \left( -t_1 \right)^{n_k-j} & \text{for } 1 \leq k \leq p < q, \\ C(n_k, j)(t')^j \left( -t_2 \right)^{n_k-j} & \text{for } 1 + p \leq k \leq q, \end{cases}
\]  \hspace{1cm} (3.6)

\( 1 \leq j \leq n_k \), belong to \( K \), are independent of \( \rho \) and satisfy
\[
A_{k,n_k}(x, h) = A_{k,n_k}(0, h) = (t')^{n_k} \neq 0 \quad \forall x \in E,
\]  \hspace{1cm} (3.7)
and for all \( 1 \leq k \leq q \). Consequently,
\[
F(P_k) = \left\{ h \in E : A_{k,n_k}(0, h) = (t')^{n_k} \neq 0 \right\},
\]  \hspace{1cm} (3.8)
\[
= \left\{ h \in E : h \notin E_0 \right\} = E - E_0 \neq \emptyset \quad \forall 1 \leq k \leq q.
\]
Now we see that $P_k \in \mathcal{P}_n(E, K)$, satisfies (3.2) and $\bigcap_{k=1}^q F(P_k) = E - E_0 \neq \emptyset$. That is, for the hyperplanes $S_1$ and $S_2$ (members of $D^*(E_0)$) there exist a.p.'s $P_k \in \mathcal{P}_n(E, K)$ satisfying all the hypotheses of Theorem (2.1). Since $S_1, S_1 \cup S_2$ and $S_1 \cup S_2 \cup S_3$ are all proper subsets of $E$, the statements of Theorem (2.1) and Theorem (2.3) are neither vacuous nor trivial.

For convenience, we fix the following notations to be used in the next example: A disk with center $c \in K$ and radius $r > 0$ is defined by

$$D = D(c; r) = \{ z \in K : |z - c| \leq r \}.$$ 

Clearly, $D$ is a nonempty $K_0$-convex proper subset of $K$ and $D \in D(K_0)$. We denote by $f : E \to K$ an arbitrary nontrivial linear functional with $f^* = \{ x \in E : f(x) = 0 \}$. Given $f$ and $D$, we define $S = f^{-1}(D)$. Since $f$ is onto, $f(S) = f(f^{-1}(D)) = D$. Obviously, $S$ is a nonempty $K_0$-convex proper subset of $E$ such that $S \in D^*(E_0)$.

**Example (3.2).** For each $i = 1, 2$, consider the disks $D_i = D(c_i, r_i)$ and the corresponding regions $S_i \in D^*(E_0)$ (via a given $f$). Next, choose arbitrary (but fixed) elements $\lambda_i \in D_i$ ($i = 1, 2$) and define

$$P_k(x) = \begin{cases} [f(x) - \lambda_1]^n_k & \text{for } 1 \leq k \leq p < q, \\ [f(x) - \lambda_2]^n_k & \text{for } p + 1 \leq k \leq q, \end{cases}$$

for all $x \in E$. Then

$$P_k(x) = 0 \iff \begin{cases} f(x) = \lambda_1 \in D_1 & \text{for } 1 \leq k \leq p, \\ f(x) = \lambda_2 \in D_2 & \text{for } p + 1 \leq k \leq q. \end{cases}$$

Since $S_i = f^{-1}(D_i)$, we have

$$Z(P_k) \subseteq \begin{cases} S_1 & \text{for } 1 \leq k \leq p, \\ S_2 & \text{for } p + 1 \leq k \leq q. \end{cases}$$

Now, for each $x, h \in E$,

$$P_k(x + \rho h) = \begin{cases} [f(x) - \lambda_1 + \rho f(h)]^n_k & \text{for } 1 \leq k \leq p, \\ [f(x) - \lambda_2 + \rho f(h)]^n_k & \text{for } p + 1 \leq k \leq q. \end{cases} \quad (3.9)$$

Let $P_k(x + \rho h)$ be represented by (3.4). To avoid duplicating a similar argument, we modify Example (3.1) to suit our present needs, by replacing the elements $t_i, t, t'$ in (3.3) respectively by $\lambda_i$, $f(x)$, and $f(h)$. Now, proceeding with similar arguments as in (3.5)–(3.8), we easily conclude that
$P_k \in P_n(E, K)$ for all $1 \leq k \leq q$ and that (3.9) holds. Further,

$$F(P_k) = \{ h \in E: A_{k,n}(0, h) = [f(h)]^{n_k} \in K - \{0\} \}$$

$$= \{ h \in E: f(h) \neq 0 \}$$

$$= E - f^\perp$$

Therefore, $\bigcap_{k=1}^q F(P_k) = E - f^\perp \neq \emptyset$. That is, for the regions $S_i \in D^*(E_w)$, not necessarily disjoint, there exist a.p.'s $P_k \in P_n(E, K)$ satisfying all the hypotheses of Theorem (2.1) and Theorem (2.3) in its full generality.

Since $f$ is onto, note that

(i) $S_1 \cup S_2 = f^{-1}(D_1) \cup f^{-1}(D_2) = f^{-1}(D_1 \cup D_2) \subseteq E \iff D_1 \cup D_2 \subseteq K$.

(ii) $S_3 = (A_2 S_2 + A_1 S_1)/(A_1 + A_2)$, where $A_i \in K_0$ such that $A_1 + A_2 \neq 0$, then $S_3 \subseteq f^{-1}(D_3)$ with

$$D_3 = (A_1 D_2 + A_2 D_1)/(A_1 + A_2) = D(c_3; r_3),$$

where

$$c_3 = \frac{A_2 c_1 + A_1 c_2}{A_2 + A_1}, \quad r_3 = \frac{|A_2| r_1 + |A_1| r_2}{|A_2 + A_1|}.$$ 

Consequently, $S_1 \cup S_2 \cup S_3 = f^{-1}(D_1 \cup D_2 \cup D_3) \subseteq E$, so that our results are neither vacuous nor trivial.

REFERENCES


