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Existence of Homoclinic Solutions for a Class of Time-Dependent Hamiltonian Systems

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1. INTRODUCTION

This paper deals with the existence of homoclinic orbits in $H^1(\mathbb{R}, \mathbb{R}^N)$ for second-order time-dependent Hamiltonian systems of the type

$$\ddot{q} - L(t)q + W_q(t, q) = 0, \qquad (HS)$$

where $q = (q_1, \dots, q_N) \in \mathbb{R}^N$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$, and $L(t) \in C(\mathbb{R}, \mathbb{R}^{N^2})$ is a positive definite symmetric matrix for all $t \in \mathbb{R}$.

The case where $L(t)$ and $W(t, q)$ are either periodic in t or independent of t, were studied by several authors (see, for instance, Ambrosetti and Bertotti [1], Coti-Zelati, Ekeland and Séré [4], Coti-Zelati and Rabinowitz [5], Rabinowitz [15–17], Séré [19], and their references).

In this kind of problem the function $L(t)$ plays an important role; for example, if $L(t) = aI_N$, where $a > 0$ and I_N denotes the identity matrix, Felmer and Silva [8] have showed the existence of homoclinic orbits as the limit of subharmonic solutions, provided that (HS) is autonomous (see also Rabinowitz and Tanaka [18]). Serra, Tarallo, and Terracini [20], using an approach introduced by [4] (see also [19]) treated the (HS) problem

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assuming that W is almost periodic in t. If L is neither a constant nor periodic, the problem is quite different from the ones just described, because of the lack of compactness of the Sobolev embedding.

The motivation for the paper comes mainly from the paper by Rabinowitz and Tanaka [18], where they have studied (HS) systems without periodicity assumption both on L and W; more precisely, they assumed that

$$\text{the smallest eigenvalue of } L(t) \to \infty \quad \text{as } |t| \to \infty, \qquad (L_\infty)$$

and using a variant of the Mountain Pass Theorem of Ambrosetti and Rabinowitz without (PS) (see, e.g., Mawhin and Willem [13, page 80]), they showed that (HS) possesses a homoclinic orbit. Omana and Willem [14], by employing a new compact embedding theorem (see Costa [3]), obtained an improvement on the latter results, in fact, they verified the (PS) condition (see also Ding [6]). More recently, Korman and Lazer [10] removed the technical coercivity condition (L_∞) and proved the existence of homoclinic orbits when L and W are even in t.

In this paper, we study (HS) systems without assuming any symmetric or coercivity conditions like (L_∞); more exactly, we assume for sake of simplicity that

$$V(t,q) = -\frac{a(t)}{2} I_N q^2 + W(t,q) \equiv -\frac{L(t)}{2} q^2 + W(t,q),$$

where $a\colon \mathbb{R} \to \mathbb{R}$ is a positive continuous function satisfying the following conditions:

$$\text{there exists } a_0 > 0 \text{ such that } a(t) \geq a_0 \quad \forall t \in \mathbb{R} \qquad (a_0)$$

and

$$\text{there is a constant } a_\infty > 0 \text{ such that } a(t) \to a_\infty \quad \text{as } |t| \to \infty, \qquad (a_\infty)$$

and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. Furthermore, W is assumed to be superquadratic at infinity and subquadratic at zero in q, that is, W verifies the following assumptions:

there exists a constant $\mu > 2$ such that

$$0 < \mu W(t,q) \leq q W_q(t,q) \quad \forall q \neq 0 \text{ and } t \in \mathbb{R}, \qquad (W_1)$$

and

$$W_q(t,q) = o(|q|) \quad \text{as } |q| \to 0, \quad \text{uniformly in } t \in \mathbb{R}. \qquad (W_2)$$

Also, we shall impose a condition of the technical nature, on behavior of W at infinity,

$$\lim_{|q|\to\infty} \frac{|W_q(t,q)|}{|q|^{s_0}} = 0 \text{ for some } s_0 + 1 > \mu, \quad \text{uniformly in } t \in \mathbb{R}. \qquad (W_3)$$

We shall study problem (HS) as a perturbation of autonomous problem $(HS)_\infty$, namely,

$$\ddot{q} - a_\infty I_N q + \overline{W}_q(q) = 0, \qquad (HS)_\infty$$

where $\overline{W}$ is a C^1 function given by

$$\lim_{|t|\to\infty} W(t,q) = \overline{W}(q) \quad \text{uniformly for } q \text{ bounded.} \qquad (W_4)$$

We are now ready to state our main result.

THEOREM 1.1. *Suppose* $(W_1),\dots,(W_4)$, (a_0), *and* (a_∞) *hold. Then there exists a homoclinic orbit* q *of* (HS) *emanating from* 0 *such that*

$$0 < \int_{-\infty}^{+\infty} |\dot{q}|^2 + L(t)q^2\,dt < \infty,$$

provided that $\overline{W}$ *satisfies the following conditions*:

$$\frac{\overline{W}_q(\tau q)}{\tau} q \text{ is nondecreasing with respect to } \tau, \quad \forall q \text{ fixed}, \qquad (W_5)$$

$\forall\epsilon > 0$, *there exist constants* $\delta > 0$ *and* $R > 0$ *such that*

$$|W(t,q) - \overline{W}(q)| \le \epsilon|q|^2, \quad |t| > R, |q| < \delta, \qquad (W_6)$$

and

$$W(t,q) \ge \overline{W}(q) \quad \text{and} \quad a(t) \le a_\infty, \qquad (W_7)$$

where the last statement at least one inequality is strict on a subset of positive measure in $\mathbb{R}$.

Remark 1.2. The result above still holds if L verifies the following conditions:

$$L(t) \in C(\mathbb{R}, \mathbb{R}^{N^2}) \text{ is a positive definite symmetric matrix for all } t \in \mathbb{R} \qquad (L_1)$$

and

$$L(t)q \cdot q \ge a(t)|q|^2, \qquad (L_2)$$

where a satisfies the conditions (a_0) and (a_∞).

2. PRELIMINARY RESULTS

In this section, we shall give some notations, definitions, and basic lemmas.

Let

$$E = \left\{ q \in H^1(\mathbb{R}, \mathbb{R}^N) \middle| \int_{-\infty}^{+\infty} \frac{1}{2}\left(|\dot{q}|^2 + L(t)q^2\right) dt < \infty \right\}$$

be with the inner product and norm given by

$$\langle q, p \rangle = \int_{-\infty}^{+\infty} (\dot{q} \cdot \dot{p} + L(t) q \cdot p)\, dt \quad \text{and} \quad \|q\|^2 = \langle q, q \rangle.$$

Note that the embeddings $E \subset H^1(\mathbb{R}, \mathbb{R}^N) \subset L^p(\mathbb{R}, \mathbb{R}^N)$ for all $p \in [2, \infty]$ are continuous.

Letting

$$I(q) = \frac{1}{2}\|q\|^2 - \int_{-\infty}^{+\infty} W(t, q)\, dt,$$

then $I \in C^1(E, \mathbb{R})$ (see, e.g., [5, page 696]), and that any critical point for I on E, is a classical solution of (HS) with $q(\pm\infty) = 0$ and $0 = \dot{q}(\pm\infty)$ (for a proof see, for instance, [17, page 5] and [15, page 339]).

In the proof of our result, we shall use the following main lemma by Lions [11] concerning the compactness of the Palais−Smale sequences, which is well known in the literature as the concentration-compactness principle.

LEMMA 2.1 (Lions [11]). *Let ρ_n be a sequence in $L^1(\mathbb{R})$ satisfying $\rho_n \geq 0$ in $\mathbb{R}$ and $\int_{-\infty}^{+\infty} \rho_n = \lambda$, where $\lambda > 0$ is a fixed constant. Then there exists a subsequence which we still denote by ρ_n, satisfying one of the three following possibilities*:

(i) (*Vanishing*):

$$\lim_{n \to \infty} \sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} \rho_n(t)\, dt = 0 \quad \textit{for all } R > 0.$$

(ii) (*Compactness*): *There exists $y_n \in \mathbb{R}$ satisfying $\forall \epsilon > 0$, $\exists R > 0$ such that*

$$\int_{y-R}^{y+R} \rho_n(t)\, dt \geq \lambda - \epsilon \quad \textit{for all } n.$$

(iii) (*Dichotomy*): *There exist* $\alpha \in (0, \lambda)$, $\rho_n^1 \geq 0$, $\rho_n^2 \geq 0$, *and* $\rho_n^1, \rho_n^2 \in L^1(\mathbb{R})$ *such that*

(a) $\| \rho_n - (\rho_n^1 + \rho_n^2)\|_{L^1} \to 0$ *as* $n \to \infty$.

(b) $\int_{-\infty}^{+\infty} \rho_n^1(t)\,dt \to \alpha$ *as* $n \to \infty$.

(c) $\int_{-\infty}^{+\infty} \rho_n^2(t)\,dt \to \lambda - \alpha$, *as* $n \to \infty$.

(d) $\operatorname{dist}(\operatorname{supp} \rho_n^1, \operatorname{supp} \rho_n^2) \to \infty$ *as* $n \to \infty$.

In the next section, in order to rule out some of the conditions above, we shall use the following lemma which is a special case of [12, Lemma I.1], due to Arioli and Szulkin [2].

LEMMA 2.2 ([2], [12]). *Let* q_n *be a bounded sequence in* $L^q(\mathbb{R})$, $1 \leq q < \infty$, *such that* q_n' *is bounded in* $L^p(\mathbb{R})$, $1 \leq p < \infty$. *If, in addition, there exists* $R > 0$ *such that*

$$\sup_{y \in \mathbb{R}} \int_{y-R}^{y+R} |q_n(t)|^q\,dt \to 0 \quad \text{as } n \to \infty,$$

then $q_n \to 0$ *in* $L^r(\mathbb{R})$, $\forall r \in (q, \infty)$.

3. PROOFS

The proof of our result is inspired partly from Jianfu and Xiping [9], but mainly from a combination of the concentration-compactness principle by Lions [11, 12] and Ambrosetti and Rabinowitz's Mountain Pass Theorem.

First, by $(W_1), (W_2)$, keeping in mind that the embedding $E \subset L^s(\mathbb{R})$ is continuous, and adapting the proof given in [18] (see also [14]), it follows that our functional I verifies the Mountain Pass Geometry, that is,

(i) there exist positive constants ρ and β such that $I(u) \geq \beta$ as $\|q\| = \rho$;

(ii) there exists $e \in E$ with $\|e\| \geq \rho$ such that $I(e) \leq 0$.

Then, applying the Mountain Pass Theorem without Palais−Smale [(PS) in short] condition, there exists a (PS) sequence $q_n \in E$ for I at level $\bar{c} > \beta$, where $\bar{c} = \inf_{h \in \Gamma} \max_{t \in [0,1]} I(h(t))$ with $\Gamma = \{h \in C([0,1], E);\ h(0) = 0 \text{ and } h(1) = e\}$.

Using (W_1), it is routine to verify that $\{q_n\}$ is bounded in E (see, e.g., [14]). Therefore there exists $q \in E$ such that $q_n \to q$ (weakly).

The following auxiliary result concerns the behavior of (PS) sequence.

LEMMA 3.1. $\|q_n\| \to l$ *with* $l > 0$.

Proof. Assume by contradiction that $\|q_n\| \to 0$. For $\epsilon > 0$ given, by using (W_2) and (W_3) we obtain

$$|W_q(t, q_n) q_n| \leq \epsilon\left(|q_n|^2 + |q_n|^{s_0+1}\right) + C_\epsilon |q_n|^r, \qquad r > 2,$$

where C_ϵ is a positive constant. Then, from Lemma 2.2 and (W_1), we conclude

$$\int_{-\infty}^{+\infty} |W(t, q_n)| \to 0 \quad \text{as } n \to \infty,$$

hence $c = 0$, which is a contradiction.

Next, we will check each one of the possible alternatives of Lemma 2.1 for $\rho_n = |\dot{q}_n|^2 + L(t) q_n^2$.

(1) **Vanishing**: From Lemma 2.2 and using the same argument as in Lemma 3.1, we can exclude this alternative.

(2) **Concentration**: Suppose that the concentration occurs. Then, there exists a sequence $y_n \in \mathbb{R}$ and $R_0 > 0$ satisfying

$$\left(\int_{-\infty}^{y_n - R_0} + \int_{y_n + R_0}^{+\infty}\right) \rho_n(t)\, dt < \epsilon. \tag{3.1}$$

Claim 1. y_n is bounded in $\mathbb{R}$.

Assuming Claim 1 has been proved, for $\epsilon > 0$ given, since E is continuously embedded in $L^s(\mathbb{R})$ $(s \geq 1)$, there exists $R_1 > 0$ such that

$$\left(\int_{-\infty}^{-R_1} + \int_{R_1}^{+\infty}\right) |q|^s < \epsilon. \tag{3.2}$$

Taking $R = \max\{R_1, R_0 + \sup|y_n|\}$, we have

$$q_n \to q \text{ in } L^s([-R, R]), \qquad s \geq 1. \tag{3.3}$$

Note that

$$\begin{aligned} \int_{-\infty}^{\infty} |q_n - q|^s &= \left(\int_{-\infty}^{-R} + \int_R^{\infty}\right) |q_n - q|^s + \int_{-R}^{R} |q_n - q|^s \\ &\leq C\left(\int_{-\infty}^{-R} + \int_R^{\infty}\right) |q_n|^s + C\left(\int_{-\infty}^{-R} + \int_R^{\infty}\right) |q|^s + \int_{-R}^{R} |q_n - q|^s, \end{aligned}$$

which, together with (3.1), (3.2), and (3.3), implies that

$$q_n \to q \text{ in } L^s(\mathbb{R}) \text{ such that } q_n \to q \quad \text{a.e. } \ t \in \mathbb{R}, \quad \text{for all } s \geq 2. \tag{3.4}$$

From (W_2) and (W_3), there exist positive constants C_1 and C_2 such that

$$\left|W_q(t,q_n)q_n\right| \le C_1|q_n|^2 + C_2|q_n|^{s_0+1}. \tag{3.5}$$

Choosing $s = 2$ and $s = s_0 + 1$ in (3.4), we can assume that there exist functions $h_2 \in L^2(\mathbb{R})$ and $h_{s_0+1} \in L^{s_0+1}(\mathbb{R})$ such that $|q_n| \le h_2$, $\forall t$ and $|q_n| \le h_{s_0+1}$, $\forall t$, so that from (3.5) we get

$$\left|W(t,q_n)q_n\right| \le H \quad \text{with } H \in L^1(\mathbb{R}),$$

which, together with the Lebesgue Dominated Convergence Theorem, implies that

$$\int_{-\infty}^{+\infty} W_q(t,q_n)q_n \to \int_{-\infty}^{+\infty} W_q(t,q)q \quad \text{as } n \to \infty. \tag{3.6}$$

Similarly, we obtain

$$\int_{-\infty}^{+\infty} W_q(t,q_n)q \to \int_{-\infty}^{+\infty} W_q(t,q)q \quad \text{as } n \to \infty. \tag{3.7}$$

Since $I'(q_n)q \to 0$, from (3.7) by passing to the limit, we have

$$0 = \|q\|^2 - \int_{-\infty}^{+\infty} W_q(t,q)q. \tag{3.8}$$

On the other hand, noting that $I'(q_n)q_n \to 0$ as $n \to \infty$ and using (3.6), (3.8), we can conclude that $\|q_n\| \to \|q\|$, that is, $q_n \to q$ in E, and from Lemma 3.1 we have $q \ne 0$. Hence, q is a weak solution of (HS).

Proof of Claim 1. Arguing by contradiction, up to a subsequence, we can assume that $|y_n| \to \infty$.

From (3.5), (W_1), and compactness conditions (ii), we have

$$\left(\int_{-\infty}^{y_n-R} + \int_{y_n+R}^{+\infty}\right)|W(t,q_n)| \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

Let $\xi \in C_0^\infty(\mathbb{R})$ be a cut-off function given by $\xi(t) = 1$ if $|t| \le 1$, and $\xi(t) = 0$ if $|t| > 2$.

Letting $\xi_n = \xi((\cdot + y_n)/R)$ and using the same argument as in (3.9), we obtain

$$\left(\int_{y_n-2R}^{y_n-R} + \int_{y_n+R}^{y_n+2R}\right)|W(t,\xi_n q_n)| \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

Now,

$$\begin{aligned}
&\left| \int_{-\infty}^{+\infty} W(t, q_n) - W(t, \xi_n q_n) \right| \\
&\quad \leq \left(\int_{y_n - 2R}^{y_n - R} + \int_{y_n + R}^{y_n + 2R} \right) |W(t, q_n) - W(t, \xi_n q_n)| \\
&\qquad + \left(\int_{-\infty}^{y_n - R} + \int_{y_n + R}^{\infty} \right) |W(t, q_n)| \\
&\quad = I_1 + I_2 .
\end{aligned}$$

From (3.9) and (3.10), we infer that $I_1 \to 0$ and $I_2 \to 0$, that is,

$$\left| \int_{-\infty}^{+\infty} (W(t, q_n) - W(t, \xi_n q_n)) \right| \to 0 \quad \text{as } n \to \infty. \tag{3.11}$$

On the other hand, from the definition of W and $\overline{W}$, we have

$$\begin{aligned}
&\left| \int_{-\infty}^{+\infty} \left(W(t, q_n) - \overline{W}(\xi_n q_n) \right) \right| \\
&\quad \leq \left| \int_{-\infty}^{+\infty} W(t, q_n) - W(t, \xi_n q_n) \right| \\
&\qquad + \left(\int_{-\infty}^{y_n - R} + \int_{y_n + R}^{+\infty} \right) |W(t, \xi_n q_n) - \overline{W}(\xi_n q_n)| \\
&\qquad + \left(\int_{y_n - R}^{y_n + R} |W(t, \xi_n q_n) - \overline{W}(\xi_n q_n)| \right. \\
&\quad \equiv J_1 + J_2 + J_3 .
\end{aligned}$$

From (3.11), we find that $J_1 \to 0$, and by (W_4) we infer that $J_3 \to 0$. Finally, since E is continuously embedded in $L^s(\mathbb{R})$, combining condition (ii) and (W_6), we conclude that $J_2 \to 0$, so that

$$\left| \int_{-\infty}^{+\infty} \left[W(t, q_n) - \overline{W}(\xi_n q_n) \right] \right| \to 0 \quad \text{as } n \to \infty. \tag{3.12}$$

Combining condition (ii), (a_∞), and definition of ξ, it is easy to check that

$$\left| \int_{y_n - 2R}^{y_n + 2R} L(t)(\xi_n q_n)^2 - a_\infty I_N(\xi_n q_n)^2 \right| \to 0 \quad \text{as } n \to \infty. \tag{3.13}$$

and

$$\left| \int_{y_n - 2R}^{y_n + 2R} \left(\dot{\overline{\xi_n q_n}} \right)^2 - \xi_n^2 |\dot{q}_n|^2 \right| \to 0 \quad \text{as } n \to \infty. \tag{3.14}$$

Now, consider the functional "at infinity" given by

$$I^\infty(q) = \frac{1}{2} \int_{-\infty}^{+\infty} \left(|\dot{q}|^2 + a_\infty I_N q^2 \right) - \int_{-\infty}^{\infty} \overline{W}(q), \qquad q \in E.$$

Using (3.12), (3.13), and (3.14), we infer that

$$I(q_n) \geq I^\infty(\xi_n q_n) - o(1) \quad \text{for all } n. \tag{3.15}$$

Similarly, we obtain

$$I'(q_n)\left(\xi_n^2 q_n\right) = I'(\xi_n q_n)\xi_n q_n + o(1),$$

$$I'(\xi_n q_n)\xi_n q_n = (I^\infty)'(\xi_n q_n)\xi_n q_n + o(1),$$

so that

$$(I^\infty)'(\xi_n q_n)\xi_n q_n \equiv \epsilon_n = o(1).$$

Define $J^\infty = \inf_{q \in M} I^\infty(q)$, where $M = \{q \in E - \{0\}\colon (I^\infty)'(q)q = 0\}$ (if $M = \emptyset$, we put $J^\infty = \infty$).

Making a scalar change $z_n(t) = \xi_n q_n(\sigma t)$, $\sigma > 0$, and arguing as in [9, page 348], we have

$$I^\infty(z_n)z_n = \sigma \int_{-\infty}^{\infty} \left| |\dot{\overline{\xi_n q_n}}| \right|^2 - \sigma^{-1}\left(\epsilon_n - \int_{-\infty}^{\infty} |\dot{\overline{\xi_n q_n}}|^2 \right)$$

$$= \sigma^{-1}\left((\sigma^2 - 1) \int_{-\infty}^{\infty} |\dot{\overline{\xi_n q_n}}|^2 + \epsilon_n \right).$$

Since there exists a positive constant C_0 such that $\int_{-\infty}^{\infty} |\dot{\overline{\xi_n q_n}}|^2 > C_0 > 0$, $\forall n$, we can choose $\sigma \equiv \sigma(n)$ such that $(\sigma^2 - 1)\int_{-\infty}^{\infty} |\dot{\overline{\xi_n q_n}}|^2 + \epsilon_n = 0$. Hence, $z_n \neq 0$ and $z_n \in M$.

Now, noting that $\sigma \equiv \sigma(n) \to 1$, as $n \to \infty$, and

$$I^\infty(z_n) = \frac{\sigma}{2} \int_{-\infty}^{\infty} |\dot{\overline{\xi_n q_n}}|^2 + \frac{\sigma^{-1}}{2} \int_{-\infty}^{\infty} a_\infty |\overline{\xi_n q_n}|^2 - \sigma^{-1} \int_{-\infty}^{\infty} \overline{W}(\xi_n q_n)$$

$$= \frac{1}{2}\sigma^{-1}(\sigma^2 - 1) \int_{-\infty}^{\infty} |\dot{\overline{\xi_n q_n}}|^2 + (\sigma^{-1} - 1) I^\infty(\xi_n q_n) + I^\infty(\xi_n q_n),$$

then from the boundedness of $I^\infty(\xi_n q_n)$ and $\int_{-\infty}^{\infty}|\dot{\widehat{\xi_n q_n}}|^2$, we obtain

$$I^\infty(\xi_n q_n) \geq I^\infty(z_n) - o(1) \geq J^\infty - o(1),$$

which together with (3.15) implies that

$$I(q_n) \geq J^\infty - o(1). \tag{3.16}$$

Therefore, $c \geq J^\infty$, which is a contradiction to the next claim.

Claim 2. $c < J^\infty$, and $J^\infty > 0$.

We postpone the proof of Claim 2 for awhile and prove that dichotomy condition (iii) can be ruled out. Assume by contradiction that this occurs. Let Q_n be the concentration function of ρ_n, given by $Q_n(t) = \sup_{y \in \mathbb{R}} \int_{y-t}^{y+t} \rho_n$.

It is well known that, for $\epsilon > 0$ given, there exist $y_n \in \mathbb{R}$, $R_0 > 0$, and $R_n \to \infty$ such that

$$\alpha - \epsilon < Q_n(R), \qquad Q_n(2R_n) < \alpha + \epsilon, \quad \forall R \geq R_0. \tag{3.17}$$

Let $\xi, \varphi \in C_0^\infty(\mathbb{R})$ be two cut-off functions given by $0 \leq \xi, \varphi \leq 1$, $\xi = 1$ if $|t| < 1$, $\xi = 0$ if $|t| \geq 2$, $\varphi = 1$ if $|t| \geq 2$, and $\varphi = 0$ if $|t| < 1$.

Letting $\xi_n = \xi((\cdot - y_n)/R_1)$ and $\varphi_n = \varphi((\cdot - y_n)/R_n)$, then

$$\begin{aligned}\left|\int_{-\infty}^{+\infty} (\xi_n \dot{q}_n)^2 \left(\dot{\widehat{\xi_n q_n}}\right)^2\right| &\leq \frac{C}{R_1^2}\int_{-\infty}^{\infty} q_n^2 + \frac{C}{R_1}\int_{-\infty}^{\infty}|q_n \dot{q}_n| \\ &\leq \frac{C}{R_1}\|q_n\|.\end{aligned}$$

so that,

$$\left|\int_{-\infty}^{+\infty} (\xi_n \dot{q}_n)^2 - \left(\dot{\widehat{\xi_n q_n}}\right)^2\right| \to 0 \quad \text{as } n \to \infty. \tag{3.18}$$

by choosing $R_1 > 0$ sufficiently large.

Similarly,

$$\left|\int_{-\infty}^{+\infty} (\xi_n \dot{q}_n)^2 - \dot{q}_n\left(\dot{\widehat{\xi_n^2 q_n}}\right)\right| \to 0 \quad \text{as } n \to \infty. \tag{3.19}$$

Now, from (3.5) and the continuous embedding $E \subset L^{s+1}$, we have

$$\left| \int_{-\infty}^{+\infty} \xi_n^2 q_n W_q(t, q_n) - \xi_n q_n W_q(t, \xi_n q_n) \right|$$
$$\leq C_1 \left(\int_{y_n - 2R_1}^{y_n + 2R_1} - \int_{y_n - R_1}^{y_n + R_1} \right) \rho_n + C_2 \left[\left(\int_{y_n - 2R_1}^{y_n + 2R_1} - \int_{y_n - R_1}^{y_n + R_1} \right) \rho_n \right]^{s+1},$$

where C_1 and C_2 are positive constants independent of R_1.

It follows from (3.17) that

$$\left| \int_{-\infty}^{+\infty} \xi_n^2 q_n W_q(t, q_n) - \xi_n q_n W_q(t, \xi_n q_n) \right| \to 0 \quad \text{as } n \to \infty. \tag{3.20}$$

Analogous inequalities to those of (3.18), (3.19), and (3.20) hold for the function φ_n.

Letting $q_n^1 = \xi_n q_n$ and $q_n^2 = \varphi_n q_n$, it follows from (3.17) and dichotomy condition (iii) that

$$\|q_n - q_n^1 - q_n^2\| \to 0 \quad \text{as } n \to \infty, \tag{3.21}$$

then proceeding as in (3.21), one gets

$$\|q_n^1\|^2 \to \alpha, \qquad \|q_n^2\|^2 \to \lambda - \alpha \quad \text{as } n \to \infty. \tag{3.22}$$

Combining (W_1), dichotomy condition (iii), and the continuous embedding $E \subset L^{s+1}$, we obtain

$$\left| W(t, q_n) - W\left(t, q_n^1\right) - W\left(t, q_n^2\right) \right| \to 0 \quad \text{as } n \to \infty. \tag{3.23}$$

It can be readily shown that

$$\left| \int_{-\infty}^{\infty} \left(|\dot{q}_n|^2 - |\dot{\widehat{q_n^1}}|^2 - |\dot{\widehat{q_n^2}}|^2 \right) \right| \to 0$$

and

$$\left| \int_{-\infty}^{\infty} \left(L(t) \left(|q_n|^2 - \left(q_n^1\right)^2 - \left(q_n^2\right)^2 \right) \right) \right| \to 0,$$

which together with (3.23) implies that

$$I(q_n) \geq I\left(q_n^1\right) + I\left(q_n^2\right) - o(1). \tag{3.24}$$

Claim 3. (a) $I(q_n^2) \geq I^\infty(q_n^2) - \theta_2(\epsilon)$, as $\{y_n\}$ is bounded,
(b) $I(q_n^1) \geq I^\infty(q_n^1) - \theta_1(\epsilon)$, as $\{y_n\}$ is unbounded,
where $\theta_i(\epsilon) \to 0$, as $\epsilon \to 0$, $i = 1, 2$.

Proof of Claim 3. First of all, by (a_∞), note that

$$\int_{-\infty}^{+\infty} |L - a_\infty I_N|\,|q_n^2|^2 = \left(\int_{-\infty}^{y_n - R_n} + \int_{y_n + R_n}^{+\infty}\right) |L - a_\infty I_N|\,|\varphi_n q_n|^2 \to 0, \tag{3.25}$$

as $\{y_n\}$ is bounded and if $\{y_n\}$ is unbounded, we obtain

$$\int_{-\infty}^{+\infty} |L - a_\infty I_N| |q_n^1|^2 = \left(\int_{-\infty}^{y_n - 2R_1} + \int_{y_n + 2R_1}^{+\infty}\right) |L - a_\infty I_N|\,|\xi_n q_n|^2 \to 0. \tag{3.26}$$

Setting $A_\delta = \{t \in \mathbb{R}\colon |q_n^2(t)| < \delta \text{ and } |t - y_n| \geq R_n\}$, we have

$$\left|\int_{-\infty}^{+\infty} \left[\overline{W}(q_n^2) - W(t, q_n^2)\right]\right| \leq \left(\int_{A_\delta} + \int_{|q_n^2| > 1/\delta} + \int_{\delta \leq |q_n^2| \leq 1/\delta}\right) |\overline{W}(q_n^2) - W(t, q_n^2)| \equiv I_1 + I_2 + I_3.$$

By (W_6) and taking $|t| > R_n - \sup|y_n|$, we infer that

$$I_1 \to 0 \quad \text{as } \{y_n\} \text{ is bounded.} \tag{3.27}$$

Now, if the sequence $\{y_n\}$ is unbounded, using (W_6), we obtain

$$\int_{A_\delta} \left|\overline{W}(q_n^1) - W(t, q_n^1)\right| \to 0 \quad \text{as } n \to \infty, \tag{3.28}$$

where $\overline{A}_\delta = \{t \in \mathbb{R}\colon |q_n^1(t)| < \delta \text{ and } |t - y_n| \leq 2R_1\}$.
From (W_4) and the boundedness of $\{y_n\}$, we conclude that

$$I_3 \to 0, \tag{3.29}$$

and if $\{y_n\}$ is unbounded, using (W_4), we get

$$\int_{\delta \leq |q_n^1| \leq 1/\delta} \left|\left(\overline{W}(q_n^1) - W(t, q_n^1)\right)\right| \to 0 \quad \text{as } n \to \infty. \tag{3.30}$$

Now, for all $\epsilon > 0$, from (W_3), one has

$$\left|W\left(t, q_n^2\right)\right| \leq \epsilon |q_n^2|^{s_0+1}, \qquad |q_n^2| > \frac{1}{\delta}, \tag{3.31}$$

for δ sufficiently small. So that, passing to the limit in (3.31), we obtain

$$|\overline{W}\left(q_n^2\right)| \leq \epsilon |q_n^2|^{s_0+1}, \qquad |q_n^2| > \frac{1}{\delta}. \tag{3.32}$$

Therefore, since E is continuously embedded in L^{s_0+1}, from (3.31) and (3.32), we conclude that

$$I_2 \to 0. \tag{3.33}$$

On the other hand, if $\{y_n\}$ is unbounded, arguing as before, we have

$$\int_{B_\delta} |\overline{W}\left(q_n^1\right) - W\left(t, q_n^1\right)| \to 0 \quad \text{as } n \to \infty, \tag{3.34}$$

where $B_\delta = \{t \in \mathbb{R}: |q_n^1(t)| > 1/\delta, \text{ and } |t - y_n| \leq 2R_1\}$. Combining (3.25), (3.27), (3.29), and (3.33), Claim 3(a) is proven, and by (3.26), (3.28), (3.30), and (3.34), Claim 3(b) follows.

This finishes the proof of Claim 3.

Next, we shall establish several estimates in order to exclude dichotomy condition (iii).

Using (W_1), we obtain

$$I\left(q_n^1\right) \geq \frac{1}{2}\|q_n^1\|^2 - \frac{1}{2}\int_{-\infty}^{\infty} q_n^1 W_q\left(t, q_n^1\right)$$
$$+ \frac{(\mu - 2)}{2\mu}\int_{-\infty}^{\infty} q_n^1 W_q\left(t, q_n^1\right). \tag{3.35}$$

Noting that $I'(q_n)(\xi_n^2 q_n) \to 0$ and applying an analogous argument to (3.12), (3.13), and (3.14), we conclude that

$$\|q_n^1\|^2 - \int_{-\infty}^{\infty} q_n^1 W_q\left(t, q_n^1\right) \to 0, \tag{3.36}$$

which, combined with (3.22) and (3.35), gives

$$I\left(q_n^1\right) \geq \left(\frac{\mu - 2}{2\mu}\right)\alpha + o(1). \tag{3.37}$$

From Claim 3(a) and arguing as in (3.16), we have

$$I\left(q_n^2\right) \geq J_\infty - o(1). \tag{3.38}$$

Similarly, if $\{y_n\}$ is unbounded, by Claim 3(b), we obtain

$$I(q_n^2) \geq \left(\frac{\mu - 2}{2\mu}\right)(\lambda - \alpha) + o(1) \tag{3.39}$$

and

$$I(q_n^1) \geq J_\infty - o(1). \tag{3.40}$$

Combining (3.24) with either (3.37) and (3.38) or (3.39) and (3.40), we have $c > J_\infty$, which is a contradiction to Claim 2 and we may exclude condition (iii).

Next we will give the proof of Claim 2.

Proof of Claim 2. First of all, we shall verify that $J^\infty > 0$. It suffices to consider the case $M \neq \emptyset$.

Note that, by (W_4) and passing to the limit in (W_1), (W_2), and (W_3), we have

$$0 < \mu \overline{W}(q) \leq q \overline{W}_q(q), \qquad \mu > 2 \tag{3.41}$$

and

$$|\overline{W}_q(q) q| \leq \epsilon |q|^2 + C_\epsilon |q|^{s_0 + 1}, \qquad C_\epsilon > 0, \tag{3.42}$$

where $0 < \epsilon < 1$ is fixed.

We claim the following.

Claim 4. There exists a positive constant C such that

$$\|q\| \geq C, \quad \forall q \in M.$$

Proof of Claim 4. Suppose by contradiction (up to subsequence) that there exists a sequence $\{q_n\}$ in M such that $\|q_n\| \to 0$.

On the other hand, from (3.42) and keeping in mind that E is continuously embedded in L^s $(s \geq 2)$, we infer that

$$(1 - \epsilon)\|q_n\|^2 \leq C \|q_n\|^{s_0 + 1},$$

where $\epsilon < 1$ is fixed and C depends only on ϵ.

Hence, since $s_0 + 1 > 2$, we reach a contradiction. This proves Claim 4.

Taking $q \in M$, by (3.41) and Claim 4, we get

$$I^\infty(q) \geq \frac{(\mu - 2)}{2\mu} \int_{-\infty}^{\infty} \overline{W}_q(q) q$$

$$= \frac{(\mu - 2)}{2\mu} \|q\|^2 \geq \frac{(\mu - 2)}{2\mu} C^2 > 0.$$

Therefore, $J^\infty > 0$.

Next, we shall prove that $c < J^\infty$.

Let $\{q_n\}$ be a minimizing sequence of I^∞ on M. From Ekeland's variational principle, there exists a sequence $\{\bar{q}_n\}$ in M such that

$$I^\infty(\bar{q}_n) \to J^\infty, \qquad (I^\infty)'|_M(\bar{q}_n) \to 0 \quad \text{and} \quad \bar{q}_n - q_n \to 0 \text{ in } E.$$

Hence, we may assume that $\{\bar{q}_n\}$ is a (PS) sequence for I^∞ at level $J^\infty \equiv \bar{c}$.

Since $J^\infty \equiv \bar{c}$, from the proof of our result, it follows that we may rule out dichotomy condition (iii). Also, using Lemma 2.2 and (3.42), we can exclude vanishing condition (i). Then, noting that I^∞ is independent of t, we obtain $\bar{q}_n \to \bar{q}$ in E (up to translation and extracting a subsequence if necessary).

Therefore, since $I^\infty \in C^1(E, \mathbb{R})$, we conclude that $I^\infty(\bar{q}) = J^\infty$ and $\bar{q} \in M$.

By (W_5) and by a similar argument to [7, page 288], we obtain $\sup_{s \geq 0} I^\infty(t\bar{q}) = I^\infty(\bar{q})$. Without loss of generality, we may assume that the Lebesgues measure of the set $\operatorname{supp} \bar{q} \cap \{t \in \mathbb{R}\colon W(t, \bar{q}) \neq \overline{W}(\bar{q})\}$ is positive.

By (W_7) and choosing $e = s\bar{q}$ (for s sufficiently large) in Mountain Pass Geometry (ii), we have

$$c \leq \sup_{s \geq 0} I(s\bar{q}) < \sup_{s \geq 0} I^\infty(s\bar{q}) = J^\infty.$$

∎

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