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On the contiguous relations of hypergeometric series

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Abstract

Contiguous relations are a fundamental concept within the theory of hypergeometric series and orthogonal polynomials. Their study goes back to Gauss who gave a list of 15 “fundamental” relations for the ${}_2F_1$ hypergeometric series. In this paper we will prove some consequences of contiguous relations of ${}_2F_1$.

Keywords: Hypergeometric function; Contiguous relations

1. Introduction

The series

$$1+\frac{ab}{c}\frac{z}{1!}+\frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!}+\frac{a(a+1)(a+2)b(b+1)(b+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+\cdots \tag{1}$$

is called the Gauss series or the ordinary hypergeometric series, where it is assumed that $c \neq 0, -1, -2, -3, \ldots$ so that no zero factors appear in the denominators of the terms of the series. This series converges absolutely for $|z| < 1$. It is usually represented by the symbol ${}_2F_1[a, b; c; z]$ and customary to use $F[a, b; c; z]$. The variable is z, and a, b and c are called the parameters of the function. If either of the quantities a or b is a negative integer $-n$, the series has only a finite number of terms and becomes in fact a polynomial ${}_2F_1[a, b; c; z]$.

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Two hypergeometric functions with the same argument z are *contiguous* if their parameters a, b and c differ by integers. For example, $F[a, b; c; z]$ and $F[a + 10, b - 7; c + 3; z]$ are contiguous. For any three contiguous ${}_2F_1$ functions there is a contiguous relation, which is a linear relation, with coefficients being rational functions in the parameters $a.b$, c and the argument z [1]. For example

$$a(z-1)F[a+1, b; c; z] + (2a - c - az + bz)F[a, b; c; z] + (c-a)F[a-1, b; c; z] = 0, \tag{2}$$

$$aF[a+1, b; c; z] - (c-1)F[a, b; c-1; z] + (c-a-1)F[a, b; c; z] = 0, \tag{3}$$

$$aF[a+1, b; c; z] - bF[a, b+1; c; z] + (b-a)F[a, b; c; z] = 0. \tag{4}$$

In these relations two hypergeometric series differ just in one parameter from the third hypergeometric series, and the difference is 1. The relations of this kind were found by Gauss, there are 15 of them. A contiguous relation between any three contiguous hypergeometric functions can be found by combining linearly a sequence of Gauss contiguous relations.

Applications of contiguous relations range from the evaluation of hypergeometric series to the derivation of summation and transformation formulas for such series, they can be used to evaluate a hypergeometric function which is contiguous to a hypergeometric series which can be satisfactorily evaluated. For example, Kummer's identity [1, Corollary 3.1.2]

$${}_2F_1[a, b; 1+a-b; -1] = \frac{\Gamma(1+a-b)\Gamma(1+a/2)}{\Gamma(1+a)\Gamma(1+(a/2)-b)} \tag{5}$$

can be generalized to

$${}_2F_1[a+n, b; a-b; -1] = P(n)\,\frac{\Gamma(a-b)\Gamma((a+1)/2)}{\Gamma(a)\Gamma((a+1)/2-b)} + Q(n)\frac{\Gamma(a-b)\Gamma(a/2)}{\Gamma(a)\Gamma((a/2)-b)}. \tag{6}$$

Here n is an integer, the factors $P(n)$ and $Q(n)$ can be expressed for $n \geqslant 0$ as

$$P(n) = \frac{1}{2^{n+1}}\,{}_3F_2\left[-\frac{n}{2}, -\frac{n+1}{2}, \frac{a}{2} - b; \frac{1}{2}, \frac{a}{2}; 1\right], \tag{7}$$

$$Q(n) = \frac{n+1}{2^{n+1}}\,{}_3F_2\left[-\frac{n-1}{2}, -\frac{n}{2}, \frac{a+1}{2} - b; \frac{3}{2}, \frac{a+1}{2}; 1\right], \tag{8}$$

and similarly for $n < 0$ [10]. In fact, formula (6) is a contiguous relation between ${}_2F_1[a+n, b; a-b; -1]$, ${}_2F_1[a-1, b; a-b; -1]$ and ${}_2F_1[a, b; 1+a-b; -1]$, where the last two terms are evaluated in terms of Γ-function using Kummer's identity (5), and the coefficients are expressed as terminating ${}_3F_2(1)$ series.

Contiguous relations are also used to make a correspondence between Lie algebras and special functions. The correspondence yields formulas of special functions [5]. For more details about hypergeometric series and their contiguous relations, see [1,3,6,10,11].

In this paper, we will prove some new formulas which are consequences of contiguous relations of the basic hypergeometric functions ${}_2F_1$. Such new formulas will be of great help in the derivations of several contiguous relations of ${}_2F_1$.

The new formulas express arbitrary shifted ${}_2F_1$ functions in terms of a "diagonally" shifted ${}_2F_1$ function. Diagonally shifted functions are proportional to the derivatives of ${}_2F_1$ functions [8]. In [8], an algorithm to obtain contiguous relations of hypergeometric functions of several variable was presented. The algorithm based on Buchberger's algorithm [2] for Gröbner basis and it expresses arbitrary shifts of ${}_2F_1$ basic hypergeometric functions in terms of a fixed ${}_2F_1$ function and its derivative.

More information about using the arbitrary shifts in contiguous relations of ${}_2F_1$ basic hypergeometric functions can be found in [4,7,9].

The paper is organized as follows. In Section 2, we use well-known forms of the Gauss function to introduce our computations to prove some simple identities relating shifted ${}_2F_1$ functions together with their derivatives. In Section 3, we combine these identities to obtain more general ones; we give their proofs and we present our main theorem to generalize our main results.

We will always denote the hypergeometric series ${}_2F_1[a, b; c; z]$ by $F[a, b; c; z]$, and we will use the well-known binomial identities

$$\binom{x}{y} = \begin{cases} \dfrac{x}{x-y}\dbinom{x-1}{y}, \\ \dfrac{x-y+1}{y}\dbinom{x}{y-1}. \end{cases} \tag{9}$$

2. Initial computations

Let us start with

$$F(z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \ldots$$

from which we can rewrite (1) in the form

$$F[a, b; c; z] = \sum_{k=0}^{\infty} \frac{\dbinom{a+k-1}{k}\dbinom{b+k-1}{k}}{\dbinom{c+k-1}{k}} z^k. \tag{10}$$

Now, replacing a, b and c by $a+\alpha$, $b+\beta$ and $c+\gamma$, respectively, we can rewrite (10) as

$$F[a+\alpha, b+\beta; c+\gamma; z] = \sum_{k=0}^{\infty} \frac{\dbinom{a+\alpha+k-1}{k}\dbinom{b+\beta+k-1}{k}}{\dbinom{c+\gamma+k-1}{k}} z^k. \tag{11}$$

Using (9), we can rewrite (11) as follows:

$$F[a+\alpha, b+\beta; c+\gamma; z] = \sum_{k=0}^{\infty} \frac{\Psi_a^{\alpha_1,\alpha-1}\Psi_b^{\beta_1,\beta-1}\dbinom{a+\alpha_1+k-1}{k}\dbinom{b+\beta_1+k-1}{k}}{\Psi_c^{\gamma_1,\gamma-1}\dbinom{c+\gamma_1+k-1}{k}} z^k, \tag{12}$$

where

$$\Psi_a^{\alpha_1,\alpha-1}=\prod_{i=\alpha_1}^{\alpha-1}\left(\frac{a+k+i}{a+i}\right) \tag{13}$$

and

$$0\leqslant\alpha_1\leqslant\alpha,\quad 0\leqslant\beta_1\leqslant\beta\quad\text{and}\quad 0\leqslant\gamma_1\leqslant\gamma.$$

Now, we are going to deduce a recurrence relation between $F[a+n,b;c;z]$ and $F[a+n-1,b;c;z]$, therefore, put $\alpha=n$, $\alpha_1=n-1$,$\beta=\beta_1=\gamma=\gamma_1=0$ in (12), we get

$$F[a+n,b;c;z]=\sum_{k=0}^{\infty}\frac{\Psi_a^{n-1,n-1}\Psi_b^{0,-1}\binom{a+n+k-2}{k}\binom{b+k-1}{k}}{\Psi_c^{0,-1}\binom{c+k-1}{k}}z^k. \tag{14}$$

From (13), we have

$$\Psi_a^{n-1,n-1}=\prod_{i=n-1}^{n-1}\left(\frac{a+k+i}{a+i}\right)=\frac{a+k+n-1}{a+n-1}$$

while

$$\Psi_b^{0,-1}=\Psi_c^{0,-1}=1$$

then (14), can be rewritten as

$$F[a+n,b;c;z]=\sum_{k=0}^{\infty}\left[1+\frac{k}{a+n-1}\right]\frac{\binom{a+n+k-2}{k}\binom{b+k-1}{k}}{\binom{c+k-1}{k}}z^k$$

that mean

$$F[a+n,b;c;z]=\sum_{k=0}^{\infty}\frac{\binom{a+n+k-2}{k}\binom{b+k-1}{k}}{\binom{c+k-1}{k}}z^k + \frac{z}{a+n-1}\sum_{k=0}^{\infty}k\,\frac{\binom{a+n+k-2}{k}\binom{b+k-1}{k}}{\binom{c+k-1}{k}}z^{k-1}$$

or

$$F[a+n,b;c;z]=F[a+n-1,b;c;z]+\frac{z}{a+n-1}F'[a+n-1,b;c;z], \tag{15}$$

where

$$F'[a,b;c;z]=\frac{\mathrm{d}}{\mathrm{d}z}F[a,b;c;z].$$

As a special case let $n=2$, we get

$$F[a+2,b;c;z]=F[a+1,b;c;z]+\frac{z}{a+1}F'[a+1,b;c;z], \tag{16}$$

but at $n=1$ and using (15), we obtain

$$F[a+1,b;c;z]=F[a,b;c;z]+\frac{z}{a}F'[a,b;c;z] \tag{17}$$

combining (16) and (17), we will have

$$F[a+2,b;c;z]=F[a,b;c;z]+\frac{z}{a}F'[a,b;c;z]+\frac{z}{a+1}F'[a+1,b;c;z]. \tag{18}$$

Moreover, differentiating (17) with respect to z, we get

$$F'[a+1,b;c;z]=\left(\frac{a+1}{a}\right)F'[a,b;c;z]+\frac{z}{a}F''[a,b;c;z]$$

from which (18), can be rewritten as

$$F[a+2,b;c;z]=F[a,b;c;z]+\frac{2z}{a}F'[a,b;c;z]+\frac{z^2}{a(a+1)}F''[a,b;c;z]$$

that means

$$F[a+2,b;c;z]=F[a,b;c;z]+\frac{2z}{a}F'[a,b;c;z]+\frac{z^2}{a(a+1)}F''[a+1,b;c;z]. \tag{19}$$

Example 1. Using the above relations one can easily show that

$$F[a+3,b;c;z]=F[a,b;c;z]+\frac{3z}{a}F'[a,b;c;z]+\frac{3z^2}{a(a+1)}F''[a,b;c;z]+\frac{z^3}{a(a+1)(a+2)}F'''[a,b;c;z].$$

Now, let us have the following theorem:

Theorem 2. *The hypergeometric series $F[a,b;c;z]$ satisfies the following recurrence relation*:

$$F[a+n,b;c;z]=\sum_{i=0}^{n}\frac{\binom{n}{i}}{\binom{a+i-1}{i}}\frac{z^i}{i!}F^{(i)}[a,b;c;z] \tag{20}$$

or

$$F[a+n,b;c;z]=(a-1)!\sum_{i=0}^{n}\frac{\binom{n}{i}}{(a+i-1)!}\,z^i F^{(i)}[a,b;c;z]. \tag{21}$$

Note that, Theorem 2 *can be also extended for b* (*but not for c*), *that is*, *also we can have the following recurrence relation*:

$$F[a,b+n;c;z]=\sum_{i=0}^{n}\frac{\binom{n}{i}}{\binom{b+i-1}{i}}\frac{z^i}{i!}\,F^{(i)}[a,b;c;z] \tag{22}$$

or

$$F[a,b+n;c;z]=(b-1)!\sum_{i=0}^{n}\frac{\binom{n}{i}}{(b+i-1)!}\,z^i\, F^{(i)}[a,b;c;z]. \tag{23}$$

Proof. The proof of the theorem can easily obtained by induction. □

Clearly, One can easily solve the last example by the use of Theorem 2, by replacing n by 3 in (20), that is

$$\begin{aligned}F[a+3,b;c;z]&=\sum_{i=0}^{3}\frac{\binom{3}{i}}{\binom{a+i-1}{i}}\frac{z^i}{i!}\,F^{(i)}[a,b;c;z]\\&=F[a,b;c;z]+\frac{3}{a}\,zF'[a,b;c;z]+\frac{3}{a(a+1)}\frac{z^2}{2!}\,F''[a,b;c;z]\\&\quad+\frac{1}{a(a+1)(a+2)}\,z^3F'''[a,b;c;z].\end{aligned}$$

Our next theorem gives an important relation between $F^{(i)}[a,b;c;z]$ and $F[a+i,b+i;c+i;z]$ for $i=0,1,2,\ldots,n$

Theorem 3.

$$F^{(i)}[a,b;c;z]=\left(\prod_{m=0}^{i-1}\frac{(a+m)(b+m)}{(c+m)}\right)F[a+i,b+i;c+i;z].$$

Proof. In order to prove the theorem, let us first prove the following identity:

$$F'[a+n-1,b;c;z]=\frac{b(a+n-1)}{c}\,F[a+n,b+1;c+1;z]. \tag{24}$$

Recall that

$$\begin{aligned}F'[a+n-1,b;c;z] &= \frac{\mathrm{d}}{\mathrm{d}z}(F[a+n-1,b;c;z])\\ &= \frac{\mathrm{d}}{\mathrm{d}z}\left[\sum_{k=0}^{\infty}\frac{\binom{a+n+k-2}{k}\binom{b+k-1}{k}}{\binom{c+k-1}{k}}z^k\right]\\ &= \sum_{k=1}^{\infty}k\,\frac{\binom{a+n+k-2}{k}\binom{b+k-1}{k}}{\binom{c+k-1}{k}}z^{k-1}.\end{aligned} \tag{25}$$

Using (9), then (25), can be rewritten as

$$F'(a+n-1,b,c,z) = \frac{(a+n-1)b}{c}\sum_{k=1}^{\infty}\frac{\binom{a+n+k-2}{k-1}\binom{b+k-1}{k-1}}{\binom{c+k-1}{k-1}}z^{k-1} \tag{26}$$

setting $l=k-1$, we get

$$F'[a+n-1,b;c;z] = \frac{(a+n-1)b}{c}\sum_{l=0}^{\infty}\frac{\binom{a+n+l-1}{l}\binom{b+l}{l}}{\binom{c+l}{l}}z^l$$

which implies that

$$F'[a+n-1,b;c;z] = \frac{(a+n-1)b}{c}F[a+n,b+1;c+1;z].$$

Now, setting $n=1$ in (24) and differentiate $(i-1)$ times with manipulation, we get

$$F^{(i)}[a,b;c;z] = \left(\prod_{m=0}^{i-1}\frac{(a+m)(b+m)}{(c+m)}\right)F[a+i,b+i;c+i;z]. \qquad \square \tag{27}$$

Corollary 4.

$$F[a+n,b;c;z] = \sum_{i=0}^{n}\binom{n}{i}z^i\left(\prod_{m=0}^{i-1}\left(\frac{b+m}{c+m}\right)\right)F[a+i,b+i;c+i;z]. \tag{28}$$

Proof. This result obtained directly by straight forward substitution using formula (21) of Theorem 2 and formula (27) of Theorem 3. □

Also, we can use (24) and (15) obtained before to have the following interesting contiguous relation:

$$F[a+n,b;c;z]=F[a+n-1,b;c;z]+\frac{b}{c}zF[a+n,b+1;c+1;z]. \tag{29}$$

Example 5. Applying Corollary 4, for $n=3$, we get

$$F[a+3,b;c;z]=\sum_{i=0}^{3}\binom{3}{i}z^i\left(\prod_{m=0}^{i-1}\left(\frac{b+m}{c+m}\right)\right)F[a+i,b+i;c+i;z]$$

simplifying, we obtain

$$\begin{aligned}F[a+3,b;c;z]=F[a,b;c;z]&+3\,\frac{b}{c}zF[a+1,b+1;c+1;z]\\&+3\,\frac{b(b+1)}{c(c+1)}\,z^2F[a+2,b+2;c+2;z]\\&+\frac{b(b+1)(b+2)}{c(c+1)(c+2)}\,z^3F[a+3,b+3;c+3;z].\end{aligned}$$

To make our next computations easier, let us have the following notations:

$$\begin{aligned}&F_{\alpha,\beta,\gamma}:=F[a+\alpha,b+\beta;c+\gamma,z],\\&F_{i,i,i}:=F[a+i,b+i;c+i,z]:=F_i,\end{aligned} \tag{30}$$

that is

$$F_{2,3,5}=F[a+2,b+3;c+5;z],$$

while

$$F[a,b;c;z]:=F_0,\ F[a+1,b+1;c+1;z]:=F_1,\ F[a+2,b+2;c+2;z]:=F_2,\ldots\ .$$

Remark 6. As a special case of (27), at $i=1$, we have

$$F'[a,b;c;z]=\frac{ab}{c}\,F[a+1,b+1;c+1;z]$$

from which and using the previous parameter shift notations, we can easily have the very well-known relation

$$F'_{\alpha,\beta,\gamma}=\frac{(a+\alpha)(b+\beta)}{(c+\gamma)}\,F_{\alpha+1,\beta+1,\gamma+1}. \tag{31}$$

Now, let us have the following theorem.

Theorem 7.

$$F_{\alpha,\beta,0}=\sum_{i=0}^{\alpha}\sum_{j=0}^{\beta}\binom{\alpha}{i}\binom{\beta}{j}\left(\prod_{m=0}^{i-1}\frac{b+\beta+m}{c+m}\right)\left(\prod_{n=0}^{j-1}\frac{a+i+n}{c+i+n}\right)F_{i+j}$$

or

$$F_{\alpha,\beta,0} = \sum_{j=0}^{\beta} \sum_{i=0}^{\alpha} \binom{\beta}{j} \binom{\alpha}{i} \left(\prod_{m=0}^{j-1} \frac{a+\alpha+m}{c+m} \right) \left(\prod_{n=0}^{i-1} \frac{b+i+n}{c+i+n} \right) F_{i+j}.$$

Proof. From (28), we have

$$F_{\alpha,0,0} = \sum_{i=0}^{\alpha} \binom{\alpha}{i} z^i \left(\prod_{m=0}^{i-1} \left(\frac{b+m}{c+m} \right) \right) F_i \tag{32}$$

also, using the symmetry of $F[a, b; c; z]$ with respect to a and b (i.e. $F[a, b; c; z] = F[b, a; c; z]$), then we will have

$$F_{0,\beta,0} = \sum_{j=0}^{\beta} \binom{\beta}{j} z^j \left(\prod_{m=0}^{j-1} \left(\frac{a+m}{c+m} \right) \right) F_j \tag{33}$$

replacing b by $b + \beta$ in (32), we will have

$$F_{\alpha,\beta,0} = \sum_{i=0}^{\alpha} \binom{\alpha}{i} z^i \left(\prod_{m=0}^{i-1} \left(\frac{b+\beta+m}{c+m} \right) \right) F[a+i, b+\beta+i; c+i; z]$$

which by using (33), and replacing a, b and c by $a + i$, $b + i$ and $c + i$, respectively, we will have

$$F_{\alpha,\beta,0} = \sum_{i=0}^{\alpha} \binom{\alpha}{i} z^i \left(\prod_{m=0}^{i-1} \left(\frac{b+\beta+m}{c+m} \right) \right) \sum_{j=0}^{\beta} \binom{\beta}{j} z^j \left(\prod_{m=0}^{j-1} \left(\frac{a+i+m}{c+i+m} \right) \right) F_{i+j}, \tag{34}$$

that is

$$F_{\alpha,\beta,0} = \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \binom{\alpha}{i} \binom{\beta}{j} z^{i+j} \left(\prod_{m=0}^{i-1} \left(\frac{b+\beta+m}{c+m} \right) \right) \left(\prod_{m=0}^{j-1} \left(\frac{a+i+m}{c+i+m} \right) \right) F_{i+j}. \tag{35}$$

Similarly, if we start with (33), by replacing a by $a + \alpha$ and using (32), we can easily get the similar formula

$$F_{\alpha,\beta,0} = \sum_{j=0}^{\beta} \binom{\beta}{j} z^j \left(\prod_{m=0}^{j-1} \left(\frac{a+\alpha+m}{c+m} \right) \right) \sum_{i=0}^{\alpha} \binom{\alpha}{i} z^i \left(\prod_{m=0}^{i-1} \left(\frac{b+j+m}{c+j+m} \right) \right) F_{i+j}, \tag{36}$$

that is

$$F_{\alpha,\beta,0} = \sum_{j=0}^{\beta} \sum_{i=0}^{\alpha} \binom{\beta}{j} \binom{\alpha}{i} z^{i+j} \left(\prod_{m=0}^{j-1} \left(\frac{a+\alpha+m}{c+m} \right) \right) \left(\prod_{m=0}^{i-1} \left(\frac{b+j+m}{c+j+m} \right) \right) F_{i+j}. \quad \square \tag{37}$$

Note that formula (28) is a special case of both formulas (35) and (37) where $\alpha = n$ and $\beta = 0$.

We can easily check that both formulas (35) and (37) are equivalent. For example, when $\alpha = 1$ and $\beta = 1$ both formulas gives

$$F_{1,1,0} := F[a+1, b+1; c; z] = F_0 + \frac{a+b+1}{c} zF_1 + \frac{(a+1)(b+1)}{c(c+1)} z^2 F_2.$$

Also, when $\alpha = 2$ and $\beta = 1$, formulas (35) and (37) gives

$$F_{2,1,0} = F_0 + \left(\frac{a+2b+2}{c}\right) zF_1 + \left(\frac{(b+1)(2a+b+4)}{c(c+1)}\right) z^2 F_2 + \left(\frac{(a+2)(b+1)(b+2)}{c(c+1)(c+2)}\right) z^3 F_3.$$

In addition, for $\alpha = 2$, $\beta = 2$ both formulas (35) and (37) gives

$$F_{2,2,0} = F_0 + 2\left(\frac{a+b+2}{c}\right) zF_1 + \left(\frac{(a+b)(a+b+9)+2ab+14}{c(c+1)}\right) z^2 F_2 + \left(\frac{2(a+2)(b+2)(a+b+4)}{c(c+1)(c+2)}\right) z^3 F_3.$$

Finally, and in addition to the previous recurrence relations, it is necessary to obtain a more general one.

Since

$$F_0 := F[a, b; c; z] = \sum_{k=0}^{\infty} \frac{\binom{a+k-1}{k}\binom{b+k-1}{k}}{\binom{c+k-1}{k}} z^k$$

then we will have

$$F_{\alpha,\beta,\gamma} = \sum_{k=0}^{\infty} \frac{\binom{a+\alpha+k-1}{k}\binom{b+\beta+k-1}{k}}{\binom{c+\gamma+k-1}{k}} z^k = 1 + \sum_{k=1}^{\infty} \frac{\binom{a+\alpha+k-1}{k}\binom{b+\beta+k-1}{k}}{\binom{c+\gamma+k-1}{k}} z^k.$$

Using (9), we will have

$$F_{\alpha,\beta,\gamma} = 1 + \frac{(a+\alpha)(b+\beta)}{(c+\gamma)(c+\gamma+1)} \sum_{k=1}^{\infty} \left(\frac{c+\gamma+k}{k}\right) \frac{\binom{a+\alpha+k-1}{k-1}\binom{b+\beta+k-1}{k-1}}{\binom{c+\gamma+k}{k-1}} z^k$$

from which by differentiating both sides with respect to z, we get

$$F'_{\alpha,\beta,\gamma}=\frac{(a+\alpha)(b+\beta)}{(c+\gamma)(c+\gamma+1)}\left[(c+\gamma)F_{\alpha+1,\beta+1,\gamma+2}\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{\binom{a+\alpha+k-1}{k-1}\binom{b+\beta+k-1}{k-1}}{\binom{c+\gamma+k}{k-1}}kz^{k-1}\right] \tag{38}$$

replacing $k-1$ by k in the second term of (38), we get

$$F'_{\alpha,\beta,\gamma}=\frac{(a+\alpha)(b+\beta)}{(c+\gamma)}\left[F_{\alpha+1,\beta+1,\gamma+2}+\frac{z}{c+\gamma+1}F'_{\alpha+1,\beta+1,\gamma+2}\right], \tag{39}$$

but from (31), we have

$$F'_{\alpha+1,\beta+1,\gamma+2}=\frac{(a+\alpha+1)(b+\beta+1)}{(c+\gamma+2)}F_{\alpha+2,\beta+2,\gamma+3}$$

using the above expression together with (39), we get

$$F_{\alpha+1,\beta+1,\gamma+1}=F_{\alpha+1,\beta+1,\gamma+2}+\frac{(a+\alpha+1)(b+\beta+1)}{(c+\gamma+1)(c+\gamma+2)}zF_{\alpha+2,\beta+2,\gamma+3} \tag{40}$$

or

$$F_{\alpha,\beta,\gamma+1}=\frac{(c+\gamma-1)(c+\gamma)}{(a+\alpha-1)(b+\beta-1)}\frac{1}{z}[F_{\alpha-1,\beta-1,\gamma-1}-F_{\alpha-1,\beta-1,\gamma}]. \tag{41}$$

Example 8. When $\alpha=2$, $\beta=3$ and $\gamma=4$, formula (41) gives,

$$F_{2,3,5}=\frac{(c+3)(c+4)}{z(a+1)(b+2)}[F_{1,2,3}-F_{1,2,4}].$$

3. Conclusion and main theorem

We start with two important remarks

(1) The hypergeometric series is symmetric with respect to the parameters a and b, i.e. $F[a,b;c;z]=F[b,a;c;z]$. Therefore, we can rewrite any contiguous relation by changing the location of the first and second parameters in $F_{\alpha,\beta,\gamma}$ and replacing $(a,\alpha;b,\beta)$ by $(b,\beta;a,\alpha)$, for example, identity (29) can be written as

$$F_{0,\beta,0}=F_{0,\beta-1,0}+\frac{a}{c}zF_{1,\beta,1}. \tag{42}$$

(2) The notation

$$F_{\alpha,\beta,\gamma} := F[a+\alpha, b+\beta; c+\gamma; z]$$

enable us to obtain additional formulas from ours that means any formula gives $F_{\alpha,0,0}$ can be modified to obtain a general one for $F_{\alpha,\beta,\gamma}$ by replacing b and c in the original formula by $b+\beta$ and $c+\gamma$ in the new one, i.e. formula (29) can be written in a general form as

$$F_{\alpha,\beta,\gamma} = F_{\alpha-1,\beta,\gamma} + \left(\frac{b+\beta}{c+\gamma}\right) zF_{\alpha,\beta+1,\gamma+1} \tag{43}$$

while, if we use our first note with (43), we will have

$$F_{\alpha,\beta,\gamma} = F_{\alpha,\beta-1,\gamma} + \left(\frac{a+\alpha}{c+\gamma}\right) zF_{\alpha+1,\beta,\gamma+1} \tag{44}$$

and so on.

Now, we are ready to establish our main theorem.

Theorem 9. *Let*

$${}_2F_1[a,b;c;z] = 1 + \frac{ab}{c.1} z + + \frac{a(a+1)b(b+1)}{c(c+1).1.2} z^2 + \dots$$

denotes the Gauss' hypergeometric function with argument z and parameters a, b and c, and $F_{\alpha,\beta,\gamma}$ is defined as in (30), *then*

$$F_{\alpha,\beta,\gamma} = F_{\alpha-1,\beta,\gamma} + \frac{b+\beta}{c+\gamma} zF_{\alpha,\beta+1,\gamma+1} \tag{45}$$

$$= F_{\alpha,\beta-1,\gamma} + \frac{a+\alpha}{c+\gamma} zF_{\alpha+1,\beta,\gamma+1} \tag{46}$$

$$= F_{\alpha,\beta,\gamma+1} + \frac{(a+\alpha)(b+\beta)}{(c+\gamma)(c+\gamma+1)} zF_{\alpha+1,\beta+1,\gamma+2} \tag{47}$$

$$= \frac{(c+\gamma-2)(c+\gamma-1)}{(a+\alpha-1)(b+\beta-1)} \frac{1}{z} [F_{\alpha-1,\beta-1,\gamma-2} - F_{\alpha-1,\beta-1,\gamma-1}] \tag{48}$$

$$= \sum_{i=0}^{\alpha} \binom{\alpha}{i} z^i \left(\prod_{m=0}^{i-1} \frac{b+\beta+m}{c+\gamma+m}\right) F_{i,\beta+i,\gamma+i} \tag{49}$$

$$= \sum_{i=0}^{\beta} \binom{\beta}{i} z^i \left(\prod_{m=0}^{i-1} \frac{a+\alpha+m}{c+\gamma+m}\right) F_{\alpha+i,i,\gamma+i}; \quad \alpha,\beta \geqslant 0. \tag{50}$$

$$= \sum_{i=0}^{\alpha} \sum_{j=0}^{\beta} \binom{\alpha}{i} \binom{\beta}{j} z^{i+j} \left(\prod_{m=0}^{i-1} \frac{b+\beta+m}{c+\gamma+m}\right) \left(\prod_{m=0}^{j-1} \frac{a+i+m}{c+\gamma+i+m}\right) \times F_{i+j,i+j,i+j+\gamma}; \quad \alpha,\beta \geqslant 0. \tag{51}$$

Proof. The proof is straightforward from Theorems 2 and 3 and Corollary 4 coupled with the above two notes. □

Example 10. To find $F_{1,1,1}$ using the formulas of Theorem 9 above, then: formulas (45), (49) gives

$$F_{1,1,1} = F_{0,1,1} + \left(\frac{b+1}{c+1}\right) zF_{1,2,2},$$

formulas (46), (50) gives

$$F_{1,1,1} = F_{1,0,1} + \left(\frac{a+1}{c+1}\right) zF_{2,1,2},$$

formula (47) gives

$$F_{1,1,1} = F_{1,1,2} + \frac{(a+1)(b+1)}{(c+1)(c+2)} zF_{2,2,3},$$

formula (48) gives

$$F_{1,1,1} = \frac{c(c-1)}{ab}\frac{1}{z}[F_{0,0,-1} - F_{0,0,0}],$$

while formula (51) gives

$$F_{1,1,1} = F_{0,0,0} + \left(\frac{a+b+1}{c+1}\right) zF_{1,1,2} + \frac{(a+1)(b+1)}{(c+1)(c+2)} z^2 F_{2,2,3}.$$

As shown in Example (10), formulas (45), (46) gives the same representations for $F_{1,1,1}$ as formulas (49), (50). The situation will be different in case of $\alpha > 1$ for formulas (45), (49) and in case of $\beta > 1$ for the formulas (46), (50).

Example 11. Formula (45) gives

$$F_{2,0,0} = F_{1,0,0} + \frac{b}{c} zF_{2,1,1}$$

while formula (49) gives

$$F_{2,0,0} = F_{0,0,0} + \frac{2b}{c} zF_{1,1,1} + \frac{b(b+1)}{c(c+1)} z^2 F_{2,2,2}.$$

If we reused formula (45) to find $F_{2,1,1}$ and $F_{1,0,0}$ we can easily see that the above two identities are identical.

It is very important to notice that our formulas can generate many other contiguous relations, for example replacing α by $\alpha+1$ in (45), and α, β and γ by $\alpha+1$, $\beta+1$ and $\gamma+1$, respectively, in (48), we get

$$F_{\alpha+1,\beta,\gamma} = F_{\alpha,\beta,\gamma} + \left(\frac{b+\beta}{c+\gamma}\right) zF_{\alpha+1,\beta+1,\gamma+1} \tag{52}$$

and

$$F_{\alpha+1,\beta+1,\gamma+1} = \frac{(c+\gamma)(c+\gamma-1)}{(a+\alpha)(b+\beta)}\,\frac{1}{z}[F_{\alpha,\beta,\gamma-1} - F_{\alpha,\beta,\gamma}] \tag{53}$$

eliminating $F_{\alpha+1,\beta+1,\gamma+1}$ from (52) and (53), we get

$$\begin{aligned} F_{\alpha+1,\beta,\gamma} &= F_{\alpha,\beta,\gamma} + \frac{c+\gamma-1}{a+\alpha}[F_{\alpha,\beta,\gamma-1} - F_{\alpha,\beta,\gamma}] \\ &= \left[1 - \frac{c+\gamma-1}{a+\alpha}\right] F_{\alpha,\beta,\gamma} + \frac{c+\gamma-1}{a+\alpha}\, F_{\alpha,\beta,\gamma-1}, \end{aligned}$$

that is

$$(a+\alpha)F_{\alpha+1,\beta,\gamma} = [(a+\alpha) - (c+\gamma) + 1]F_{\alpha,\beta,\gamma} + (c+\gamma-1)F_{\alpha,\beta,\gamma-1}$$

or

$$(a+\alpha)F_{\alpha+1,\beta,\gamma} - (c+\gamma-1)F_{\alpha,\beta,\gamma-1} + [(c+\gamma) - (a+\alpha) - 1]F_{\alpha,\beta,\gamma} = 0, \tag{54}$$

which by replacing α, β, γ by zeros is exactly the same as (3).

Moreover, replacing α by $\alpha+1$ in (45) and β by $\beta+1$ in (46), we get

$$F_{\alpha+1,\beta,\gamma} = F_{\alpha,\beta,\gamma} + \left(\frac{b+\beta}{c+\gamma}\right) zF_{\alpha+1,\beta+1,\gamma+1} \tag{55}$$

and

$$F_{\alpha,\beta+1,\gamma} = F_{\alpha,\beta,\gamma} + \left(\frac{a+\alpha}{c+\gamma}\right) zF_{\alpha+1,\beta+1,\gamma+1} \tag{56}$$

again, if we eliminate $F_{\alpha+1,\beta+1,\gamma+1}$ from (55) and (56), we will have

$$[(a+\alpha) - (b+\beta)]F_{\alpha,\beta,\gamma} = (a+\alpha)F_{\alpha+1,\beta,\gamma} - (b+\beta)F_{\alpha,\beta+1,\gamma},$$

that is

$$(a+\alpha)F_{\alpha+1,\beta,\gamma} - (b+\beta)F_{\alpha,\beta+1,\gamma} + [(b+\beta) - (a+\alpha)]F_{\alpha,\beta,\gamma} = 0, \tag{57}$$

which again, by replacing α, β, γ by zeros is exactly the same as (4).

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