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$\mathfrak{F}$-Sets and Permutation Groups

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Let $\mathfrak{F}$ be an isomorphism class of groups and G a finite group. Following Fischer, an $\mathfrak{F}$-*set* of G is a collection Ω of subgroups; such that Ω is normalized by and generates G, and such that any pair of distinct members of Ω generates a subgroup in $\mathfrak{F}$.

If G is a doubly transitive permutation group on a set Ω', $\alpha \in \Omega'$, and Q is a normal subgroup of G_α, then $\Omega = Q^G$ is an $\mathfrak{F}$-set of $\langle Q^G \rangle$ where $\mathfrak{F} = \{\langle Q, Q^g \rangle\}$ for any $g \in G - H$. This fact can be used to investigate the following conjecture.

CONJECTURE. *Let G^Ω be a doubly transitive permutation group, $\alpha \in \Omega$, and $1 \neq Q \trianglelefteq G_\alpha$, with Q semiregular on $\Omega - \alpha$. Then one of the following holds*:

(1) *G^Ω has a regular normal subgroup,*

(2) *$L = \langle Q^G \rangle$ is isomorphic to $L_2(q)$, $Sz(q)$, $U_3(q)$, or $R(q)$, and acts in its natural doubly transitive representation on Ω.*

(3) *$L \cong L_2(8)$ and $G = R(3)$ has degree* 28.

Hering has established this conjecture if Q has even order [13]. Further O'Nan has recently shown that if G^Ω is a doubly transitive permutation group then either (i) G^Ω is a known group, (ii) the socal of G_α is simple, or (iii) G^Ω satisfies the hypothesis of the conjecture.

It is the purpose of this paper to prove the following.

THEOREM 1. *Let Ω be an $\mathfrak{F}$-set of G with G^Ω transitive, $Q \in \Omega$, and $\mathfrak{F} = \{L\}$ with L quasisimple. Then*

(1) *If $|Q| = q = 3^n > 3$ and $L/Z(L) \cong L_2(q)$, with $Z(L)$ a $3'$-group, then $G/Z(G) \cong L_2(q)$ or $U_3(q)$.*

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(2) *If Q is a 3-group and $L \simeq L_2(8)$ then $G = L$.*

(3) *If Q is strongly closed in a Sylow 3-subgroup of L and $L/Z(L) \simeq R(q)$, then $G = L$.*

$R(q)$ denotes a group of Ree type and order $q^3(q^3 + 1)(q - 1)$.

Theorem 1 together with work of the author and M. Hall yield the following corollaries.

THEOREM 2. *Let G^Ω be a minimal counterexample to the conjecture. Then*

(1) $G = \langle Q, Q^g \rangle$, *for each* $g \notin G_\alpha$.

(2) $C_G(Q)$ *is semiregular on* $\Omega - \alpha$.

Actually something stronger than Theorem 2 is proved. See Section 4.

THEOREM 3. *Let G^Ω be a doubly transitive permutation group, $\alpha \in \Omega$, and assume $1 \neq Q$ is a normal cyclic subgroup of G_α. Then one of the following hold:*

(1) *G^Ω has a regular normal subgroup.*

(2) *$L = \langle Q^G \rangle$ is isomorphic to $L_2(q)$ or $U_3(q)$ in its natural 2-transitive representation.*

(3) *$L \simeq L_2(8)$ and $G = R(3)$ has degree 28.*

Theorem 1 also has application in the study of rank 3 permutation groups in which the stabilizer of a point has a nonfaithful orbit.

1. NOTATION

Let G be a permutation group on a set Ω, $X \subseteq G$, and $\Delta \subseteq \Omega$. Let $F(X)$ be the set of fixed points of X on Ω. Let $G(\Delta)$ and G_Δ be the global and pointwise stabilizer of Δ in G, respectively. Let $G^\Delta = G(\Delta)/G_\Delta$ with induced permutation representation.

The group theoretic notation is standard and taken from [12].

$O_\infty(G)$ denotes the largest normal solvable subgroup of G.

Given positive integral n and prime p, n_p is the largest power of p dividing n.

2. PRELIMINARY RESULTS

THEOREM 2.1. *Let p be an odd prime, Q an elementary abelian p-subgroup of G, $\Omega = Q^G$, and $\mathfrak{F} = \{L\}$. Assume Ω is an $\mathfrak{F}$ set of G. Then*

(1) *If* $|Q| = q$ *and* L *is quasisimple with* $L/Z(L) \cong L_2(q)$ *and* $Z(L)$ *is a* $3'$-*group, then* $G/Z(G) \cong L_2(q)$, $U_3(q)$, *or is Frobenius of order* $3 \cdot 4^n$.

(2) *If* $L/Z(L)$ *is Frobenius with compliment* Q *and kernel* $L'/Z(L)$ *of odd order, then* $G/Z(G)$ *has the same form.*

(3) *If* $L/Z(L)$ *is Frobenius with compliment* Q *and elementary* 2-*group kernel* $L'/Z(L)$ *and either* $Z(L) = 1$ *or* $\Omega_1(L') \leqslant Z(L)$ *with* $|L'| > 8$, *then* G *has the same form.*

(4) *If* Q *is strongly closed in a Sylow* 3-*group of the quasisimple group* L *and* $L/Z(L) \cong R(q)$, *then* $G = L$.

(5) *If* $|Q| = 3$ *and* $L \cong L_2(8)$ *then* $G = L$.

Proof. This is a restatement of Theorem 1 and results in [4, 6, 7].

THEOREM 2.2 O'Nan, [17]. *Let* G^{Ω} *be a doubly transitive permutation group,* $\alpha, \beta \in \Omega$, *and* X *a normal subgroup of* G_α. *Then* $G^{F(X_\beta)}$ *is doubly transitive.*

LEMMA 2.3. *Let* Q *be a subgroup of prime order* p *in* G, R *a* 2-*subgroup of* G *and* $Z = C_R(Q)$. *Assume* $RQ \trianglelefteq G$, $R = [Q, R]$, $Z \trianglelefteq G$, *and* G *is transitive on* $(R/Z)^{\#}$. *Then either* $\Omega_1(R) \leqslant Z$ *or* R *is elementary abelian.*

Proof. Let G be a minimal counter example, and $X = \Phi(Z)$. $R = [Q, R]$ and $Z = C_R(Q) \trianglelefteq G$, so $Z \leqslant Z(R)$.

Suppose $X \neq 1$, and set $\bar{G} = G/X$. Then minimality of G implies $\Omega_1(\bar{R}) \leqslant Z(\bar{R})$. Therefore $\Omega_1(R) \leqslant Z$, contradicting the choice of G as a counter example.

So $X = 1$ and Z has exponent 2. Now if a is an involution in $R - Z$ then every element in Za is an involution. Then as G is transitive on $(R/Z)^{\#}$, R has exponent 2 and therefore is elementary abelian.

LEMMA 2.4. *Let* L *be a quasisimple group with* $L/Z(L) \cong R(q)$. *Then the center of* L *and the outer automorphism group of* L *are of odd order, and* $Z(L)$ *is a* $3'$-*group.*

Proof. Suppose $Z = Z(L)$ is of even order. We may assume Z has order 2. Let $S \in \mathrm{Syl}_2(L)$. Then S/Z is elementary of order 8, so $S - Z$ contains an involution. As G has one class of involutions each coset of Z in S consists of involutions, so S is abelian. Now Gaschuetz's theorem yields a contradiction.

Next assume L is simple and x is a 2-element inducing an outer automorphism on L. Let S be an x-invariant Sylow 2-group of L. We may assume

$x^2 \in S$. Let a be an involution in $C_S(x)$. Then $C_L(a) = \langle a \rangle \times K$ with $K \simeq L_2(q)$. Then x centralizes an involution $b \in S \cap K$.

Let $T = \langle x, S \rangle$. If x induces an outer automorphism on K then $\langle b \rangle = T'$. But as a and b are in the center of T and conjugate in L, a is conjugate to b in the normalizer of T, a contradiction. So x induces an inner automorphism on K and T is abelian. If $x^2 = a$ then a is not conjugate to b in $N(T)$. So T is elementary and we find an isolated involution in $T - S$, contradicting the Z^*-theorem.

In [2] Alperin and Gorenstein prove the Ree groups have a trivial multiplier. The same proof shows if $L/Z(L)$ is of Ree type, then $Z(L)$ is a $3'$-group.

LEMMA 2.5. *Let L be quasisimple with $L/Z(L) \simeq L_2(q)$ and $Z(L)$ a $3'$-group. Then $L \simeq L_2(q)$ or $SL_2(q)$.*

Proof. See [11].

LEMMA 2.6. *Let $G \simeq R(q)$, $P \in \mathrm{Syl}_p(G)$, a an involution normalizing P and $Q \neq 1$ a strongly closed subgroup of P in G. Then*

(1) *If a fixes P^g then a induces the same automorphism on P and P^g.*

(2) *$|C_P(a)| = q = |Z(P)|$. a inverts $Z(P)$.*

(3) *Q equals P, $C_P(a)\, Z(P)$, or $Z(P)$.*

Proof. All these facts can be easily derived from [19].

LEMMA 2.7. *Let $L = L_2(q)$, $q = 3^n > 3$, $G = \mathrm{Aut}(L)$, $Q \in \mathrm{Syl}_3(L)$, $H = N_G(Q)$, t an involution in $L - H$, $D = H \cap H^t$, $\Omega = Q^G$ and $E = C_D(t)$. Then*

(1) *If $e \in E$ is of order m then $\langle C_Q(e)^{C(e)} \rangle \simeq L_2(3^{n/m})$, and is 2-transitive on $F(e)$.*

(2) *For each prime divisor p of order $(q-1)/2$, D contains a unique subgroup X of order p with $C_Q(X) = 1$.*

(3) *If $n \equiv 0 \bmod 4$ and X is a cyclic 2-subgroup of D semiregular on $Q^{\#}$ then X is normalized by an involution $a \in D$ with $C_Q(a) \neq 1$.*

(4) *$[D, t]$ is contained in a cyclic subgroup regular on $\Omega - \{Q, Q^t\}$, inverted by t and containing $L \cap D$ of index 2.*

(5) *Q is regular on $\Omega - Q$.*

Proof. (1), (4), and (5) are well known. Let p be a prime divisor of $(q-1)/2$. As Q is self centralizing and there is a unique automorphism inverting Q, in proving (2) we may take p odd. Let $P \in \mathrm{Syl}_p(D)$. If P is cyclic, (2) follows from (4), so we may assume P is not cyclic. Then $P = (P \cap L)(P \cap E)$ is metacyclic. By 5.4.10 in [12], P contains a normal noncyclic subgroup Y of order p^2. As P is metacyclic, $Y = \Omega_1(P)$. By (1),

$q = r^p$ and $|C_{L\cap D}(x)|$ divides $r - 1$, where $\langle x \rangle = Y \cap E$. So $r \equiv 1 \bmod p$. Then $(q-1)/(r-1) \equiv 0 \bmod p$, so x does not centralize P. Thus all cyclic subgroups in $Y - (Y \cap L)$ are conjugate in P. This proves (2).

Let $n \equiv 0 \bmod 4$ and $X = \langle x \rangle$ a cyclic 2-subgroup of D semiregular on $Q^{\#}$. Let $X \leqslant S \in \mathrm{Syl}_2(D)$ and a an involution in S with $C_Q(a) \neq 1$. Then X contains the involution u inverting Q, and $\langle u, a \rangle \trianglelefteq S$. So $[x, a] = u$ or 1 and then $a \in N(X)$.

Lemma 2.8. *Let G be a simple group, $Q \leqslant G$, and assume the pair (G, Q) satisfies the hypothesis of Theorem 1, part (1). Then*

(1) *$H = N_G(Q)$ does not contain a proper 2-generated core of G.*

(2) *If G has semidihedral Sylow 2-subgroups and $|\Omega|$ is even then $G \cong U_3(q)$.*

Proof. Let $L = \langle Q, Q^g \rangle$. Then $L \cong L_2(q)$ or $SL_2(q)$, $q = 3^n > 2$. Assume H contains a proper 2-generated core of G. Then by [5], $O(H) = 1$, a contradiction.

Next assume G has semidihedral Sylow 2-subgroups. By [1] $G = M_{11}$, $L_3(r)$ or $U_3(r)$, r odd. Notice $U_3(r) \leqslant L_3(r^2)$. Now $L_3(r)$ contains a subgroup isomorphic to L only if r is a power of 3. (e.g., [10]). Let $Q \leqslant P \in \mathrm{Syl}_p(G)$. As Q is a strongly closed abelian subgroup of P and $Z(P)^{\#}$ is fused in G, we get $Z(P) \leqslant Q$. If $G = U_3(r)$ then $Z(P)$ is the unique strongly closed abelian subgroup of P, so $Q = Z(P)$ and $r = q$. If $G = L_3(r)$ then $|Z(P)| = r$, and choosing K to be the stabilizer of a point in the action of G on the projective plane, with $Z(P) \leqslant O_3(K)$, we get $\langle Z(P)^K \rangle = O_3(K) \leqslant Q$. Then $Q = O_3(K)$ has order r^2, and G contains no subgroup isomorphic to L. Finally if $G = M_{11}$ then $Q = P$ and $|P^G|$ is odd.

Lemma 2.9. *Let q be a prime power, $q \equiv -1 \bmod 4$ with $(q^2+1)/2$ a power of a prime. Then q is prime.*

Proof. Let $r^e = (q^2+1)/2$, $q = p^{ab}$. As $q \equiv -1 \bmod 4$, a and b are odd and $2r^e = (p^{2a}+1)\,A(a, b)$, with $(p^{2a}+1, A(a, b))$ dividing b. Thus if $ab \neq 1$ we must have $2r = p^2 + 1$ and $A(1, r) = r$, which is impossible.

3. Semiregular Groups

In this section we operate under the following hypothesis.

Hypothesis 3.1. *$Q \neq 1$ is a subgroup of odd order of the group G, $\Omega = Q^G$, and $H = N_G(Q)$. Represent G by conjugation on Ω and assume $H \neq G$ and Q acts semiregularly on $\Omega - Q$.*

LEMMA 3.2. *Let $K \leqslant G$, p prime, and $P \in \mathrm{Syl}_p(Q)$. Then*

(1) *P is strongly closed in S with respect to G for any $P \leqslant S \in \mathrm{Syl}_p(G)$.*

(2) *K acts transitively on the set $\Delta = \{Q^g : |K \cap Q^g|_p \neq 1\}$.*

(3) *If $1 \neq K \cap Q$ and $K \not\leqslant H$ then the pair $(K, K \cap Q)$ has Hypothesis* 3.1.

(4) *If $K \trianglelefteq G$ either $G = HK$ or $K \cap Q = 1$ and the pair $(G/K, QK/K)$ has Hypothesis* 3.1.

Proof. (1) P fixes the unique point $Q \in \Omega$. Therefore $S \leqslant H$. Now if $x \in P^{\#}$ and $x^g \in S$, then x^g fixes Q and Q^g, so as Q is semiregular on $\Omega - Q$, $Q = Q^g$. Then $x^g \in S \cap Q^g = S \cap Q = P$.

(2) Assume $Q \in \Delta$ and let $P \in \mathrm{Syl}_p(K \cap Q)$. P fixes the unique point $Q \in \Omega$, so letting $P \leqslant S \in \mathrm{Syl}_p(K)$, $S \leqslant H$. Similarly if $Q^g \in \Delta$, then $K \cap H^g$ contains $S_1 \in \mathrm{Syl}_p(K)$. Now $S_1 = S^k$, for some $k \in K$, so $Q^g = F(S_1) = F(S^k) = F(S)^k = Q^k$.

(3) Trivial.

(4) Assume $G \neq HK$. It suffices to show that if x is a p-element in $Q^{\#}$ and $Q^{gx} \leqslant Q^g K$, then $Q^g K = QK$. But if $Q^{gx} \leqslant Q^g K$ then x normalizes $Q^g K$. Let $P \in \mathrm{Syl}_p(Q^g)$. We may assume $\langle x, P\rangle$ is a p-group. Then by (1), $x \in P \leqslant Q^g$, so $Q^g = Q$ and $QK = Q^g K$.

LEMMA 3.3. *Let p be a prime divisor of the order of Q, K a normal subgroup of G contained in H, and assume $G = \langle \Omega \rangle$. Then*

(1) *$K \leqslant Z(G)$.*

(2) *$O_p(G) \leqslant Z(G)$.*

(3) *If $m(Q) > 1$ then $O_\infty(G) = Z(G)$.*

Proof. By 3.2.4, $[K, Q] \leqslant K \cap Q = 1$, so as $G = \langle \Omega \rangle$, $K \leqslant Z(G)$. Let $P \in \mathrm{Syl}_p(Q)$. Then $PO_p(G)$ is a p-group, so by 3.2.1, $O_p(G) \leqslant N(P) \leqslant H$. Similarly if $m(P) > 1$ and $X = O_{p'}(G)$, then $X = \langle C_X(y) : y \in P^{\#}\rangle \leqslant H$. Now (1) implies (2) and (3).

LEMMA 3.4. *Assume $G = \langle \Omega \rangle$, some Sylow q-subgroup of Q is not cyclic, and let $K \trianglelefteq G$ with $K \cap Q = 1$. Then $K \leqslant Z(G)$.*

Proof. We may assume Q is an elementary abelian q-group and K is a minimal normal subgroup of G. Then K is the direct product of isomorphic simple groups and by 3.3 we may assume K is not solvable. As $K \not\leqslant Z(G)$ and $G = \langle \Omega \rangle$, $[K, Q] \neq 1$, so we may assume $G = KQ = \langle Q, Q^g \rangle$ for any $g \in G - H$. As Q is abelian and $K \cap Q = 1$, $H = C_G(Q)$. Also if $g \in G - H$

then $H \cap H^g \leqslant C(\langle Q, Q^g \rangle) = Z(G) = 1$. Finally by 3.2.1, H is self-normalizing. It follows that G is a Frobenius group with compliment Q. But this is impossible as Q is not cyclic.

LEMMA 3.5. *Assume $G = \langle \Omega \rangle$ and some Sylow group of Q is not cyclic. Then $G = QG'$, G' is quasisimple, and $Q \cap G' \neq 1$.*

Proof. We may assume $Z(G) = 1$. Let K be a minimal normal subgroup of G. By 3.2.4 and 3.4, $Q \cap K \neq 1$ and then $G = HK$. As $G = \langle \Omega \rangle$, $G = KQ$. Let P be a noncyclic subgroup of order p^2 in Q. We may assume $G = KP$.

Let L be a component of K. Then $K = \langle L^P \rangle$. We may assume $K \neq L$, so $P \not\leqslant N(L)$. If $N_P(L) = 1$ then for each prime r and each $R \in \mathrm{Syl}_r(L)$, $R_1 = \langle R^P \rangle \in \mathrm{Syl}_r(K)$. But as P is not cyclic, $R_1 = \langle C_{R_1}(x) : x \in P^\# \rangle \leqslant H$, so $G = PK \leqslant H$, a contradiction.

Let $x \in P - N(L)$. It follows that $K = LL^x \cdots L^{x^{p-1}}$, and $C_K(x) = J = \{ll^x \cdots l^{x^{p-1}} : l \in L\}$. Then $J \leqslant H$. By 3.4 there exists a prime r such that $1 \neq R \in \mathrm{Syl}_r(Q \cap K)$. Let $R \leqslant R_1 \in \mathrm{Syl}_r(K)$. Then $R_1 \leqslant H$ by 3.2.1, so $K = \langle R_1, J \rangle \leqslant H$, a contradiction.

4. THEOREMS 2, 3, AND 4

We now prove something slightly stronger than Theorem 2.

THEOREM 4. *Let $\mathscr{C}$ be a class of groups satisfying the hypothesis of the conjecture and possessing the following closure property:*

(*) *If $\Delta \subseteq \Omega$ with G^Δ doubly transitive and $Q \leqslant G(\Delta)$, then $G^\Delta \in \mathscr{C}$. Let G be of minimal order subject to belonging to $\mathscr{C}$ and not satisfying the conjecture. Then*

(1) *$G = \langle Q, Q^g \rangle$, for each $g \notin G_\alpha$.*

(2) *$C_G(Q)$ is semiregular on $\Omega - \alpha$.*

Proof. Fix $g \notin G_\alpha = H$ and set $L = \langle Q, Q^g \rangle$. We may take $\Omega = Q^G$, $\alpha = Q$. [13] implies Q is of odd order. We may assume Q is a minimal normal subgroup of H, so Q is an elementary abelian p-group for some odd prime p.

Now double transitivity of G^Ω implies Ω is an $\mathfrak{F}$-set of $\langle \Omega \rangle$ where $\mathfrak{F} = \{L\}$. Let $\Delta = \Omega \cap L$. If (β, γ) is a pair of distinct points in Δ, then there exists $x \in G$ with $(\beta^x, \gamma^x) = (Q, Q^g)$. Then $L^x = \langle \beta, \gamma \rangle^x = \langle \beta^x, \gamma^x \rangle = L$. So $N_G(L) = G(\Delta)$ is doubly transitive on Δ. Then by hypothesis $G^\Delta \in \mathscr{C}$.

Assume $L \neq G$. Then minimality of G implies G^Δ satisfies the conjecture. So $L^\Delta \cong L_2(q)$, $R(q)$, $L_2(8)$, or L^Δ is a Frobenius group whose kernel R^Δ is an elementary abelian r-group for some prime r. Now $L_\Delta \leqslant H$, so by 3.3, $L_\Delta = Z(L)$, and L is the central extension of L^Δ by $Z(L)$.

Suppose $L^\Delta \simeq L_2(q)$ and a Sylow 3-group Z of $Z(L)$ is nontrivial. By [11] $q = 9$. Then $QZ \in \mathrm{Syl}_3(L)$ and by 3.2.4, $Q \cap Z = 1$. Now Gaschuetz's theorem implies a contradiction. If R^Δ is a regular normal subgroup for L^Δ then by 2.3, either $RQ \simeq SL_2$ or RQ satisfies the hyothesis of 2.1.2 or 2.1.3. But now 2.1 implies G satisfies the conjecture.

So $G = L$. Next let $X = C_G(Q)$, and assume X is not semiregular on $\Omega - Q$. Let $D = H \cap H^g$ and $Y = X \cap D$. Then $Y \neq 1$ and by 2.2, $G^{F(Y)}$ is 2-transitive. But then $L \leqslant G(F(Y)) < G$, a contradiction. This establishes Theorem 4.

Next the proof of Theorem 3. Let G be a minimal counter example. We may assume Q has prime order. As the class $\mathscr{C}$ of doubly transitive groups with Q cyclic has the closure property (*) of Theorem 4, that theorem implies $C_G(Q)$ is semiregular on $\Omega - Q$. Recall $D = H \cap H^g$. Then $C_D(Q) = 1$, so D is isomorphic to a subgroup of the automorphism group of Q. As Q is of prime order, D is cyclic. But now a theorem of Kantor, O'Nan, and Seitz [15], yields a contradiction.

5. Parts (2) and (3) of Theorem 1

In this section assume the hypothesis of part (2) or part (3) of Theorem 1. Let $H = N_G(Q)$ and $I = C_G(Q)$. We may take $Q \in \mathrm{Syl}_3(L)$.

Suppose a is an involution in I. If a fixes but does not centralize $Q^g \in \Omega$ then a acts on $L = \langle Q, Q^g\rangle$. By 2.4 a induces an inner automorphism on L, so by 2.6, a induces the same automorphism on Q and Q^g, a contradiction. So $F(a) = C_\Omega(a)$. Let (P, P^a) be a cycle of a on Ω and set $L = \langle P, P^a\rangle$. Then again a induces an inner automorphism on L, so by 2.6, a fixes but does not centralize some member of $L \cap \Omega$, a contradiction. So I has odd order.

Next let a be an involution in G, (Q, Q^a) a cycle in a, and $L = \langle Q, Q^a\rangle$. By 2.4 a induces an inner automorphism on L, so as I has odd order, $a \in L$. Suppose $x^2 = a$. a fixes two points of $L \cap \Omega$, so there exist points P and R of Ω, which we may choose in L, either fixed or permuted in a cycle of length 2. So x normalizes L, and by 2.4 induces an inner automorphism on L. But a Sylow 2-subgroup of L is of exponent 2.

So a Sylow 2-subgroup of G is of exponent 2.

Now the pair (G, Q) has hypothesis 3.1, so by 3.5, either G' is quasisimple or $L \simeq L_2(8)$. In the latter case one easily shows G to be quasisimple. (e.g., 3.1.5 in [4]). As G' has abelian Sylow 2-subgroups and contains L, $G' \simeq L_2(2^n)$ or $R(q_0)$ [18].

Suppose $G' \cong R(q_0)$ and let $Q \leqslant P \in \mathrm{Syl}_3(G)$. As Q is strongly closed in P, 2.6 implies that $Z(P) \leqslant Z(Q)$ is of order q_0 and a inverts $Z(P)$. As $R(3)' = L_2(8)$, $q_0 > 3$, and then $Z(P)$ is not cyclic. So $Z(Q)$ is not cyclic and then $L \not\cong L_2(8)$. So $C_Q(a) \neq 1$ and then as Q is strongly closed in P, 2.6 implies that $q = |C_Q(a)| = q_0$. So $L = G$.

So $G' \cong L_2(2^n)$. Then $\langle a^G \cap C(a)\rangle = O_2(C(a))$, so $L \cong L_2(8)$ and G is simple. If n is even then Q normalizes a Sylow 2-group T of G and $\langle Q, Q^t\rangle$ is Frobenius for $t \in T^{\#}$. So n is odd and $N_G(Q)$ is dihedral of order $2(2^n + 1)$. Then $v = |\Omega| = (2^n - 1)\, 2^{n-1}$. L is self normalizing in G, so $b = |L^G| = (2^{2n} - 1)\, 2^n/504$. Counting the number of pairs of distinct elements in Ω in two ways we get $v(v-1) = 28.27.b$. Then $(2^n - 1)\, 2^{n-1} - 1 = v - 1 = 3(2^n + 1)$, so $-1 \equiv (v-1) = 3(2^n + 1) \equiv 3 \bmod 2^{n-1}$. Thus $2^{n-1} = 4$ and $G = L$.

6. Theorem 1, Part (1)

In this section assume G is a minimal counter example to Theorem 1, part (1). Let $H = N_G(Q)$, $I = C(Q)$, t a 2-element with cycle (Q, Q^t) in $L = \langle Q, Q^t\rangle$, $K = C(L)$, $D = H \cap H^t$, $D^* = D\langle t\rangle$, and Γ the union of conjugacy classes with a representative in $Q^{\#}$.

Notice that 2.5 implies $L \cong L_2(q)$ or $SL_2(q)$.

The proof involves a long series of reductions.

Lemma 6.1. *Let $x \in G$ fix $P, Q \in \Omega$. Then $|C_Q(x)| = |C_P(x)|$.*

Proof. x acts on $\langle P, Q\rangle \cong SL_2(q)$ or $L_2(q)$. Now apply 2.7.

Lemma 6.2. (1) *The pair (G, Q) satisfies hypothesis* 3.1.

(2) *G is simple.*

(3) *If $X \leqslant H$ and $G_0 = \langle C_Q(X)^{C(X)}\rangle \neq 1$ then letting $\Omega_0 = C_Q(X)^{G_0}$, $q_0 = |C_Q(X)|$, $L_0 \leqslant L$ isomorphic to $L_2(q_0)$ or $SL_2(q_0)$, and $\mathfrak{F}_0 = \{L_0\}$, then Ω_0 is an $\mathfrak{F}_0$-set for G_0.*

Proof. (1) is easy. Minimality of G implies $Z(G) = 1$. So by 3.5, $G = QG'$ with G' simple. As L is perfect, $G = G'$. This yields (2). (3) follows from 6.1 and 2.7.

Lemma 6.3. *Let $X \leqslant H$ with $C_Q(X)$ of order $q_0 \neq 1$. Then one of the following holds*:

(1) $Q = F(X)$.

(2) $|F(X)| = q_0 + 1$ *or* $q_0^3 + 1$, $\langle C_\Gamma(X)\rangle \cong L_2(q_0)$, $SL_2(q_0)$ *or* $U_3(q_0)$, *and* $C(X)^{F(X)}$ *is 2-transitive.*

(3) $|F(X)| = 4^i$, $q_0 = 3$, *and* $\langle C_\Gamma(X)\rangle$, *modulo its center, is Frobenius of order* 3.4^i.

Proof. By 6.1, if $Q^g \in F(X)$ then X acts on $\langle Q, Q^g\rangle \cong L_2(q)$ or $SL_2(q)$ and by 2.7, $C_{Q^g}(X) \in C_Q(X)^{C_L(X)}$. So if $q_0 > 3$ then 6.3 follows from 6.2.3 and minimality of G. If $q_0 = 3$, 6.2.3 and [7] imply 6.3.

LEMMA 6.4. (1) *G^Ω has even degree.*

(2) *If a is an involution then* $|\Omega| \equiv |F(a)| \bmod 4$.

Proof. Assume $|\Omega|$ is odd. If for each choice of $g \in G - H$, $m(D) \leqslant 1$, then 2.8 yields a contradiction. So assume D contains a 4-group U. Then U contains an involution u with $C_Q(u) \neq 1$. So by 6.3, $|F(u)|$ is even, a contradiction.

Let a be an involution in G. By 6.2, G has no subgroup of index 2, so a induces an even permutation on Ω. This yields (2).

LEMMA 6.5. *Let $p \neq 3$ be prime and $1 \neq X \leqslant P \in \mathrm{Syl}_p(K)$. Then*

(1) $P \in \mathrm{Syl}_p(I)$.

(2) *Either $L = \langle C_\Gamma(X)\rangle$ or $K \cap L$ has even order and $X \neq K \cap L$.*

(3) $H = IN_H(L) = ID$.

Proof. Let $L_0 = \langle C_\Gamma(X)\rangle$. By 6.3, $L_0 = L$ or $L_0 \cong U_3(q)$. In the latter case $L \cong SL_2(q)$ contains a central involution u. Then $u \in L \leqslant \langle C_\Gamma(u)\rangle = L_1$, so $L_1 \not\cong U_3(q)$. So $L = L_1$ and $X \neq K \cap L$. In any event $|F(P)| = q^i + 1 \not\equiv 1 \bmod p$, so a Sylow p-subgroup of $N_I(P)$ fixes 2 points of $F(P)$. Then by 6.3, P is Sylow in $N_I(P)$, yielding (1). Choosing $p = 2$ if necessary we may assume $L = \langle C_\Gamma(P)\rangle$. By a Frattini argument, $H = IN_H(P) = IN_H(L) = IQD = ID$.

LEMMA 6.6. *Let $x \in G$ fixes 2 or more points and assume for each $\alpha, \beta \in F(x)$ that $x \in (L \cap D)K$, $|xK/K| > 2$, and x is not inverted in D. Then*

(1) *There exists no $y \in Ht$ with $[x, y] \in I^t$, and*

(2) $|F(x)| = 2$.

Proof. Assume $y \in Ht$ with $[x, y] \in I^t$. Then y centralizes $x \bmod I^t$. As $x \in (L \cap D)K$, t inverts $x \bmod K$. So yt inverts $x \bmod I$.

Suppose $H = IN_H(L)$. Then by hypothesis x is not inverted in H/I, a contradiction. So by 6.5, K is a 3-group. We may assume x is a p-element where p divides $q - 1$, so K is a p'-group. Then as $x \in (L \cap D)K$, $x \in L \cap D$, and then $\langle x \rangle \trianglelefteq D$.

If $Q^{y^{-1}} = Q^t$ then $y \in Dt$, so as $\langle x \rangle \trianglelefteq D$ and t inverts x but x is not inverted in D, $[y, x] \neq 1$. This is impossible as $[x, y] \in I^t$. Now $[x, y^{-1}] \in I$, so as above $x \in (L \cap D)^{y^{-1}}$ and there exists a 2-element s with cycle $(Q, Q^{y^{-1}})$ inverting x. Then $ts \in C(x)$ and $(Q^t)^{ts} = Q^{y^{-1}}$, so again there exists r with cycle $(Q^y, Q^{y^{-1}})$ inverting x. Then $trs \in H$ inverts x, so there exists a 2-element $h \in H$ inverting x.

If h fixes a second point $Q^u \in F(x)$ then x is inverted in $H \cap H^u$ contrary to hypothesis. So $F(x) \cap F(h) = Q$ and thus $|F(x)|$ is odd.

So t fixes a point $Q_1 \in F(x)$. If $q \equiv 1 \bmod 4$ then t inverts $Q_2 \in L \cap \Omega$, so by 6.1, t inverts Q_1. Then $x^2 = [t, x] \in C(Q_1)$, a contradiction. So $q \equiv -1 \bmod 4$ and t does not invert Q_1. But by 6.4, $|\Omega|$ is even, so t fixes a point $Q_3 \neq Q_1$ of Ω. Then t acts on $\langle Q_1, Q_3 \rangle$, so as $q \equiv -1 \bmod 4$, t inverts Q_1, a contradiction. This yields (1).

Next assume $Q^s \in F(x) - \{Q, Q^t\}$. Let P be an x-invariant Sylow p-subgroup of I. (Recall x is a p-element.) If $1 \neq P \leqslant K$ then by 6.3, $F(\langle x \rangle C_P(x)) = \{Q, Q^t\}$, so $C_P(x)$ moves Q^s, contrary to (1). So by 6.5, K is a p'-group and then $x \in L \cap D$. So t inverts x. Similarly we may pick s with cycle (Q, Q^s) to invert x. Then $ts \in C(x)$ moves Q, contradicting (1).

LEMMA 6.7. *Let $x \in L \cap D$ and set $q = 3^n$ where $n = 2^e m$, m odd. Assume either $|x|$ is an odd prime divisor of $3^m - 1$ or $e > 0$ and $|x| = |L \cap D|_2$. Then $|F(x)| = 2$ and if $e = 1$ then I has even order.*

Proof. Unless $e = 1$ and $|x| = |L \cap D|_2$, x is not inverted in $\mathrm{Aut}_H(L)$. Further by 6.1 and 2.7, the same holds for each pair $\alpha, \beta \in F(x)$.

If x has odd order then by 6.1 and 2.7, $x \in (L \cap D)K$ for each $\alpha, \beta \in F(x)$. Assume $e > 1$ and $|x| = |L \cap D|_2$. Choose t so that D contains a Sylow 2-subgroup S of $H = G_\alpha$. If $\gamma \in F(x) - \alpha$ and $C(\langle \alpha, \gamma \rangle)$ has even order then x centralizes an involution $a \in C(\langle \alpha, \gamma \rangle)$. By 6.3, $|F(x) \cap F(a)| = 2$. So assume $C(\langle \alpha, \gamma \rangle)$ has odd order. If a is an involution in $G_{\alpha\gamma}$ with $C_Q(a) \neq 1$ then by 2.7, a normalizes $X = \langle x \rangle$ and again by 6.3, $|F(x) \cap F(a)| = 2$. So $N_H(X)$ is transitive on the set Δ of points $\gamma \in F(x) - \alpha$ with $m(G_{\alpha\gamma}) > 1$. By the choice of β, if $\Delta \neq \varnothing$, then $\beta \in \Delta$. Let $\theta = F(x) - \Delta - \alpha$. Then $x \in \langle \alpha, \gamma \rangle_{\alpha\gamma}$ for each $\gamma \in \theta$, so there exists a 2-element s with cycle (α, γ) in $\langle \alpha, \gamma \rangle$ inverting x. Thus $N(X)$ is transitive on $\theta \cup \alpha$. Suppose $N(X)^{F(x)}$ is not transitive. Then $\theta \cup \alpha$ and Δ are the orbits of $N(X)^{F(x)}$. But $\beta \in \Delta$ and as $x \in L \cap D$, t inverts x and has cycle (α, β), a contradiction. So $N(X)^{F(x)}$ is transitive. Thus $X^G \cap H = X^H$.

Let $\bar{H} = H/I$. $\bar{S} = \bar{Y}\bar{W}$ with $\bar{X} \leqslant \bar{Y}$ inducing a cyclic group of automorphism in $PGL_2(q)$ on $\bar{L}$ and $\bar{W}$ inducing field automorphisms on $\bar{L}$. If $\bar{a}$

is an involution in $\bar{W}$ then $C_Q(a) \neq 1$ so $\bar{a}^{\bar{H}} \cap \bar{Y}$ is empty. Then transfer implies $\bar{W}$ has a normal compliment in $\bar{H}$. Thus $X^G \cap S = X^H \cap S \subseteq YI$. In particular it follows that for all $\alpha, \beta \in F(x)$, $x \in (L \cap D)K$. Now by 6.6, $|F(x)| = 2$.

So assume $e = 1$. If $L \cong SL_2(q)$, 6.3 yields the conclusion, so $L \cong L_2(q)$. Assume $|F(x)| > 2$ and let $u = x^2$. Suppose $a \neq u$ is an involution centralizing x. If a fixes 2 points $Q_i \in F(x)$ then as a acts on $\langle Q_1, Q_2\rangle = L_1$, either $a \equiv u \bmod C(L_1)$ or $C_{Q_1}(a) \neq 1$. But if $Q_1 \neq C_{Q_1}(a) \neq 1$ then $[a, x] \neq 1$, so $a \in K$. Thus $\langle u, a\rangle \cap K \neq 1$ and by 6.3, $|F(\langle x, a\rangle)| = 2$. Then a inverts or centralizes Q_1 and has a cycle $(Q_3, Q_3{}^a)$ in $F(x)$. As a centralizes x and acts on $L_3 = \langle Q, Q^a\rangle$, a induces a field automorphism on L_3. So $Q_3 \neq C_{Q_3}(a) \neq 1$ for some $Q_3 \in L_3 \cap \Omega$, contradicting 6.1.

So letting $x \in S \in \mathrm{Syl}_2(H)$, if $|F(x)| > 2$ then $\langle u\rangle = \Omega_1(C_S(X))$, and in particular I has odd order.

So we may assume I has odd order. $|\Omega|$ is even, so S fixes a second point Q_1 and then acts on $L_1 = \langle Q, Q_1\rangle$. So S is quaternion, dihedral, or semidihedral. Let $s \in L_1$ have cycle (Q, Q_1) with $S^* = \langle S, s\rangle$ a 2-group.

Assume S is quaternion. Then L_1 admits no field automorphism and arguing as above $\langle u\rangle = \Omega_1(C_T(x))$ where $x \in T \in \mathrm{Syl}_2(G)$. Now by 5.4.8 in [12], T is semidihedral. So 2.8 yields a contradiction.

So S contains an involution a inducing a field automorphism on L_1. Then letting $q = q_0{}^2$, by 6.3, $L_0 = \langle C_\Gamma(a)\rangle \cong L_2(q_0)$ or $q_0 = 3$ and L_0 is Frobenius of order $3 \cdot 4^i$. Further u induces an outer automorphism on L_0. In any event $C(a)^{F(a)}$ is transitive, so $a \cap H = a^H$.

Suppose S is dihedral and let $u \in R \in \mathrm{Syl}_2(\langle u\rangle L_0)$. If $L_0 \cong L_2(q_0)$, then $R \leqslant L_1\langle a\rangle$ and $R = \langle a\rangle \times (R \cap L_1)$. Also all involutions in $R \cap L_1$ are in u^G, and as $a^G \cap H = a^H$, a is not fused to au or u. Now if $R \notin \mathrm{Syl}_2(G)$ then there exists a 2-element in $N(R)$ moving a. But $a^G \cap Z(R) = \{a\}$. Thus $R \in \mathrm{Syl}_2(G)$. But now considering the transfer of G to $R/(R \cap L_1)$, G has a subgroup of index 2, a contradiction. Similarly if L_0 is Frobenius let R_1 be the kernel of L_0. $R_1 = \langle u^G \cap R_1\rangle$ and in S^* the product of any two commuting conjugates of u is in u^G. Also each $r \in R_1{}^\#$ acts fixed point free on $F(a)$, so ar is not fused to a. Now we argue as above to the same contradiction.

So S is semidihedral. Then S^* is the holomorph of Z_8. Let Y be the cyclic subgroup of order 8 in S. Y is weakly closed in S and not inverted in D, so we may argue as in 6.6 to show $|F(Y)| = 2$. Now $X = \langle x\rangle = S^{*\prime}$ is characteristic in S^*. Set $\bar{S}^* = S^*/X$. Let a be an involution in S with $C_Q(a) \neq 1$. Then Xya contains all elements in $S^* - X$ of order 4, so $\langle X, ya\rangle \trianglelefteq N(S^*)$. Also all elements of order 8 in $S^* - X$ are in Xy and $Xyat$. So if $Y \ntrianglelefteq N(S^*)$ then $\bar{y} \to \bar{y}\bar{a}\bar{t}$. So $\bar{a} = \bar{y}(\bar{y}\bar{a}) \to \bar{y}\bar{a}\bar{t}(\bar{y}\bar{a}) = \bar{t}$. But $Xa \subseteq a^G$ while $Xt \subseteq t^G \neq a^G$. So $Y \trianglelefteq N(S^*)$. Thus $S^* \in \mathrm{Syl}_2(G)$, so

$|\Omega| \equiv 2 \bmod 4$. But $|C_Q(a)| = 3^m$, so $|F(a)| \equiv 0 \bmod 4$ by 6.3. This contradicts 6.4.

LEMMA 6.8. *G^Ω is doubly transitive.*

Proof. By 6.7 there exists a prime p such that for each $\alpha, \beta \in \Omega$ there exists a p-subgroup X with $F(X) = \{\alpha, \beta\}$. It follows that $G_{\alpha\beta}$ contains a Sylow p-subgroup P of G_α and $F(P) = \{\alpha, \beta\}$. Now Sylow's theorem implies G_α is transitive on $\Omega - \alpha$.

LEMMA 6.9. *Let $a \in D$ with $|F(a)| > 2$ and $|aK/K| = p$ prime. Then*

(1) *Either $C(a)^{F(a)}$ is double transitive or $\langle C_\Gamma(a)\rangle$ is a Frobenius group of order $3 \cdot 4^i$, modulo its center.*

(2) *If a is an involution in $Z^*(D)$ then $a \in Z^*(H)$.*

Proof. (1) By 6.3 we may assume $C_Q(a) = 1$. If $P \in F(a)$ then by 6.1 and 2.7, a centralizes $\langle Q, P\rangle_{P,Q}$. Now 6.7 and the argument in 6.8 implies $C(a)^{F(a)}$ is 2-transitive.

(2) Let $b \in C_H(a) \cap a^G$. As $|\Omega|$ is even, b fixes 2 points of $F(a)$. As $a \in Z^*(D)$ the last case of (1) cannot occur, so $C(a)^{F(a)}$ is 2-transitive. Thus we may take $b \in a^G \cap C_D(a) = a^D \cap C(a) = \{a\}$. Then the Z^*-theorem yields the result.

LEMMA 6.10. *Let π be the set of primes dividing $|\Omega| - 1$, and assume $H = O(I)D$. Then $H = O_\pi(I)D$, $O_\pi(I)$ is not nilpotent, and $O_3(K) = O_\pi(I) \cap D \neq 1$.*

Proof. Set $P = O_\pi(I)$ and let $R/P \leqslant O(I)/P$ be a minimal normal subgroup of H/P. Then R/P is a r-subgroup for some prime $r \notin \pi$, so $R = R_1P$, $R_1 \in \mathrm{Syl}_r(R)$. By a Frattini argument, $H = PN_H(R_1) = PD$ by 6.3. By [14], $P \cap D \neq 1$, while by [16], P is not nilpotent. By 6.5, $P \cap D = O_3(K)$.

LEMMA 6.11. *If $q \equiv 1 \bmod 4$ then I has even order.*

Proof. Assume $q \equiv 1 \bmod 4$ and I has odd order. Let $S \in \mathrm{Syl}_2(D)$ and $S^* = \langle S, t\rangle$. If S is cyclic, 6.6 implies $|F(S)| = 2$, so as S is characteristic in S^*, $S^* \in \mathrm{Syl}_2(G)$. Then $|\Omega| \equiv 2 \bmod 4$ and [3] yields a contradiction. If $q = 3^{2m}$, m odd, then I has even order by 6.7. Finally, if $q = 3^{4m}$ then $S \cap L$ is a characteristic subgroup of S^* with $|F(S \cap L)| = 2$ by 6.7. Thus $S^* \in \mathrm{Syl}_2(G)$ and $|\Omega| \equiv 2 \bmod 4$. S is not cyclic so there exists an involution $a \in S$ with $C_Q(a) \neq 1$. Every involution in $S^* \cap L = T$ is fused in L and as $C_Q(a) \neq 1$, $a^G \cap T$ is empty. If S does not contain an element inducing an outer automorphism r in $PGL_2(q)$ on L, let $T = R$. If r exists let $R = \langle T, ra\rangle$. In this case R is semidihedral so again $a^G \cap R$ is empty. In any event S^*/R is cyclic, so considering the transfer of G to S^*/R, G has a subgroup of index 2, a contradiction.

LEMMA 6.12. *Suppose* $q \equiv -1 \bmod 4$ *and* u *is an involution inverting* Q. *Then either* $|F(u)| = 2$, *or* $C(u)^{F(u)}$ *is an extension of* $L_2(q)$ *on* $q + 1$ *letters,* $|C_K(u)| \leqslant 2$, *and* $\langle u\rangle = C(u)_{F(u)}$.

Proof. $q \equiv -1 \bmod 4$, so q is an odd power of 3. Assume $|F(u)| > 2$ and let $S \in \mathrm{Syl}_2(C_D(u))$. Then as $q \equiv -1 \bmod 4$, $S = (S \cap K)\langle u\rangle$, so by 6.3, $S^{F(u)}$ acts semiregularly on $F(u) - \{Q, Q^t\}$. So either $C(u)^{F(u)}$ has a 2-transitive subgroup $X^{F(u)}$ (consisting of even permutations on $F(u)$) of index at most 2 and with $(X \cap H)^{F(u)}$ of odd order, or $|F(u)| \equiv 2 \bmod 4$ and K has even order. But in the latter case letting k be an involution in K, $0 \equiv |F(k)| \equiv |\Omega| \equiv |F(u)| \equiv 2 \bmod 4$, by 6.3 and 6.4.

As $[(D \cap L)^{F(u)}, t] \neq 1$, it follows from a result of Bender [8] that $C(u)^{F(u)}$ is an extension of $L_2(m)$ on $m + 1$ letters. If $m \equiv 1 \bmod 4$ then $(K \cap S)^{F(u)} \neq 1$ and $|F(u)| \equiv 2 \bmod 4$, which we have shown is not the case. So $m \equiv -1 \bmod 4$ and $(m-1)/2 = |[D^{F(u)}, t]| = (q-1)/2$. So $m = q$. By 6.3, $C_K(u)$ acts semiregularly on $F(u) - \{Q, Q^t\}$, so t inverts $K^{F(u)}$. Thus $|C_K(u)| \leqslant 2$.

LEMMA 6.13. *Let* a *be an involution in* D *with* $C(a)^{F(a)}$ *2-transitive. Let* $c = |a^D|$, $e = |a^G \cap D^* - D|$ *and* $m = |F(a)|$. *Then* $|\Omega| = m(m-1)\,e/c + m$.

Proof. Let S be the set of pairs (b, x) where $b \in a^G$ and x is a cycle in b. Set $n = |\Omega|$. Then $|a^G|(n-m)/2 = |S| = n(n-1)\,e/2$. Further as $C(a)^{F(a)}$ is 2-transitive, $|a^G| = n(n-1)\,c/m(m-1)$.

LEMMA 6.14. (1) *Let* S *be a 2-subgroup of* H *with* $C_Q(S) \neq 1$. *Then* $m(S) \leqslant 1$.

(2) $H = O(I)D$.

(3) *I has even order.*

Proof. Suppose I has odd order. By 6.11, $q \equiv -1 \bmod 4$. By [8], D has even order, so there exists an involution $u \in D$ inverting Q, and $\langle u\rangle$ is Sylow in H. If $|F(u)| = 2$ then by 6.4, $|\Omega| \equiv 2 \bmod 4$, contradicting [3]. So by 6.12, $C(u)$ has a characteristic subgroup X with $X/\langle u\rangle \cong L_2(q)$ and $|F(u)| = q + 1$. Let R be the subgroup of order q in $X \cap H$. Then $H = IC_H(u) = IN_H(R) = IRD$. As $L \cap D \leqslant N(QR)$ acts irreducibly on Q, $IR = O(I)$. Now by 6.10 there exists a nontrivial u-invariant Sylow 3-subgroup P of K. By 6.5, $F(P) = L \cap \Omega$, so t acts fixed point free on $F(P)$. Thus as $[P, t] = 1$,

$|F(t)| \equiv 0 \bmod 3$. So $t \notin u^G$. Therefore $u^G \cap D^* - D = (ut)^D$. By 6.12, u inverts K, so $e = |D: C_D(ut)| = |K|(q-1)/2$ and $c = |D: C_D(u)| = |K|$. Now by 6.13, $|\Omega| - 1 = q(q^2+1)/2$. Therefore $QP \in \mathrm{Syl}_3(I)$. Now u inverts Q and P, so u inverts QP. But $1 \neq R$ is a 3-subgroup of I centralized by u, a contradiction. This yields (3).

Now 6.5 implies $H = ID$. Assume (2) and let S be a 2-subgroup of H with $C_Q(S) \neq 1$. By 6.10, $H = PD$ where $P = O_\pi(I)$. By 6.3 and 6.10, $C_P(s)$ is a 3-group for each $s \in S^\#$, unless $q = 9$ and $|F(s)| = 4^i$. In the latter case choosing k to be an involution in K, and in $K \cap L$ if possible, $10 = |F(k)| \equiv |\Omega| \equiv |F(s)| = 4^i \bmod 4$, by 6.4, a contradiction. It follows that $P = \langle C_P(s) : s \in S^\# \rangle$ is a 3-group, contradicting 6.10.

So it remains to show (2). If $K \cap L = \langle z \rangle \neq 1$ then $z \in Z^*(D)$, so by 6.9, $z \in Z^*(H)$. Then $H = IC_H(z) = O(I)\, N_H(L) = O(I)D$. So we may assume $K \cap L = 1$. Then by 6.5, $N_H(X) \leqslant N_H(L)$ for each nontrivial subgroup X of K. Thus $QK = N_I(L)$ is strongly embedded in I. As $[L \cap D, K] = 1$, [9] implies that $I = O(I)\, C_I(L \cap D) = O(I)K$ by 6.7.

For the remainder of this paper define $P = O_\pi(I)$ as in 6.10 and set $P_0 = P \cap D$.

LEMMA 6.15. (1) *$F(x) = L \cap \Omega$ for each $1 \neq X \leqslant K$.*

(2) *$L \cap K = 1$.*

(3) *Let u be an involution in K and let $v \in u^G \cap C(u)$ have cycle (Q, Q^t). Let P_1 be a $\langle u, v \rangle$-invariant Sylow 3-subgroup of $O(K)$. Then v inverts P_1, $[u, P_1] \neq 1$ and v acts fixed point free on $F(u)$.*

Proof. If $F(X) \neq L \cap \Omega$ then by 6.5, $L \cap K \neq 1$. So (2) implies (1). Suppose $1 \neq X \leqslant P_1$ with $Y = \langle C_\Gamma(X) \rangle \cong U_3(q)$. As $N_K(X)^{F(X)}$ is a $3'$-group we may take $X = P_1$. Let $\langle u \rangle = L \cap K$ and let (Q_1, Q_2) be a cycle in u. Then u centralizes its conjugate v in the center of $\langle Q_1, Q_2 \rangle$ and so v acts on $L = \langle C_\Gamma(u) \rangle$. Then v also acts on $P_0 = O_3(K)$. So v induces an automorphism of $Y \cong U_3(q)$ and fixes points $Q_i \in F(X)$, $i = 3, 4$. Then $Q_1 \in F(v) = \langle Q_3, Q_4 \rangle \cap \Omega \subseteq F(X)$. So $\Omega = F(X)$, a contradiction.

Now choose u, v, and P_1 as in (3) with $F(u) = L \cap \Omega$. Then v acts fixed point free on $F(u) = F(x)$ for each $x \in P_1^\#$, so $C_{P_1}(v)$ acts semiregularly on $F(v)$. Thus if $1 \neq C_{P_1}(v)$ then $0 \equiv |F(v)| = (q+1) \bmod 3$, a contradiction. So v inverts P_1. Suppose $[P_1, u] = 1$. Notice in particular this occurs if $u \in L \cap K$. Define e and c as in 6.13. As v inverts P_1, $e \equiv 0 \bmod 3$. By 6.14, $u \in Z^*(H)$, so as $[P_1, u] = 1$, $c \not\equiv 0 \bmod 3$. Now by 6.13, $|\Omega| - 1 = q[(q+1)\, e/c + 1] \equiv q \bmod 3q$. So P_0Q is Sylow in P. But u centralizes P_0Q and inverts a Hall $3'$-subgroup P_2 of P. So $P_2 = [P, u] \trianglelefteq P$ and then $P_2 = O_{3'}(H)$. So $QP_2 \trianglelefteq H$ and QP_2 is regular on $\Omega - Q$, contradicting [15].

LEMMA 6.16. $q \equiv 1 \bmod 4$.

Proof. Assume $q \equiv -1 \bmod 4$. By 6.14, I contains a unique class of involutions u^I. Let $v \in u^G$ have cycle (Q, Q^t). As $m(K) = 1$ we may take $[u, v] = 1$. v acts fixed point free on $F(u)$, so $v \in tK$. As $m(K) = 1 = [t, K]$, $v = t$ or ut. By 6.15, v inverts P_1, so $v \neq t$. Then defining e and c as in 6.13, $e = (q - 1)\, c/2$. So 6.13 implies $|\Omega| - 1 = q(q^2 + 1)/2$. Let R be a $\langle u \rangle(L \cap D)$-invariant sylow r-subgroup of P, $r \neq 3$. As $q \equiv -1 \bmod 4$, no element of $L \cap D^{\#}$ is inverted in D, so $F(x) = \{Q, Q^t\}$ for each $x \in L \cap D^{\#}$ by 6.6 and 2.7. Also by 6.5, R acts semiregularly on $\Omega - Q$. So $\langle u \rangle(L \cap D)$ acts semiregularly on $R^{\#}$ and thus $|R| > q$. As a $3'$-Hall group of P has order $(q^2 + 1)/2$, $|R| = (q^2 + 1)/2$ is a prime power. Then by 2.9, $q = 3$, a contradiction.

LEMMA 6.17. $q \equiv -1 \bmod 4$.

Proof. Assume $q \equiv 1 \bmod 4$. By 6.14, I contains a unique class of involutions u^G. Let $v \in u^G \cap C(u)$ have cycle (Q, Q^t). By 6.15, v inverts P_1 and $[u, P_1] \neq 1$, so $C_{P_1}(uv) \neq 1$. As v acts fixed point free on $F(u) = F(C_{P_1}(uv))$, $|F(uv)| \equiv 0 \bmod 3$. So by 6.3 and 6.4, uv is conjugate to the involution $x \in L \cap D$, or to xu. Now $[x, P_1] = 1$, so $|F(x)| \equiv 2 \bmod 3$. Thus $uv \in (ux)^G$. As $|F(ux)| \equiv |F(uv)| \equiv 0 \bmod 3$, $C_{P_1}(u) = C_{P_1}(ux) = 1$. So u inverts P_0. Therefore $Q = C_P(u)$, so u inverts P/Q. as $Q \leqslant Z(P)$, it follows that P is nilpotent, contradicting 6.10.

Now 6.16 and 6.17 yield a contradiction, establishing Theorem 1, part (1).

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