An Asymptotic Numerical Method for Singly Perturbed Third-Order Ordinary Differential Equations of Convection-Diffusion Type

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(Received January 2001; revised and accepted June 2001)

Abstract—Singly perturbed two-point boundary value problems (SPBVPs) for third-order ordinary differential equations (ODEs) with a small parameter multiplying the highest derivative are considered. A numerical method is suggested in this paper to solve such problems. In this method, the given BVP is transformed into a weakly coupled system of two ODEs subject to suitable initial and boundary conditions. Then, the computational method, presented in this paper, is applied to this system. In this method, we reduce the weakly coupled system into a decoupled system. Then, to solve this decoupled system numerically, we apply a ‘boundary value technique (BVT)’, in which the domain of definition of the differential equation is divided into two nonoverlapping subintervals called inner and outer regions. Then, we solve the decoupled system over these regions as two point boundary value problems. An exponentially fitted finite difference scheme is used in the inner region and a classical finite difference scheme, in the outer region. The boundary conditions at the transition point are obtained using the zero-order asymptotic expansion approximation of the solution of the problem. This computational method is distinguished by the facts that the decoupling reduces the computational time very much and it is well suited for parallel computing. This method can be extended to a system of two ordinary differential equations, of which, one is of first order and the other is of second order. Numerical examples are given to illustrate the method. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords—Singular perturbation, Third-order differential equation, Non-self-adjoint boundary value problem, Asymptotic expansion approximation, Boundary layer, Exponentially fitted finite difference scheme.

1. INTRODUCTION

For the past 20 years an extensive research has been made on numerical methods for singularly perturbed ordinary differential equations (SPODEs). Robust techniques have been developed for second-order ordinary differential equations but for higher-order equations only few results are reported in the literature. Analytical treatment of SPBVPs for higher-order nonlinear ordinary differential equations, which have important applications in fluid dynamics, is available in [1–16]. The classification of singularly perturbed higher-order problems depend on how the order of the

The authors wish to express their sincere thanks for the referee's constructive suggestions.

*Author wishes to acknowledge the support of the U.G.C., New Delhi, India, to carry out the research work under Minor Research Project.
original equation is affected if one set \( \varepsilon = 0 \). We say that the problem is of convection-diffusion type if the order is reduced by one and of reaction-diffusion type if the order is reduced by two. Here, \( \varepsilon \) is a small positive parameter multiplying the highest derivative of the differential equation. Niederdrenk et al. [10] have considered convection-diffusion type problems and derived conditions for the uniform stability of the discrete and continuous problems. Gartland [11] has shown that the uniform stability of the discrete boundary value problem follows from the uniform stability of the associated discrete initial value problem and uniform consistency of the scheme. Some results connected with the exponentially fitted HODIE method [11] and defect corrections with piecewise constant coefficients are available in the literature. Feckan [9] has considered higher-order problems and his approach is based on the nonlinear analysis involving fixed-point theory, Leray-Schauder theory, etc. In [4], an iterative method is described. Further, if the order of the equation is even then a finite element method (FEM) based on standard \( C^{m-1} \) splines on a Shishkin mesh is reported [12]. In [4, 13], an FEM for convection-reaction type problems is described. Also Semper [14], Roos [15], and O'Malley [16] have considered fourth-order equation and applied a standard FEM. As far as authors' knowledge goes, only few results are reported in the case of third-order differential equations, that too on the analytical behaviour of the solution. In fact, Howes [6] has considered problems of type

\[
e^2 y''' = f(y)y' + g(x, y), \quad y(a) = A, \quad y'(b) = C, \quad y(b) = B,
\]

and discussed the existence and asymptotic estimates of the solution by the method of descent. He has also reported results [5] on problems of the form

\[
e y^n = f\left(t, y, y', \ldots, y^{(n-2)}\right), \quad a < t < b, \quad n \geq 3,
\]

\[
y^{(j)}(a, \varepsilon) = A_j, \quad 0 \leq j \leq n - 2,
\]

\[
y^{(n-2)}(b, \varepsilon) = B_{n-2},
\]

which include existence, uniqueness, and asymptotic behaviour of the solution. Roberts [7] has suggested a method of finding approximate solution for third-order ordinary differential equations. Zhao [8] has considered a more general class of third-order nonlinear SPBVPs of the form

\[
e y''' = f(x, y, y', \varepsilon), \quad \text{with} \quad y'(0) = 0, \quad y(1) = 0, \quad y'(1) = 0,
\]

or

\[
e y''(0) = 0, \quad y(1) = 0, \quad y'(1) = 0,
\]

and discussed the existence, uniqueness of the solution and obtained asymptotic estimates using the theory of differential inequalities. Motivated by the works of Roberts [17], Zhao [8], Howes [5], Jayakumar et al. [18,19], Natesan et al. [20], and Khadlalbajoo [21–23] we, in this paper, suggest a computational method, which makes use of the zero-order asymptotic expansion approximation and a boundary value technique (BVT) to find a numerical solution for third-order singularly perturbed ordinary differential equations subject to certain boundary conditions. As mentioned earlier, the BVP is transformed into a weakly coupled system of two equations, one of which is first order and the other a second-order equation, subject to initial and boundary conditions, respectively. Then, this weakly coupled system is decoupled by replacing one of the unknowns by its zero-order asymptotic expansion. Finally, the BVT is applied to the decoupled system. By this procedure of decoupling the computational time is very much reduced. Also, the BVT gives excellent portrait of the solution, especially within the boundary layers. This method is easy to apply and further we could give a full-fledged theory (consistency, stability, convergence and error estimates) for the same. Also, the material presented in Section 7 is entirely new. The present method is well suited for parallel computing. Thus, the present technique is more advantageous when compared with the other methods available in the literature.
In this paper, we consider the following problem

$$-\varepsilon y''(x) + a(x)y''(x) + b(x)y'(x) + c(x)y(x) = f(x), \quad x \in D,$$

$$y(0) = p, \quad y'(0) = q, \quad y'(1) = r,$$

where $\varepsilon$ is a small positive parameter, $a(x)$, $b(x)$, $c(x)$, and $f(x)$ are sufficiently smooth functions,

$$a(x) \leq -\alpha, \quad \alpha > 0, \quad (1.3)$$

$$b(x) \geq 0, \quad (1.4)$$

$$0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \quad (1.5)$$

$$\delta = \alpha - \gamma(1 + 3\eta) \geq \lambda > 0, \quad \text{for some } \eta > 0 \text{ and } \lambda, \quad (1.6)$$

$D = (0, 1)$, $D_0 = (0, 1]$, $\bar{D} = [0, 1]$, and $y \in C^3(D) \cap C(D)$. It may not be difficult to extend the present method to solve the more general system of equations of the form

$$y' = g(x, y, z), \quad y(0) = p, \quad \varepsilon z'' = f(x, y, z, z'), \quad z(0) = q, \quad z(1) = r.$$ 

REMARK 1.1. For problem (1.1),(1.2) one boundary layer occurs at $x = 0$, which is less severe [4]. Condition (1.3) says that problem (1.1),(1.2) is a nonturning point problem. Condition (1.5) ensures that the equivalent system (2.1),(2.2) of (1.1),(1.2) is a weakly coupled system of equations. The other two conditions help to establish the maximum principle for system (2.1),(2.2) and using this principle, we establish a uniform stability result. In the following, $C$ is a constant independent of the nodes, mesh size, and the small parameter $\varepsilon$.

2. SOME ANALYTICAL RESULTS

This section presents the maximum principle and stability result on the solution for problem (2.1),(2.2). A result on the asymptotic expansion approximation of the solution of (2.1),(2.2) is presented. Further, estimates for the derivatives of the solution are given. The maximum principle is stated in a form similar to that given in [24].

Problem (1.1),(1.2) can be transformed into an equivalent problem of the form

$$Ay = F \iff \begin{cases} P_1y := y_1'(x) - y_2(x) = 0, \\ P_2y := -\varepsilon y_2''(x) + a(x)y_2''(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), \quad x \in D, \\ P_3y := y_1(0) = p, \\ P_4y := y_2(0) = q, \\ R_3y := y_2(1) = r, \end{cases}$$

where $y = (y_1, y_2)$.

\subsection*{2.1. Maximum Principle and Stability Result}

THEOREM 2.1. MAXIMUM PRINCIPLE. Consider the BVP (2.1),(2.2). If $u_1(0) \geq 0$, $u_2(0) \geq 0$, $u_2(1) \geq 0$, $P_1u \geq 0$, and $P_2u \geq 0$, then $u(x) \geq 0$ in $\bar{D}$.

PROOF. Defining

$$s(x) = (s_1(x), s_2(x)) > 0, \quad \text{as}$$

$$s_1(x) = (1 + 2\eta)x + \eta,$$

$$s_2(x) = 1 + \eta - x,$$

$\eta$ as given in condition (1.6) and $x \in \bar{D}$, we can easily prove that $P_1s > 0$ and $P_2s > 0$. Assume that the theorem is not true, and define

$$\xi = \max \left\{ \max_{x \in \bar{D}} \left( -\frac{u_1}{s_1} \right)(x), \max_{x \in \bar{D}} \left( -\frac{u_2}{s_2} \right)(x) \right\}.$$
Then, $\xi > 0$. Further, $u_1 + \xi s_1 \geq 0$ and $u_2 + \xi s_2 \geq 0$. Then, there exists a point $x_0 \in \partial D$ such that $-(u_1/s_1)(x_0) = \xi$ for $x_0 \in D_0$, or $-(u_2/s_2)(x_0) = \xi$ for $x_0 \in D$.

CASE 1. $-(u_1/s_1)(x_0) = \xi$, for $x_0 \in D_0$. That is, $(u_1 + \xi s_1)(x_0) = 0 \Rightarrow u_1 + \xi s_1$ attains its minimum at $x_0$. Therefore, $0 \leq P_1(u + \xi s)(x_0) = (u_1 + \xi s_1)'(x_0) - (u_2 + \xi s_2)(x_0) < 0$. It is a contradiction.

CASE 2. $-(u_2/s_2)(x_0) = \xi$, for $x_0 \in D$. That is, $(u_2 + \xi s_2)(x_0) = 0 \Rightarrow u_2 + \xi s_2$ attains its minimum at $x_0$. Therefore,

$$0 \leq P_2(u + \xi s)(x_0) = -\varepsilon(u_2 + \xi s_2)''(x_0) + a(x)(u_2 + \xi s_2)'(x_0) + b(x)(u_2 + \xi s_2)(x_0) + c(x)(u_1 + \xi s_1)(x_0) < 0.$$ 

It is a contradiction. Hence, we conclude that $u(x) \geq 0$ for all $x \in \partial D$.

**LEMMA 2.1. STABILITY.** Consider the BVP (2.1),(2.2). If $y_1 \in C^3(D) \cap C^1(\partial D)$,

$$\|y(x)\| \leq C \max \left\{ |y_1(0)|, |y_2(0)|, |y_2(1)|, \max_{x \in D} |P_1 y|, \max_{x \in D} |P_2 y| \right\},$$

where $\|y(x)\| = \max \{|y_1(x)|, |y_2(x)|\}$, for all $x \in \partial D$.

**PROOF.** Defining two functions $W^\pm(x)$ as

$$W_1^\pm(x) = C((1 + 2\eta)x + \eta) \max \left\{ |y_1(0)|, |y_2(0)|, |y_2(1)|, \max_{x \in D} |P_1 y|, \max_{x \in D} |P_2 y| \right\} \pm y_1(x),$$

$$W_2^\pm(x) = C(1 + \eta - x) \max \left\{ |y_1(0)|, |y_2(0)|, |y_2(1)|, \max_{x \in D} |P_1 y|, \max_{x \in D} |P_2 y| \right\} \pm y_2(x),$$

with $\eta$ as given in condition (1.6) and $x \in \partial D$, by a proper selection of the constant $C$, we can prove that $W_1^\pm(0) \geq 0$, $W_2^\pm(0) \geq 0$, $W_2^\pm(1) \geq 0$, $P_1 W^\pm(x) \geq 0$, for $x \in D_0$ and $P_2 W^\pm(x) \geq 0$, for $x \in D$. Then, the desired result follows from Theorem 2.1.

### 2.2. Asymptotic Expansion Approximation

One can look for the asymptotic expansion solution of (2.1),(2.2) in the form

$$y(x, \varepsilon) = (u_0 + v_0) + \varepsilon(u_1 + v_1) + O(\varepsilon^2).$$

By the method of stretching variable [3] one can obtain the zero-order asymptotic expansion $y_{as} = (u_0 + v_0)$ where $u_0$ is solution of the reduced problem of (2.1),(2.2) given by

$$u_0' = u_0 = 0, \quad a(x)u_0'' + b(x)u_0 + c(x)u_0 = f(x), \quad u_0(0) = p, \quad u_0(1) = r, \quad (2.3)$$

and $v_0$ is the layer correction given by, $v_0 = (v_{01}, v_{02})$, with

$$v_{01} = 0, \quad v_{02} = [q - u_0(0)] \exp \left( \frac{a(0)x}{\varepsilon} \right). \quad (2.5)$$

**THEOREM 2.2.** The zero-order asymptotic approximation $y_{as}$ of the solution $y$ of (2.1),(2.2) satisfies the inequality

$$\|y - y_{as}\| \leq C \varepsilon.$$

**PROOF.** We have

$$|P_1 y - P_1 y_{as}| \leq C \exp \left( \frac{-\alpha x}{\varepsilon} \right).$$
We use the fact that
\[ t \exp(-t) \leq \exp \left( \frac{-t}{2} \right), \quad \forall t \geq 0, \]
to obtain
\[ |P_2y - P_2y_\text{as}| \leq C \varepsilon \left[ 1 + \varepsilon^{-1} \exp \left( -\frac{\alpha \varepsilon}{2\varepsilon} \right) \right]. \]

Now, defining barrier functions,
\[ \phi_1 = \gamma_1 \varepsilon \left[ (1 + 2\eta) x + \eta \right] + \gamma_2 \varepsilon^2 \exp \left( -\frac{\alpha \varepsilon}{2\varepsilon} \right) \pm (y_1 - y_\text{as}), \]
\[ \phi_2 = \gamma_1 \varepsilon \left[ 1 + \eta - x \right] + \gamma_2 \varepsilon \exp \left( -\frac{\alpha \varepsilon}{2\varepsilon} \right) \pm (y_2 - y_\text{as}), \]
\[ \eta \text{ as given in condition (1.6) and } x \in \bar{D}, \] it is easy to verify that \( P_1 \Phi \geq 0, \ P_2 \Phi \geq 0, \ \Phi_1(0) = 0, \ \Phi_2(0) > 0, \) and \( \Phi_2(1) \geq 0, \) for a proper choice of \( \gamma_1 \) and \( \gamma_2. \) Then, by Lemma 2.1, we have \( \|y - y_\text{as}\| \leq C \varepsilon. \)

### 2.3. Estimates for Derivatives

**Theorem 2.3.** If \( a(x), \ b(x), \ c(x), \) and \( f(x) \) belong to \( C^j(\bar{D}) \) then, \( y_1^{(k)}(x) \) and \( y_2^{(k)}(x) \) satisfy
\[ \left| y_1^{(k)}(x) \right| \leq C \left[ 1 + \varepsilon^{-(k-1)} \exp \left( -\frac{\alpha x}{\varepsilon} \right) \right], \]
\[ \left| y_2^{(k)}(x) \right| \leq C \left[ 1 + \varepsilon^{-k} \exp \left( -\frac{\alpha x}{\varepsilon} \right) \right], \quad \text{for all } k \leq j \text{ and } x \in \bar{D}, \]

where \( y = (y_1, y_2) \) is the solution of (2.1), (2.2).

**Proof.** The proof is by induction on \( k. \) The result is obvious for the case \( k = 0 \) (Lemma 2.1). Assume that
\[ \left| y_1^{(m)}(x) \right| \leq C \left[ 1 + \varepsilon^{-(m-1)} \exp \left( -\frac{\alpha x}{\varepsilon} \right) \right], \]
\[ \left| y_2^{(m)}(x) \right| \leq C \left[ 1 + \varepsilon^{-m} \exp \left( -\frac{\alpha x}{\varepsilon} \right) \right], \quad \text{for all } m \leq k. \]

Consider the differential equation
\[ -\varepsilon y_2''(x) + a(x)y_2'(x) + b(x)y_2(x) + c(x)y_1(x) = f(x). \]

Differentiating this \( k \) times, we are with the equation
\[ \varepsilon \left( y_2^{(k)}(x) \right) + a(x) y_2^{(k)}(x) = g_k(x), \]
where
\[ |g_k(x)| \leq C \left[ 1 + \varepsilon^{-k} \exp \left( -\frac{\alpha x}{\varepsilon} \right) \right]. \]

Then, following the method of proof given in [25] and using Lemma 2.1, we obtain
\[ \left| y_2^{(k+1)}(x) \right| \leq C \left[ 1 + \varepsilon^{-(k+1)} \exp \left( -\frac{\alpha x}{\varepsilon} \right) \right]. \]

Using \( y_1(x) = y_2(x), \)
\[ \left| y_1^{(k+1)}(x) \right| \leq C \left[ 1 + \varepsilon^{-(k)} \exp \left( -\frac{\alpha x}{\varepsilon} \right) \right]. \]
Hence, the proof of the theorem is complete.

To prove the uniform convergence of the numerical solution, we need the following stronger result on the estimates of the derivatives of the components of the solution of (2.1),(2.2).

**Theorem 2.4.** If \( a(x), b(x), c(x), \) and \( f(x) \in C^1(D) \), then the solution \( y = u + v \) satisfies

\[
\left\| u^{(k)}(x) \right\| \leq C \left[ 1 + \epsilon^{1-k} \exp \left( \frac{-\alpha x}{\epsilon} \right) \right],
\]

\[
\left\| v^{(k)}(x) \right\| \leq C \epsilon^{-k} \exp \left( \frac{-\alpha x}{\epsilon} \right),
\]

where \( u \) is the smooth component given by \( u = u_0 + \varepsilon u_1 \), \( u_0 \) is the solution of the reduced problem (2.3),(2.4) and \( u_1 \) is the solution of the problem,

\[
u_{11} - u_{12} = 0, \quad -\varepsilon u_{12}' + a(x)u_{12}' + b(x)u_{12} + c(x)u_{11} = u_{02}, \quad (2.6)\]

\[
u_{11}(0) = 0, \quad u_{12}(0) = 0, \quad u_{12}(1) = -\epsilon^{-1}u_{02}(1), \quad (2.7)\]

and \( v \) is the singular component given by (2.5).

**Proof.** Following the method of proof adopted in [25] and using Lemma 2.1, we can get the desired estimates.

### 3. SOME RESULTS ON NUMERICAL SCHEMES

In this section, we present a classical finite difference scheme and an exponentially fitted finite difference scheme to solve (2.1),(2.2) numerically. As in the case of continuous problem, the maximum principle theorem and the stability result for the discrete case are presented. Theorems giving the error estimates for the difference schemes are also presented.

As presented earlier, the outer region problem is solved by a classical finite difference scheme (see [26]), whereas the inner region problem is solved by an EFD scheme [26]. These schemes for the problem (2.1),(2.2) are, respectively, as given below

\[
P_1^h y_i := D^- y_{1,i} - y_{2,i} = 0, \quad \quad i = 1(1)n, \quad (3.1)
\]

\[
P_2^h y_i := -\varepsilon D^+ D^- y_{2,i} + a(x_i)D^+ y_{2,i} + b(x_i)y_{1,i} + c(x_i)y_{1,i} = f(x_i), \quad \quad i = 1(1)n - 1, \quad (3.2)
\]

and

\[
P_1^h y_i := D^- y_{1,i} - y_{2,i} = 0, \quad \quad i = 1(1)n, \quad (3.3)
\]

\[
P_2^h y_i := -\varepsilon \sigma(\rho) D^+ D^- y_{2,i} + a(x_i)D^+ y_{2,i} + b(x_i)y_{2,i} + c(x_i)y_{1,i} = f(x_i), \quad \quad i = 1(1)n - 1, \quad (3.4)
\]

where \( i \in \{0, 1, \ldots, n\}, 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1, \)

\[
x_i = x_0 + ih, \quad y_i = (y_{1,i}, y_{2,i}), \quad \quad i = 0(1)n,
\]

\[
D^- y_{1,i} = \frac{y_{1,i} - y_{1,i-1}}{h}, \quad \quad D^+ y_{2,i} = \frac{y_{2,i+1} - y_{2,i}}{h},
\]

\[
D^+ D^- y_{2,i} = \frac{y_{2,i+1} - y_{2,i}}{h^2} - \frac{y_{2,i} - y_{2,i-1}}{h^2},
\]

\[
\sigma(\rho) = \frac{\rho a(0)}{(1 - \exp(-\rho a(0)))}, \quad \quad \rho = \frac{h}{\varepsilon}.
\]

Analogues of the results for the continuous problem (2.1),(2.2) can be given for the discrete problems also.
3.1. Maximum Principle and Stability Result

**Theorem 3.1. Maximum Principle.** Consider the discrete problem (3.1),(3.2) or (3.3),(3.4). If \( y_{1,0} \geq 0, y_{2,0} \geq 0, y_{2,n} \geq 0, P_{1}^{h}y_{i} \geq 0 \) for \( 0 \leq i \leq n - 1 \) and \( P_{2}^{h}y_{i} \geq 0 \) for \( 0 \leq i \leq n \), then \( y_{i} \geq 0 \) for \( i = 0(1)n \).

**Proof.** Define \( s_{i} = (s_{1,i}, s_{2,i}) > 0 \) as \( s_{1,i} = (1 + 2\eta)x_{i} + \eta, s_{2,i} = 1 + \eta - x_{i}, i = 0(1)n, \) and \( \eta \) is as in condition (1.6). Then, we can easily prove that

\[
P_{1}^{h}s_{i} > 0, \quad i = 1(1)n, \quad P_{2}^{h}s_{i} > 0, \quad i = 1(1)n - 1.
\]

Using the basic idea underlying in the method of proof adopted for the continuous problem, with the definition that

\[
\xi = \max \left\{ \max_{0 \leq i \leq n} \frac{-y_{1,i}}{s_{1,i}}, \max_{0 \leq i \leq n} \frac{-y_{2,i}}{s_{2,i}} \right\},
\]

one can prove the present theorem.

**Lemma 3.1. Stability.** Consider problem (3.1),(3.2) or (3.3),(3.4). If \( y_{i} \) is any mesh function, then

\[
\|y_{i}\| \leq C \max \left\{ |y_{1,0}|, |y_{2,0}|, |y_{2,n}|, \max_{0 \leq i \leq n} |P_{1}^{h}y_{i}|, \max_{0 \leq i \leq n} |P_{2}^{h}y_{i}| \right\}, \quad i = 0(1)n,
\]

where \( \|y_{i}\| = \max\{|y_{1,i}|, |y_{2,i}|\} \) for \( 0 \leq i \leq n \).

**Proof.** Define two mesh functions \( W_{1,i}^{\pm} \) by

\[
W_{1,i}^{\pm} = C((1 + 2\eta)x_{i} + \eta) \max \left\{ |y_{1,0}|, |y_{2,0}|, |y_{2,n}|, \max_{0 \leq i \leq n} |P_{1}^{h}y_{i}|, \max_{0 \leq i \leq n} |P_{2}^{h}y_{i}| \right\} \pm y_{1,i},
\]

\[
W_{2,i}^{\pm} = C(1 + \eta - x_{i}) \max\{|y_{1,0}|, |y_{2,0}|, |y_{2,n}|, \max_{0 \leq i \leq n} |P_{1}^{h}y_{i}|, \max_{0 \leq i \leq n} |P_{2}^{h}y_{i}| \} \pm y_{2,i},
\]

\( i = 0(1)n \).

Following the method of proof of Lemma 2.1, for a proper choice of the constant \( C \) and using Theorem 3.1 we can obtain the desired bounds for \( y_{i} \).

3.2. Error Estimates

**Theorem 3.2.** The error in using the classical scheme (3.1),(3.2) to solve problem (2.1),(2.2), at the inner grid points \( \{x_{i}, i = 1, 2, \ldots, n - 1\} \) satisfies

\[
\|y(x_{i}) - y_{i}\| = \begin{cases} 
Ch \left[ 1 + \epsilon^{-1} \exp \left( \frac{-\alpha x_{i}}{\epsilon} \right) \right], & \text{if } h \leq \epsilon, \\
Ch + C \exp \left( \frac{-\alpha x_{i}}{\epsilon} \right), & \text{if } h \geq \epsilon.
\end{cases}
\]

**Proof.** The consistency errors due to \( P_{1}^{h} \) and \( P_{2}^{h} \) are given by

\[
|P_{1}^{h}(y(x_{i}) - y_{i})| = \left| (P_{1}^{h} - P_{1})(y(x_{i})) \right| \leq C \int_{x_{i-1}}^{x_{i+1}} \left| y_{1}^{(2)}(t) \right| dt
\]

and

\[
|P_{2}^{h}(y(x_{i}) - y_{i})| = \left| (P_{2}^{h} - P_{2})(y(x_{i})) \right| \leq C \int_{x_{i-1}}^{x_{i+1}} \left( \epsilon \left| y_{2}^{(2)}(t) \right| + \left| y_{2}^{(2)}(t) \right| \right) dt.
\]

Moreover, \( y(x_{i}) \) and \( y_{i} \) agree at the end points. Following the method of proof given in [4] and Lemma 3.1, we can complete the proof of this theorem.
REMARK 3.1. The above theorem ensures the convergence of the numerical solutions in the outer region but not in the inner region, whereas the following theorem proves that the EFD scheme (3.3)(3.4) converges in the whole interval \([0, 1]\).

**Theorem 3.3.** The error in using the EFD scheme (3.3)(3.4) to solve (2.1)(2.2) satisfies

\[ \| (y(x_i) - y_i) \| \leq Ch^{2/3}. \]

**Proof.** Define \(y(x) = u(x) + v_0(x)\), where \(u\) is the smooth component, \(u = u_0 + \varepsilon u_1\) and \(v_0\) is the singular component as given in Theorem 2.4. The local truncation errors for \(u(x)\) due to \(P_1^h\) and \(P_2^h\) are given by

\[ |P_1^h (u(x_i) - u_i)| = |(P_1^h - P_1) (u(x_i))| \leq Ch, \]
\[ |P_2^h (u(x_i) - u_i)| = |(P_2^h - P_2) (u(x_i))| \leq -\varepsilon\sigma(\rho) + \varepsilon - \frac{1}{2} ha(0) \left| u_{02} (x_i) \right| + \frac{1}{2} h |a(x_i) - a(0)| \left| u_{02} (x_i) \right| + O \left( \frac{h^2}{\varepsilon^2} + \frac{h^3}{\varepsilon^3} \right). \]

\(u(x_i)\) and \(u_i\) agree at the end points. By Lemma 3.1, we get

\[ \| (u(x_i) - u_i) \| \leq C \left( h + \left( \frac{h^2}{\varepsilon^2} + \frac{h^3}{\varepsilon^3} \right) \right). \]  

(3.5)

Again, following the method of proof adopted in [26], we get

\[ \| (v_0(x_i) - v_0_i) \| \leq Ch, \quad \text{where } C \text{ is independent of } i, h, \text{ and } \varepsilon. \]  

(3.6)

Combining (3.5) and (3.6), we get the classical estimate

\[ \| (y(x_i) - y_i) \| \leq C \left( h + \left( \frac{h^2}{\varepsilon^2} + \frac{h^3}{\varepsilon^3} \right) \right). \]

On the other hand, one can check that

\[ \| (u_0(x_i) + v_0(x_i)) - y_i \| \leq C(h + \varepsilon). \]

(3.7)

But, from Theorem 2.2,

\[ \| y(x_i) - (u_0(x_i) + v_0(x_i)) \| \leq C\varepsilon. \]

(3.8)

From (3.7) and (3.8), we conclude that

\[ \| (y(x_i) - y_i) \| \leq C(h + \varepsilon), \]

which is a nonclassical estimate of the error. We make use of the classical estimate when \(h^{2/3} \leq \varepsilon\) and the nonclassical estimate when \(h^{2/3} \geq \varepsilon\) to have the estimate that is claimed.

4. BOUNDARY VALUE TECHNIQUE

In general, classical finite difference schemes fail to yield good approximation to singular perturbation problems throughout the domain of definition of differential equation. Therefore, one has to look for new, efficient methods for solving these problems. The 'boundary value technique' is one such method. Consider the problem (2.1), (2.2). Let \(k > 0\) be such that \(ke \ll 1\). We divide the domain into two subintervals \([0, k]\) and \([k, 1]\). From (2.1), (2.2), we derive two problems namely, the 'Inner region problem' in \([0, k]\), and the 'Outer region problem' in \([k, 1]\). To obtain boundary conditions at the transition point \(x = ke\), we use the zero-order asymptotic expansion...
or a suitable asymptotic expansion of the solution of (2.1), (2.2). In fact, we take the boundary values at \( x = k\varepsilon \) as
\[
p_1 = u_{01}(k\varepsilon) + v_{01}(k\varepsilon), \quad q_1 = u_{02}(k\varepsilon) + v_{02}(k\varepsilon).
\]
The inner region problem is given by
\[
y'_{1}(x) - y_{2}(x) = 0, \quad x \in (0, k\varepsilon],
-\varepsilon y''_{1}(x) + a(x)y'_{1}(x) + b(x)y_{1}(x) + c(x)y_{1}(x) = f(x), \quad x \in (0, k\varepsilon),
y_{1}(0) = p, \quad y_{2}(0) = q, \quad y_{2}(k\varepsilon) = q_1.
\]
(4.1)
(4.2)
To solve (4.1), (4.2) we use the following exponentially fitted difference scheme
\[
D^{-}y_{1,i} - y_{2,i} = 0, \quad i = 1(1)n,
-\varepsilon\sigma(\rho)D^{+}y_{2,i} + a(x_{i})D^{+}y_{2,i} + b(x_{i})y_{2,i} + c(x_{i})y_{1,i} = f(x_{i}), \quad i = 1(1)n - 1,
y_{1,0} = p, \quad y_{2,0} = q, \quad y_{2,n} = q_1,
\]
(4.3)
(4.4)
with
\[
\sigma(\rho) = \frac{\rho a(0)}{1 - \exp(-\rho a(0))}, \quad \rho = \frac{h_1}{\varepsilon}, \quad h_1 = \frac{k\varepsilon}{n},
D^{-}y_{1,i} = \frac{y_{1,i} - y_{1,i-1}}{h_1},
D^{+}y_{2,i} = \frac{y_{2,i+1} - y_{2,i}}{h_1},
D^{+}D^{-}y_{2,i} = \frac{y_{2,i+1} - 2y_{2,i} + y_{2,i-1}}{h_1^2}, \quad \text{and} \quad x_{i} = ih_1.
\]
The outer region problem is given by
\[
y'_{1}(x) - y_{2}(x) = 0, \quad x \in (k\varepsilon, 1],
-\varepsilon y''_{1}(x) + a(x)y'_{1}(x) + b(x)y_{1}(x) + c(x)y_{1}(x) = f(x), \quad x \in (k\varepsilon, 1),
y_{1}(k\varepsilon) = p_1, \quad y_{2}(k\varepsilon) = q_1, \quad y_{2}(1) = r.
\]
(4.5)
(4.6)
To solve (4.5), (4.6) numerically, we apply the following classical finite difference scheme
\[
D^{-}y_{1,i} - y_{2,i} = 0, \quad i = 1(1)n,
-\varepsilon D^{+}D^{-}y_{2,i} + a(x_{i})D^{+}y_{2,i} + b(x_{i})y_{2,i} + c(x_{i})y_{1,i} = f(x_{i}), \quad i = 1(1)n - 1,
y_{1,0} = p_1, \quad y_{2,0} = q_1, \quad y_{2,n} = r,
\]
(4.7)
(4.8)
where
\[
D^{-}y_{1,i} = \frac{y_{1,i} - y_{1,i-1}}{h_2},
D^{+}y_{2,i} = \frac{y_{2,i+1} - y_{2,i}}{h_2},
D^{+}D^{-}y_{2,i} = \frac{y_{2,i+1} - 2y_{2,i} + y_{2,i-1}}{h_2^2}, \quad h_2 = \frac{1 - k\varepsilon}{n} \text{ and } x_{i} = k\varepsilon + ih_2.
\]
The algebraic systems (4.3), (4.4) and (4.7), (4.8) are linear and are solved by the self-correcting LU decomposition procedure [28].
After solving the inner and outer region problems, we consider their solutions to obtain an approximate solution to problem (2.1),(2.2) in the whole interval \([0,1]\). We repeat this process by increasing the value of \(k\) (thus widening the inner region) until the solution profiles do not differ much from iteration to iteration. For computational purposes we use an absolute error criterion, 
\[
\|y(x)^{m+1} - y(x)^m\| \leq \delta
\]
where \(y(x)^m\) is the \(m\)th iterative value of \(y\) and \(\delta\) is a prescribed tolerance bound.

**Remark 4.1.** Since the inner and outer region problems are independent of each other the present method is suitable for parallel computing.

**Remark 4.2.** According to Farrell [27], the exponentially fitted difference scheme is more effective inside the layer. Our numerical experiments show that, in general, the exponentially fitted schemes yield better results than the classical schemes inside the boundary layers.

**Remark 4.3.** This self-correcting LU decomposition algorithm has a special feature that it works even when the coefficient matrix is *nearly singular*, in which case, the usual LU decomposition scheme fails or unstable. Another interesting thing about this algorithm is that, while solving the system \(Ax = b\), inverse of the matrix \(A\) and the solution vector \(x\) are stored in \(A\) and the right-hand side column vector \(b\), respectively. This saves the memory space of the computer.

5. **ERROR ESTIMATES**

In this section, we derive error estimates for the numerical solutions obtained by the method described in the previous section. The method of proof is the same as that adopted by [18]. The discussion is carried out in an arbitrary interval \([A, B]\). Consider the following BVPs:

\[
\begin{align*}
y_1(x) - y_2(x) &= 0, \quad x \in (A, B], \\
-\varepsilon y_1''(x) + a(x)y_1'(x) + b(x)y_1(x) + c(x)y_1(x) &= f(x), \quad x \in (A, B), \\
y_1(A) &= \mu, \quad y_2(A) = \nu, \quad y_2(B) = \lambda,
\end{align*}
\]

and

\[
\begin{align*}
y_1'(x) - y_2(x) &= 0, \quad x \in (A, B], \\
-\varepsilon y_1''(x) + a(x)y_1'(x) + b(x)y_1(x) + c(x)y_1(x) &= f(x), \quad x \in (A, B), \\
y_1(A) &= \mu, \quad y_2(A) = \nu, \quad y_2(B) = \lambda + O(\varepsilon).
\end{align*}
\]

Let \(y(x)\) and \(y^1(x)\) be the solutions of (5.1),(5.2) and (5.3),(5.4), respectively. Further, let \(w(x) = y(x) - y^1(x)\). Then, \(w\) satisfies

\[
\begin{align*}
w_1'(x) - w_2(x) &= 0, \quad x \in (A, B], \\
-\varepsilon w_1''(x) + a(x)w_1'(x) + b(x)w_1(x) + c(x)w_1(x) &= 0, \quad x \in (A, B), \\
w_1(A) = 0, \quad w_2(A) = 0, \quad w_2(B) = O(\varepsilon).
\end{align*}
\]

Applying Lemma 2.1, we have \(\|w(x)\| \leq C\varepsilon\). That is,

\[
\|y(x) - y^1(x)\| \leq C\varepsilon, \quad \text{for} \ x \in [A, B].
\]

Discretizing (5.3),(5.4) using the exponentially fitted scheme described in the last section, we have

\[
\begin{align*}
D^-y_{1,i} - y_{2,i} &= 0, \quad i = 1(1)n, \\
-\varepsilon\sigma(\rho)D^+D^-y_{2,i} + a(x_i)D^+y_{2,i} + b(x_i)y_{2,i} + c(x_i)y_{1,i} &= f(x_i), \quad i = 1(1)n - 1, \\
y_{1,0} &= \mu, \quad y_{2,0} = \nu, \quad y_{2,n} = \lambda + O(\varepsilon).
\end{align*}
\]
Let \( y(x_i) \) and \( y_i \) be the solutions of (5.1),(5.2) and (5.6),(5.7), respectively. Then, using Theorem 3.3 and the inequality (5.5), we see that

\[
\| (y(x_i) - y_i) \| \leq C \left( h^{2/3} + \varepsilon \right), \quad \text{for } x_i \in [A, B], \quad i = 0(1)n.
\]

In particular, we have the following result.

**THEOREM 5.1.** If \( y(x_i) \) and \( y_i \) are the solutions of (2.1),(2.2) and (4.3),(4.4), respectively, then

\[
\| (y(x_i) - y_i) \| \leq C \left( h^{2/3} + \varepsilon \right), \quad \text{for } x_i \in [0, k\varepsilon], \quad i = 0(1)n.
\]

Consider the BVP

\[
y_1'(x) - y_2(x) = 0, \quad x \in (A, B],
\]

\[
-\varepsilon y_2''(x) + a(x)y_2'(x) - b(x)y_2(x) + c(x)y_1(x) = f(x), \quad x \in (A, B),
\]

\[
y_1(A) = \mu + O(\varepsilon), \quad y_2(A) = \nu + O(\varepsilon), \quad y_2(B) = \lambda.
\]

Discretize this using the classical scheme described in (4.7),(4.8),

\[
D^r y_{1,i} - y_{2,i} = 0, \quad i = 1(1)n,
\]

\[
-\varepsilon D^r D^s y_{2,i} + a(x_i)D^r y_{2,i} + b(x_i)y_{2,i} + c(x_i)y_{1,i} = f(x_i), \quad x \in (A, B),
\]

\[
y_{1,0} = \mu + O(\varepsilon), \quad y_{2,0} = \nu + O(\varepsilon), \quad y_{2,n} = \lambda.
\]

Hence, the discretization error is given by (Theorem 3.2)

\[
\| y(x_i) - y_i \| \leq \begin{cases} Ch \left[ 1 + \varepsilon^{-1} \exp \left( \frac{-\alpha x_i}{\varepsilon} \right) \right], & \text{if } h \leq \varepsilon, \\ Ch + C \exp \left( \frac{-\alpha x_{i-1}}{\varepsilon} \right), & \text{if } h \geq \varepsilon, \quad i = 1(1)n - 1, \quad x_i \in (A, B). \end{cases}
\]

Let \( y(x) \) and \( y^2(x) \) be the solutions of (5.1),(5.2) and (5.8),(5.9), respectively. Then, by following the earlier arguments, we get an inequality similar to (5.5), that is,

\[
\| y(x) - y^2(x) \| \leq C\varepsilon, \quad \text{for } x \in [A, B]. \quad (5.12)
\]

These discussions lead to the following theorem.

**THEOREM 5.2.** Let \( y(x) \) and \( y^*_i \) be the solutions of (5.1),(5.2) and (5.10),(5.11), respectively. Then,

\[
\| y(x_i) - y^*_i \| \leq \begin{cases} C \left[ (h + \varepsilon) + h\varepsilon^{-1} \exp \left( \frac{-\alpha x_i}{\varepsilon} \right) \right], & \text{if } h \leq \varepsilon, \\ C \left[ (h + \varepsilon) + \exp \left( \frac{-\alpha x_{i-1}}{\varepsilon} \right) \right], & \text{if } i = 1(1)n - 1, \quad x_i \in (A, B). \end{cases}
\]

**PROOF.**

\[
\| y(x_i) - y^*_i \| \leq \| y(x_i) - y^2(x_i) \| + \| y^2(x_i) - y^*_i \|
\]

\[
\leq \begin{cases} C\varepsilon + Ch \left[ 1 + \varepsilon^{-1} \exp \left( \frac{-\alpha x_i}{\varepsilon} \right) \right], & \text{if } h \leq \varepsilon, \\ C\varepsilon + Ch + C \exp \left( \frac{-\alpha x_{i-1}}{\varepsilon} \right), & \text{if } h \geq \varepsilon, \quad i = 1(1)n - 1. \end{cases}
\]
That is,

\[\|y(x_i) - y^*_i\| \leq \begin{cases} 
C \left[ (h + \varepsilon) + h\varepsilon^{-1} \exp \left( \frac{-\alpha x_i}{\varepsilon} \right) \right], & \text{if } h \leq \varepsilon, \\
C \left[ (h + \varepsilon) + \exp \left( \frac{-\alpha x_i - 1}{\varepsilon} \right) \right], & \text{if } h \geq \varepsilon, 
\end{cases} \]

\[i = 1(1) n - 1, \quad x_i \in (A, B).\]

In particular, we have the following statement.

**Theorem 5.3.** If \(y(x_i)\) and \(y^*_i\) are the solutions of (2.1),(2.2) and (4.7),(4.8), respectively, then

\[\|y(x_i) - y^*_i\| \leq \begin{cases} 
C \left[ (h_2 + \varepsilon) + h_2\varepsilon^{-1} \exp \left( \frac{-\alpha x_i}{\varepsilon} \right) \right], & \text{if } h_2 \leq \varepsilon, \\
C \left[ (h_2 + \varepsilon) + \exp \left( \frac{-\alpha x_i - 1}{\varepsilon} \right) \right], & \text{if } h_2 \geq \varepsilon, 
\end{cases} \]

\[i = 1(1) n - 1, \quad x \in (k\varepsilon, 1),\]

at the inner grid points of \([k\varepsilon, 1]\).

### 6. A COMPUTATIONAL PROCEDURE

In [29], the BVT discussed in the last section is applied to obtain numerical solution of the differential equation (1.1) subject to the boundary conditions (1.2) under assumptions (1.3)-(1.5). The same technique is applied to solve the following BVPs (convection-diffusion problems) [30]:

\[-\varepsilon y'' + b(x)y'(x) + c(x)y(x) = f(x), \quad x \in D,\]

\[y(0) = p, \quad y'(0) = q, \quad y'(1) = r,\]

where \(\varepsilon\) is a small positive parameter, \(b(x), c(x), \text{ and } f(x)\) are sufficiently smooth functions, \(b(x) \geq \beta > 0, 0 \geq c(x) \geq -\gamma, \gamma > 0, 0 = \gamma(1 + 2\delta) \geq \eta > 0, D = (0, 1), D_0 = (0, k\varepsilon), \bar{D} = [0, 1],\) and \(y \in C^3(D) \cap C(\bar{D}).\) In this section, a computational method which uses both BVT and an a priori information of the solution (zero-order asymptotic expansion of the solution is assumed to be known) is suggested to solve the BVP (1.1),(1.2). The method is described below.

Consider the BVP (2.1),(2.2). Replacing \(y_1\) by \(y_{1as}\) (refer to Section 2 for \(y_{1as}\)) in the second differential equation, (2.1),(2.2) becomes a decoupled system as shown below:

\[y_1'(x) - y_2(x) = 0, \quad x \in D_0, \quad y_1(0) = p, \quad y_2(0) = q, \quad y_2(1) = r, \quad (6.1)\]

\[-\varepsilon y_2''(x) + a(x)y_2'(x) + b(x)y_2(x) = f(x) - c(x)y_{1as}(x), \quad x \in D, \quad (6.2)\]

We now apply the BVT, described in the last section, to this decoupled system. That is, the domain is divided into two nonoverlapping intervals namely, \([0, k\varepsilon]\) and \([k\varepsilon, 1]\). Then, the classical scheme of Section 4 is applied to the BVP (6.1),(6.2) on the interval \([k\varepsilon, 1]\). On the interval \([0, k\varepsilon]\) the EFD scheme of Section 4 is applied to the BVP (6.1),(6.2). Combining these solutions we get a numerical solution of (6.1),(6.2) on the whole interval \([0, 1]\). It may be noted that, on each interval, first a numerical solution for \(y_2\) is obtained by solving the second equation and then using this solution we solve the first equation for \(y_1\). More precisely, to solve for \(y_1\) on the interval \([0, k\varepsilon]\), we first solve for \(y_2\) in \([0, k\varepsilon]\) and then use this solution and the initial value of \(y_1\) at \(x = 0\). For the interval \([k\varepsilon, 1]\), we use the value of the zero-order asymptotic expansion approximation at the transition point \(x = k\varepsilon\) (initial point for \(y_1\)) and the calculated value of \(y_{2,i}\).

If decoupling is not done, then the number of unknowns for the self-correcting LU decomposition procedure would be \(2n - 2\), and the size of the matrix to be inverted would be \((2n - 2) \times (2n - 2)\). But, after decoupling the LU decomposition procedure is applied to a system (corresponding to \(y_2\) alone) with only \(n - 1\) unknowns. In this case, the size of the matrix is reduced to half. This
reduces the computation time very much (this was experienced by us when we performed numerical experiments). This is the main advantage of our new method. Also, the problem $\text{(6.1),(6.2)}$ can be solved on the two subintervals said above simultaneously. That is, the method is well suited for parallel computing. The error estimates for the numerical solution of $\text{(6.1),(6.2)}$ are discussed as follows.

**Remark 6.1.** The relation between the solutions of the BVP $\text{(2.1),(2.2)}$ and $\text{(6.1),(6.2)}$ is given in the following theorem.

**Theorem 6.1.** If $y(x)$ and $y^1(x)$ are the solutions of $\text{(2.1),(2.2)}$ and $\text{(6.1),(6.2)}$, then $\| y(x) - y^1(x) \| \leq C \varepsilon$.

**Proof.** Subtracting $\text{(6.1),(6.2)}$ from $\text{(2.1),(2.2)}$, we have

$$
\begin{align*}
    w_1'(x) - w_2(x) &= 0, \\
    -\varepsilon w_2''(x) + a(x)w_2'(x) + b(x)w_2(x) &= c(x)[y_1(x) - y_{\text{as}}(x)], \\
    w_1(0) &= 0, \quad w_2(0) = 0, \quad w_2(1) = 0,
\end{align*}
$$

where $w(x) = (w_1(x),w_2(x)) = y(x) - y^1(x)$. Applying Lemma 2.1 for the above system, we have,

$$
\| w(x) \| \leq C \| y - y_{\text{as}} \| \leq C \varepsilon.
$$

That is, $\| y(x) - y^1(x) \| \leq C \varepsilon$.

**Remark 6.2.** The error committed in decoupling $\text{(2.1),(2.2)}$ is of order $O(\varepsilon)$. It may be noted that, the form of the error estimates presented in Theorems 5.1–5.3 remain the same for the BVP $\text{(2.1),(2.2)}$, irrespective of the fact that system $\text{(2.1),(2.2)}$ is weakly coupled ($c(x) \neq 0$ for some $x \in [0,1]$) or decoupled ($c(x) = 0$ for all $x \in [0,1]$). This follows from Theorems 5.1–5.3 and 6.1. Therefore, Theorems 5.2, 5.3, and 6.1 will yield the necessary error estimates for problem $\text{(6.1),(6.2)}$.

### 7. Augmented System [31–33]

Consider the BVP $\text{(2.1),(2.2)}$ and suppose that condition

$$
0 \geq c(x) \geq -\gamma, \quad \gamma > 0, \tag{7.1}
$$

is not satisfied, that is, system $\text{(2.1),(2.2)}$ is not quasi-monotone [32, Definition 2.1]. Then, we augment system $\text{(2.1),(2.2)}$ as the following system:

$$
A\hat{y} = F \Leftrightarrow
\begin{cases}
    \hat{y}'(x) - \hat{y}_2(x) = 0, & x \in D_0, \\
    -\varepsilon \hat{y}_2''(x) + a(x)\hat{y}_2'(x) + b(x)\hat{y}_2(x) - c^+(x)\hat{y}_3(x) + c^-(x)\hat{y}_1(x) = f(x), & x \in D, \\
    \hat{y}_3'(x) - \hat{y}_4(x) = 0, & x \in D_0, \\
    -\varepsilon \hat{y}_4''(x) + a(x)\hat{y}_4'(x) + b(x)\hat{y}_4(x) - c^+(x)\hat{y}_1(x) + c^-(-x)\hat{y}_3(x) = f(x), & x \in D, \\
    \hat{y}_1(0) = -p, \quad \hat{y}_2(0) = -q, \quad \hat{y}_2(1) = -r, \\
    \hat{y}_3(0) = p, \quad \hat{y}_4(0) = q, \quad \hat{y}_4(1) = r,
\end{cases} \tag{7.2}
$$

where $c^+(x) := c(x)$, if $c(x) \geq 0$ and zero otherwise. $c^-(x) := c(x) - c^+(x)$ and $\hat{y} = (\hat{y}_1,\hat{y}_2,\hat{y}_3,\hat{y}_4)$.

It is easy to verify that if $y = (y_1,y_2)$ is a solution of $\text{(2.1),(2.2)}$ then $\hat{y} = (-y_1,-y_2,y_1,y_2)$ is a solution of the above problem $\text{(7.2),(7.3)}$. It may be proved that all the results derived earlier for the BVP $\text{(2.1),(2.2)}$ are still valid even if the condition (7.1) is not met. For the sake of illustration, an example is provided in Section 9. We now conclude that the computational method can be applied to problems $\text{(2.1),(2.2)}$ irrespective of the fact that whether condition (7.1) is met or not.
8. NONLINEAR PROBLEMS

Consider the semilinear BVP
\[ ey^{(3)}(x) = F(x, y, y', y''), \quad x \in D, \]  
\[ y(0) = p, \quad y'(0) = q, \quad y'(1) = r, \]  
where \( F(x, y, y', y'') \) is a smooth function such that
\[ F_x, (x, y, y', y'') > 0, \quad \alpha > 0, \]
\[ F_v(x, y, y', y'') > -\gamma, \quad \gamma > 0, \]
for some \( \eta \) and \( \lambda \).

Assume that the reduced problem
\[ F(x, y, y', y'') = 0, \quad y'(1) = r, \]  
\[ y(0) = p, \]  
\[ y'(0) = q, \]  
\[ y'(1) = r, \]  
has a solution \( y_0 \in C^3(\bar{D}) \). Then, (8.1),(8.2) has a unique solution and has boundary layer of width \( O(\varepsilon) \) near \( x = 0 \) [5]. Analytical results such as existence, uniqueness, and asymptotic behaviour of the solution of (8.1),(8.2) can be found in [5,8,9]. In order to obtain numerical solution of (8.1),(8.2), the Newton’s method of quasilinearization [26] is applied. Consequently, we get a sequence \( \{y[m]\}_m^\infty \) of successive approximations with a proper choice of initial guess \( y[0] \) (mentioned above, will be a proper initial guess). Then, define \( y[m+1] \), for each fixed nonnegative integer \( m \), to be the solution of the following linear problem:
\[ -\varepsilon (y^{(3)})^{[m+1]} + a^m(x)(y'')^{[m+1]} + b^m(x)(y')^{[m+1]} + c^m(x)y^{[m+1]} = F^m(x), \]
\[ y^{[m+1]}(0) = p, \quad (y')^{[m+1]}(0) = q, \quad (y')^{[m+1]}(1) = r, \]
where
\[ a^m(x) = F_{y'}(x, y, y', y''), \]
\[ b^m(x) = F_y(x, y, y', y''), \]
\[ c^m(x) = F''(x, y, y', y''). \]

**Remark 8.1.** If the initial guess \( y[0] \) is sufficiently close to the solution \( y(x) \) of (8.1),(8.2) then, following the method of proof given in [26], one can prove that the sequence \( \{y[m]\}_m^\infty \) converges to \( y(x) \).

**Remark 8.2.** Problem (8.5),(8.6) for each fixed \( m \), is a linear BVP of third order and is of form (1.1),(1.2). Hence, it can be solved by the computational method given in Section 6.

**Remark 8.3.** The zero-order asymptotic approximation of the solution of (8.1),(8.2) is usually taken as the initial guess \( y[0] \) to generate the successive approximations \( \{y[m]\} \).

**Remark 8.4.** For the above Newton’s quasilinearization process, the following convergence criterion is used,
\[ |y^{[m+1]}(x_j) - y^{[m]}(x_j)| \leq \tau, \quad x_j \in \bar{D}, \quad m \geq 0. \]
9. ILLUSTRATIONS

In this section, we present three examples to illustrate the method described in this paper.

**Example 1.** Consider the BVP

\[-\varepsilon y^{(3)}(x) - 2y''(x) + 4y'(x) - y(x) = 4 \left[ \frac{1 - e^{-2x/\varepsilon}}{2} - \frac{x}{2} \right] + \frac{\varepsilon (1 - e^{-2x/\varepsilon})}{4 (1 - e^{-2/\varepsilon})} + \frac{x^2}{4} - x \left( 1 + \frac{1}{2} \left( 1 - e^{-2/\varepsilon} \right) \right),\]

\[y(0) = 1,\ y'(0) = 1,\ y'(1) = 1.\] It can be reduced to a weakly coupled system of equations. Table 1 gives the numerical results obtained for this BVP using the computational method presented in this paper.

<table>
<thead>
<tr>
<th>Nodes</th>
<th>Error</th>
<th>Solution</th>
<th>Derivative</th>
</tr>
</thead>
<tbody>
<tr>
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<td>0.0000000E+00</td>
<td>0.0000000E+00</td>
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</tr>
<tr>
<td>0.2*eps</td>
<td>0.8214481E-08</td>
<td>0.2847837E-09</td>
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<td>0.6*eps</td>
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<td>0.6506715E-08</td>
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<td>eps</td>
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<td></td>
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<tr>
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</tr>
<tr>
<td>3*eps</td>
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<td></td>
</tr>
<tr>
<td>0.3002100E+00</td>
<td>0.7468972E-03</td>
<td>0.4678244E-06</td>
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</tr>
<tr>
<td>0.5001500E+00</td>
<td>0.1246543E-02</td>
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<tr>
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</tr>
</tbody>
</table>

\(\varepsilon = 1.0E-05.\)

**Example 2.** Consider the BVP

\[-\varepsilon y^{(3)}(x) - 2y''(x) + 4y'(x) + y(x) = 2 + 4 \left( 1 - \frac{x}{2} + \frac{1 - e^{-2x/\varepsilon}}{2} \right) \]

\[\frac{\varepsilon (1 - e^{-2x/\varepsilon})}{4 (1 - e^{-2/\varepsilon})} + \frac{x^2}{4} - x \left( 1 + \frac{1}{2} \left( 1 - e^{-2/\varepsilon} \right) \right),\]

\[y(0) = 1,\ y'(0) = 1,\ y'(1) = 1.\] Since this system does not satisfy the condition that \(0 \geq c(x) \geq -\gamma, \gamma > 0,\) one has to go for an augmented system, as mentioned in Section 7. The numerical results are presented in Table 2.

**Example 3.** Consider the nonlinear BVP

\[-\varepsilon y^{(3)}(x) - 2y''(x) + 4y'(x) - y^2(x) = 1 + 4 \left[ 1 - \frac{x}{2} + \frac{1 - e^{-2x/\varepsilon}}{2} \right] \]

\[\frac{1}{2} \left[ 1 - \frac{\varepsilon (1 - e^{-2x/\varepsilon})}{4 (1 - e^{-2/\varepsilon})} - \frac{x^2}{4} + x \left( 1 + \frac{1}{2} \left( 1 - e^{-2/\varepsilon} \right) \right) \right]^2,\]

\[y(0) = 1,\ y'(0) = 1,\ y'(1) = 1.\] Table 3 displays the numerical results.

**Table 1.**

<table>
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\(\varepsilon = 1.0E-05.\)
Tables 1–3 present numerical results in terms of errors for the solution and its derivative of the problems considered in Examples 1–3, respectively.

Remark 9.1. It may be noted, that the derivative error (error in $y_2$) is much smaller than the solution ($y_1$) error. The solution error is the sum of two errors. One is due to the derivative error and the other is the local truncation error due to Euler scheme, which is of order $O(h)$. This can be improved by using higher-order schemes.

10. CONCLUSIONS

As mentioned in the introduction of this paper, second-order singularly perturbed differential equations have been studied extensively from the computational point of view but only few results on the higher-order problems have been reported in the literature. In this paper, we presented a computational method, which made use of asymptotic expansion approximation and also the
An Asymptotic Numerical Method

boundary value technique. In turn, the BVT combined a classical finite difference scheme and an EFD scheme to obtain a numerical solution. The EFD scheme is used only in the boundary layer region to get better approximations. An important aspect of this computational method is that, because of the incorporation of the asymptotic expansion approximation of the solution in the second equation of the system, the given weakly coupled system gets reduced to a decoupled system. This has reduced the computational time because instead of finding the inverse of $n \times n$ matrix, it is enough to find the inverse of $(n/2 \times n/2)$ matrix. The idea of introducing the adjoint system presented in Section 7 is a new and novel approach. The problems discussed in this paper belong to the category known as nonturning point problems. In the future, we plan to extend our method to turning point problems, various boundary conditions and to systems of the form

$$y' = g(x,y,z), \quad y(0) = p, \quad \varepsilon z'' = f(x,y,z,z'), \quad z(0) = q, \quad z(1) = r.$$ 

REFERENCES


30. S. Valarmathi and N. Ramanujam, Boundary value technique for finding numerical solution to boundary value problems for third order singularly perturbed ordinary differential equations (revised and resubmitted).

