K-Functionals and Moduli of Smoothness of Functions Defined on Compact Metric Spaces

C. BADEA
Mathématique, Bât. 425, Université de Paris-Sud
91405 Orsay cedex, France
badea@anh.matups.fr

Abstract—One proves that Peetre's K-functional for the couple \((C(X), \text{Lip}(X))\) and the first order modulus of continuity are equivalent for every Peano continuum. Some inequalities for the first order modulus of continuity and the mixed modulus of smoothness of functions defined on some compact metric spaces are proved and some possible applications are indicated.

Keywords—Moduli of smoothness, K-Functional, Metric spaces, Quantitative Korovkin theorems, Jackson-type theorems.

1. INTRODUCTION

Let \(X\) be a compact metric space with metric \(d = d_X\) and containing at least two distinct points. All metric spaces will be separable. Let \(C(X)\) denote the Banach space of all continuous real-valued functions defined on \(X\) endowed with the supremum norm \(\|f\| = \sup\{|f(x)| : x \in X\}\). The (first) modulus of continuity of a function \(f \in C(X)\) is defined by

\[
\omega(f,t) = \sup \{ |f(x) - f(y)| : x, y \in X, d_X(x,y) \leq t \}
\]

for every positive number \(t\). For a linear subspace \(E\) of \(C(X)\) endowed with the seminorm \(\|E\|\), Peetre's K-functional is defined as

\[
K(t, f; C(X), E) = \inf \{\|f - g\| + \|g\| : g \in E\}.
\]

The fact that \(\lim_{t \to 0^+} \omega(f,t) = 0\) expresses the continuity of \(f\) on \(X\). The relation \(\lim_{t \to 0^+} K(t, f; C(X), E) = 0\) holds for any \(f \in C(X)\) if and only if \(E\) is dense in \(C(X)\) with respect to the sup-norm. We refer to [1] for several other properties of the K-functionals. In particular, for each fixed \(f\) in \(C(X)\), \(K(\cdot, f) : \mathbb{R}_+ \to \mathbb{R}_+\) is a continuous, monotonically increasing, and concave function and it verifies the inequality

\[
K(\lambda t, f) \leq \max\{1, \lambda\} \cdot K(t, f),
\]

for arbitrary \(\lambda, t\) in \(\mathbb{R}_+\).

Consider the subspace \(\text{Lip}(X)\) of all Lipschitz functions in \(C(X)\), that is, the set of all functions \(g\) such that

\[
|g|_{\text{Lip}} = \sup \left\{ \frac{|g(x) - g(y)|}{d(x,y)} : d_X(x,y) > 0 \right\} < \infty.
\]

Then \(| \cdot |_{\text{Lip}}\) is a seminorm on \(\text{Lip}(X)\) and \(\text{Lip}(\star)\) is dense in \(C(X)\). Thus, we can consider the K-functional \(K(t, f; C(X), \text{Lip}(X))\) as a measure of the continuity of the function \(f\); if \(f\) is continuous, we have \(\lim_{t \to 0^+} K(t, f; C(X), \text{Lip}(X)) = 0\).
For an interval \([a, b]\) of the real axis, it is well known [1,2] that the modulus of continuity is equivalent to the \(K\)-functional:
\[
C^{-1} \omega(f, t) \leq K(t, f; C[a, b], C^1[a, b]) \leq C \omega(f, t).
\]
Here, \(C > 0\) is a constant not depending on \(f\), and \(C^1[a, b]\) is the space of all continuously differentiable functions on \([a, b]\). A similar result holds in the multivariate setting for (higher order) moduli of smoothness and domains \(\Omega\) in \(\mathbb{R}^n\) such that \(\partial \Omega\) is minimally smooth (cf., [2,3] for proofs and definitions). Note also that the condition that \(\partial \Omega\) is minimally smooth is equivalent to the strong local Lipschitz property (4.5) of [4]. For \(X = [a, b]\), we have
\[
K(t, f; C(X), \text{Lip}(X)) = K(t, f; C[a, b], C^1[a, b]),
\]
(see [5,6]) and thus, in this case, \(K(t, f; C(X), \text{Lip}(X))\) is equivalent to \(\omega(f, t)\). It is natural to ask for which metric spaces \(X\) the \(K\)-functional \(K(t, f; C(X), \text{Lip}(X))\) is equivalent to \(\omega(f, t)\).

By a result due to Brudnyi (cf., [5; 6, p. 301; 7], we have
\[
\bar{\omega}(f, t) = \min\{2t, \omega(f, t)\},
\]
where \(\bar{\omega}(f, t)\) denotes the least concave majorant of \(\omega(f, t)\). Therefore, the above problem reduces to the one of the equivalence between \(\bar{\omega}(f, 2t)\) and \(\omega(f, t)\). Since we have \(\omega(f, t) \leq \omega(f, 2t) \leq \bar{\omega}(f, 2t)\), this is equivalent to the existence of an inequality of the form \(\bar{\omega}(f, 2t) \leq C \omega(f, t)\) with an absolute constant \(C > 0\).

The aim of this note is to discuss inequalities of this type. An example due to H. H. Gonska [8] (see Example 2.1) shows that there are metric spaces and continuous functions such that this inequality does not hold. However, it holds for every Peano continuum, that is for any locally connected, connected, compact metric space (Theorem 2.2). Corresponding inequalities are also proved for the mixed modulus (or \((1,1)\)-modulus) of smoothness. In the last section, one indicates some possible applications of these inequalities to quantitative Korovkin-type results and to abstract Jackson-type theorems for blending approximation.

2. THE CASE OF PEANO CONTINUA

We begin with the following example which shows that we cannot expect to have a positive answer to the above equivalence problem for all compact metric spaces. It is the same example used by Gonska [8] to show that there are metric spaces \(X\) and continuous functions \(f\) such that the inequality \(\bar{\omega}(f, t) \leq C \omega(f, t)\) does not hold for all \(t \geq 0\), for any \(C > 0\).

**Example 2.1.** ([8, Remark 3.5]) Let \(X = [0, 0.25] \cup [0.75, 1]\). This is a compact metric space with respect to the restriction of the Euclidean metric in \(\mathbb{R}\) to it. The function \(f\) given by \(f(x) = 1\) for \(x \in [0, 0.25]\) and \(f(x) = 2\) for \(x \in [0.75, 1]\) is continuous on \(X\), and its metric modulus of continuity is given by \(\omega(f, t) = 0\) for \(0 \leq t < 0.5\) and \(\omega(f, t) = 1\) for \(0.5 \leq t \leq 1\). Then, the least concave majorant \(\bar{\omega}(f, t)\) is the function given by \(\bar{\omega}(f, t) = \min\{2t, 1\}\). The inequality \(\bar{\omega}(f, 2t) \leq C \omega(f, t)\) cannot hold for all \(t \geq 0\) for any \(C > 0\). This shows that for \(X\) given above, the \(K\)-functional for \((C(X), \text{Lip}(X))\) is not equivalent to the first modulus of continuity.

We will prove that if \(X\) is a Peano continuum, then the situation in Example 2.1 cannot happen. We refer, for instance, to the recent book by Nadler [9] for the terminology we use and for the basic theory of Peano continua. We recall that a continuum is a nonempty, compact and connected metric space. A Peano space is a metric space \(X\) such that for every \(p \in X\) and every neighborhood \(N\) of \(p\), there is a connected open subset \(U\) of \(X\) such that \(p \in U \subset N\). A Peano continuum is a Peano space and a continuum. By the Hahn-Mazurkiewicz theorem [9, p. 126], Peano continua coincides with continuous images of the unit interval \([0,1]\). This explains why
Peano continua are sometimes called continuous curves. A compact metric space \( X \) is Peano if and only if [9, Section 8.3] \( X \) has Property \( S \) due to Sierpinski, that is, for each positive number \( \varepsilon \), \( X \) is the union of finitely many connected subsets having the diameter less than \( \varepsilon \). We will prove the following theorem.

**Theorem 2.2.** Let \( X \) be a Peano continuum, that is, a locally connected, connected, compact metric space. Then, the \( K \)-functional \( K(t, f; C(X), \operatorname{Lip}(X)) \) is equivalent to \( \omega(f, t) \).

In view of Brudnyi’s lemma mentioned in the Introduction, it is sufficient to prove an inequality like \( \bar{\omega}(f, 2t) \leq C\omega(f, t) \) for the least concave majorant of \( \omega \) and \( \omega \) itself. It was shown by N. P. Kornéjuk [10, p. 670], that if for a function \( f \in C(X) \), the modulus of continuity \( \omega(f, \cdot) \) is continuous, nondecreasing and subadditive, then for any \( t \geq 0 \) and \( \xi > 0 \), we have

\[
\bar{\omega}(f, \xi t) \leq (1 + \xi)\omega(f, t),
\]

and this inequality cannot be improved for each \( t > 0 \) and \( \xi = 1, 2, \ldots \). This implies \( \bar{\omega}(f, 2t) \leq 3\omega(f, t) \). The following generalization was proved by H. H. Gonska [8, Lemma 3.4].

**Lemma 2.3.** [8] Let \( (X, d) \) be a compact metric space such that for all \( \xi, t > 0 \), all \( f \in C(X) \), and some fixed \( \eta > 0 \), the inequality

\[
\omega(f, \xi t) \leq (1 + \eta\xi)\omega(f, t)
\]

(2.1)

holds, then for \( f \in C(X) \), and any real numbers \( \xi, t \geq 0 \), we have

\[
\bar{\omega}(f, \xi t) \leq (1 + \eta\xi)\omega(f, t).
\]

There are several examples of metric spaces for which the above inequality (2.1) holds. For instance, all compact convex subsets of a linear metric space with translation invariant metric (i.e., \( d(x + z, y + z) = d(x, y) \) for all \( x, y, z \) in \( X \)) and starshaped \( d(., 0) \) (i.e., \( d(flx, 0) \leq \beta d(x, 0) \) for all \( f \) in \([0,1]\) and all \( x \) in \( X \)) [11] and all compact metric spaces being metrically convex in the sense of K. Menger [12] verify (2.1) with \( \eta = 1 \) (see [11]). A metric space is said to be metrically convex if for every distinct point \( x, y \) in \( X \), there exists another one \( z \), different from \( x \) and \( y \) such that \( d(x, z) + d(z, y) = d(x, y) \). Any normed space is a metrically convex space and any metrically convex space is [12, p. 98] locally connected. Note that this notion of compact metrically convex spaces coincides with the notion of (compact) length spaces (“espaces de longueur”) due to M. Gromov [13]. These are defined as those spaces for which the intrinsic metric coincides with the given metric. The intrinsic metric associates to a pair of two points, the infimum of the lengths of all rectifiable arcs joining the points. If \( (Z, d_Z) \) is a metric space, \( \varphi : [0,1] \to Z \) is a parametrization of the simple Jordan arc \( \Gamma = \Gamma[\varphi(0), \varphi(1)] \), and if \( S = \{0 = t_0 < t_1 < \cdots < t_n = 1\} \) is a subdivision of \([0,1]\), then we define

\[
\text{length}(\varphi, S) = \sum_{i=1}^{n} d_Z(\varphi(t_{i-1}), \varphi(t_i)).
\]

The arc \( \Gamma \) is said to be rectifiable with length \( (\Gamma) \) if

\[
\text{length}(\Gamma) := \sup\{\text{length}(\varphi, S) : S \text{ a subdivision of } [0,1]\} < \infty.
\]

Length spaces are also called inner or internal metric spaces. It was recently proved [14] that the set of compact metrically convex spaces is the closure in the topology induced by the Gromov-Hausdorff metric (cf., [13,15]) of the subset of all two-dimensional Riemannian manifolds which can be isometrically embedded in the Euclidean three-dimensional space.
Other examples of compact metric spaces verifying (2.1) are spaces possessing a finite coefficient of convex deformation $\rho = \rho(X)$. In this case, the inequality (2.1) holds for $\eta = \rho$ [16]. A (rectifiable pathwise connected) metric space $(X, d)$ is said to have a coefficient of convex deformation $\rho = \rho(X)$ if the intrinsic metric $d_\rho$ is equivalent to $d$; that is, there exists $M > 0$ such that $d_\rho(x, y) \leq Md(x, y)$ for all $x, y \in X$. Then, $\rho = \rho(X)$ is defined as the infimum of such $M$. Therefore, these spaces are those for which the metric topology coincides with that induced by the intrinsic metric. With this definition, these spaces had appeared even in 1928 in the same article [12] by Menger in which the metrically convex spaces were introduced. In [17], spaces with a finite convex deformation coefficient are called "geometrically acceptable spaces," while "quasiconvex spaces" is the name used in [13,18].

In order to prove Theorem 2.2, we will use a deep result (conjectured by Menger) proved independently by R. H. Bing [19] and E. E. Moise [20]: for every Peano continuum, there is an equivalent metric with respect to which the space is metrically convex. A generalization of the Bing-Moise theorem for locally compact spaces is given in [21]. Theorem 2.2 will follow from Lemma 2.3 and part (iii) in the following result which collects some properties of the first modulus of continuity for functions defined on Peano continua. It shows that for Peano continua, the inequality (2.1) is true for a certain $\eta \geq 1$.

**Theorem 2.4.** Let $(X, d)$ be a Peano continuum. Then, there is a positive constant $C \geq 1$ (depending only on $X$) such that for any function $f \in C(X)$, the following are true.

(i) $\omega(f, 0) = 0$.

(ii) $\omega(f, \cdot)$ is a positive and nonincreasing function on $\mathbb{R}^+$. 

(iii) $\omega(f, \lambda \delta) \leq (1 + |C\lambda|) \omega(f, \delta)$ for $\lambda, \delta \in \mathbb{R}^+$.

(iv) $\omega(f, \delta_1 + \delta_2) \leq \omega(f, C\delta_1) + \omega(f, C\delta_2) \leq (1 + |C^2|) \omega(f, \delta_1) + \omega(f, \delta_2)$.

Here, $\lfloor x \rfloor$ is the least integer strictly smaller than $x$.

**Proof.** (i) and (ii) are immediate consequences of the definition of the modulus of continuity. Suppose that $X$ is a Peano continuum. To prove (iii), observe first that if $\delta = 0$, then (iii) is obvious. Suppose that $\delta > 0$ and consider $x, y \in X$ such that $0 < d(x, y) \leq \lambda \delta$. By the above mentioned result of Bing and Moise, there exists a metric $d'$ on $X$ such that $(X, d')$ is metrically convex and $C_1d(x, y) \leq d'(x, y) \leq C_2d(x, y)$ for all $x, y \in X$. Denote $C = C_2/C_1$ and set $n = 1 + |C\lambda|$. Since $(X, d')$ is compact, it is also finitely compact, and thus, a theorem of Hilbert (see [22, p. 141]) implies, since $X$ is rectifiable pathwise connected, that there is a shortest rectifiable arc $\Gamma_0[x, y]$ connecting $x$ and $y$ such that the infimum in the definition of the intrinsic metric is attained for $\Gamma_0[x, y]$. Because of the metric convexity of $(X, d')$, we obtain $d'(x, y) = \text{length}(\Gamma_0[x, y])$. Therefore, we have

$$\text{length}(\Gamma_0[x, y]) = d'(x, y) \leq C_2d(x, y) \leq C_2\lambda \delta \leq C_1n \delta.$$ 

Choosing $t_i = \delta/n$, $0 \leq i \leq n$, there exists (see [13,16]) a parametrization $\Psi$ such that, with $z_i = \Psi(t_i)$, we have

$$d(\Psi(t_i), \Psi(t_{i+1})) = \frac{1}{C_1}d'(\Psi(t_i), \Psi(t_{i+1})) \leq \frac{1}{C_1}\text{length}(\Gamma_0[\Psi(t_i), \Psi(t_{i+1})]) = (t_{i+1} - t_i) \frac{1}{C_1}\text{length}(\Gamma_0[x, y]),$$

which, combined with the above estimates, leads to $d(z_i, z_{i+1}) \leq \delta$ for $0 \leq i \leq n - 1$. Therefore,

$$|f(x) - f(y)| \leq \sum_{i=0}^{n-1} |f(z_i) - f(z_{i+1})| \leq \sum_{i=0}^{n-1} \omega(f, d(z_i, z_{i+1})) \leq n \omega(f, \delta).$$

This proves that

$$\omega(f, \lambda \delta) \leq n \omega(f, \delta) = (1 + |C\lambda|) \omega(f, \delta) \leq (1 + C\lambda) \omega(f, \delta).$$
The second inequality in (iv) follows from (iii), so it is sufficient to prove the first one. If \( \delta_1 = 0 \) or \( \delta_2 = 0 \), the inequalities in (iv) are true, since \( C \geq 1 \). Let \( \delta_1 > 0 \) and \( \delta_2 > 0 \) be two fixed positive numbers and let \((x, y) \in X\) such that \( d(x, y) \leq \delta_1 + \delta_2 \). Consider \( t_0 = 0, t_1 = \delta_1 / (\delta_1 + \delta_2), t_2 = 1 \). As above, there exists an arc \( \Gamma_0 \) and a parametrization \( \Psi \) such that, for \( i = 0, 1, 2 \),

\[
d(\Psi(t_i), \Psi(t_{i+1})) \leq \frac{1}{C_1} d'(\Psi(t_i), \Psi(t_{i+1})) \leq \frac{1}{C_1} \text{length}(\Gamma_0[\Psi(t_i), \Psi(t_{i+1})])
\]

\[
= (t_{i+1} - t_i) \frac{1}{C_1} \text{length}(\Gamma_0[x, y]) = (t_{i+1} - t_i) \frac{1}{C_1} d'(x, y)
\]

\[
\leq C(t_{i+1} - t_i) d(x, y) \leq C(t_{i+1} - t_i)(\delta_1 + \delta_2).
\]

Hence,

\[
|f(x) - f(y)| = |f(\Psi(t_0)) - f(\Psi(t_2))| \\
\leq |f(\Psi(t_0)) - f(\Psi(t_1))| + |f(\Psi(t_1)) - f(\Psi(t_2))| \\
\leq \omega(f, d(\Psi(t_1), \Psi(t_2))) + \omega(f, d(\Psi(t_1), \Psi(t_2))) \\
\leq \omega(f, C(t_1 - t_0)(\delta_1 + \delta_2)) + \omega(f, C(t_2 - t_1)(\delta_1 + \delta_2)) \\
\leq \omega(f, C\delta_1(\delta_1 + \delta_2)^{-1}(\delta_1 + \delta_2)) + \omega(f, C\delta_2(\delta_1 + \delta_2)^{-1}(\delta_1 + \delta_2)) \\
= \omega(f, C\delta_1) + \omega(f, C\delta_2).
\]

This implies \( \omega(f, \delta_1 + \delta_2) \leq \omega(f, C\delta_1) + \omega(f, C\delta_2) \) completing the proof of (iv).

**Remark 2.5.** The above proof gives us the inequality \( \omega(f, \lambda \delta) \leq (1 + C^* \lambda) \omega(f, \delta) \) with the constant

\[
C^* = \inf \left\{ \frac{C_2}{C_1} : \text{there exists a convex metric } d' \text{ with } C_1 \leq d' \leq C_2 d \right\}.
\]

If \( X \) has a coefficient of convex deformation \( \rho = \rho(X) \), then \( C^* \leq \rho \), since we have \( d_i \geq d \) and \( d_i \) is metrically convex. Therefore, in Theorem 2.4, (iii) implies one of Jiménez Pozo’s results [16].

In view of the applications for blending approximation which will be given in the next section, we present some similar results for the mixed modulus of continuity. For two compact metric spaces \((X, d_X)\) and \((Y, d_Y)\) and a bounded function \( f \) defined on the product \( X \times Y \), the **mixed modulus of smoothness** \( \omega_{1,1}(f; \cdot, \cdot) \) is defined by

\[
\omega_{1,1}(f; \delta_1, \delta_2) = \sup \{ |f(x_1, y_1) - f(x_1, y_2) - f(x_2, y_1) + f(x_2, y_2)| : d_X(x_1, x_2) \leq \delta_1, d_Y(y_1, y_2) \leq \delta_2 \},
\]

where the quantity in the sup represents a mixed difference. For some properties of the mixed modulus of smoothness, we refer to [8] and, for the product of two compact intervals of the real line, to [23, p. 516], where \( \omega_{1,1} \) is called the \((1,1)\)-modulus of smoothness. Clearly, if \( f \) is continuous on \( X \times Y \) with the product topology, we have \( \lim_{\delta_i \to 0} \omega_{1,1}(f; \delta_1, \delta_2) = 0, \ i = 1, 2 \). The class of functions \( f \) for which the above relation holds coincides with the class of so-called **Bögel functions** (strictly larger than \( C(X \times Y) \)) introduced by K. Bögel in 1934. We refer to [24] for further references and details. Answering a problem of G. Freud, it was proved in [24] that the space of bounded Bögel functions is isometrically isomorphic with the completion of the blending space \( C(X) \otimes M(Y) + M(X) \otimes C(Y) =: B(M(X), M(Y); C(X), C(Y)) \) with respect to a suitable norm. Here, \( M(Z) \) is the space of all bounded functions with the sup-norm on the compact metric space and \( \otimes \) is the tensor product associating to, for instance, \( f \in C(X) \) and \( g \in M(Y) \) the function \( f \otimes g \in C(X) \otimes M(Y) \) given by \( f \otimes g(x, y) = f(x)g(y) \). The mixed modulus of smoothness is equivalent in some cases to the mixed K-functional introduced by Cottin [25]. This is a variant of the K-functional suitable for blending approximation. However, we do not yet

have an analogue of Brudnyi’s result for the mixed $K$-functional (but see the conjecture in [25]). Nevertheless, the following analogue of Theorem 2.4 holds for the mixed modulus of smoothness. Particular cases were previously proved in [24,26] for spaces with finite coefficients of convex deformation.

**THEOREM 2.6.** Let $(X, d_X)$ and $(Y, d_Y)$ be Peano continua. Then, there are positive constants $C_X, C_Y ≥ 1$ (depending only on $X$ and $Y$) such that for any function $f$ bounded on $X \times Y$, the following are true for all $λ, μ, δ_1, δ_2, η_1, η_2 ∈ \mathbb{R}_+$.

(i) $ω_{1,1}(f; λ δ_1, μ δ_2) \leq (1+|C_X λ|)(1+|C_Y μ|)ω_{1,1}(f; δ_1, δ_2)$

(ii) $ω_{1,1}(f; δ_1 + δ_2, η_1 + η_2) ≤ ω_{1,1}(f; C_X δ_1, C_Y η_1) + ω_{1,1}(f; C_X δ_2, C_Y η_2)

\[ + ω_{1,1}(f; C_Y δ_1, C_Y η_2) + ω_{1,1}(f; C_Y δ_2, C_Y η_2) \]

\[ ≤ (1+|C_X|^2)(1+|C_Y|^2) \left( ω_{1,1}(f; δ_1, η_1) + ω_{1,1}(f; δ_1, η_2) \right) \]

\[ + ω_{1,1}(f; δ_2, η_1) + ω_{1,1}(f; δ_2, η_2) \].

**PROOF.** A proof similar to that presented above for the univariate modulus can be given. A shorter one can be given using some inequalities for $ω_{1,1}$ already proved in [26] for metrically convex spaces. We present such a proof for inequality (i). Let $r_X$ and $r_Y$ be two convex metrics on $X$ and $Y$, respectively, such that $C_1 d_X(x, y) ≤ r_X(x, y) ≤ C_2 d_X(x, y)$ and $c_1 d_Y(x, y) ≤ r_Y(x, y) ≤ c_2 d_Y(x, y)$. We will denote by $ω(f, δ_1, δ_2; d)$ and $ω(f, δ_1, δ_2; r)$ the mixed modules of smoothness with respect to metrics $d_X$ and $d_Y$ (as in the given definition) and, respectively, with respect to the metrics $r_X$ in $X$ and $r_Y$ in $Y$. Since $(X, r_X)$ and $(Y, r_Y)$ are convex, we have [26]

\[ ω(f, λ δ_1, μ δ_2; r) ≤ (1+|λ|)(1+|μ|)ω(f; δ_1, δ_2; r). \]

Because of the implications

\[ d_X(x, y) ≤ \frac{1}{C_2} δ_1 ⇒ r_X(x, y) ≤ δ_1 ⇒ d_X(x, y) ≤ \frac{1}{C_1} δ_1, \]

and the similar ones for $Y$, we have

\[ ω \left( f, \frac{1}{C_2} δ_1, \frac{1}{c_2} δ_2; d \right) ≤ ω(f, δ_1, δ_2; r) ≤ ω \left( f, \frac{1}{C_1} δ_1, \frac{1}{c_1} δ_2; d \right). \]

Therefore, denoting $C_X = C_2/C_1$ and $C_Y = c_2/c_1$, we can write

\[ ω(f, λ δ_1, μ δ_2; d) = ω \left( f, \frac{1}{C_2} λ C_2 δ_1, \frac{1}{c_2} μ C_2 δ_2; d \right) ≤ ω(f, λ C_2 δ_1, μ C_2 δ_2; r) \]

\[ = ω \left( f, \frac{C_2}{C_1} C_1 δ_1, \frac{C_2}{c_1} C_1 δ_2; r \right) ≤ (1+|C_X λ|)(1+|C_Y μ|)ω(f, C_1 δ_1, C_1 δ_2; r) \]

\[ ≤ (1+|C_X λ|)(1+|C_Y μ|)ω(f, δ_1, δ_2; d) \]

proving the inequality (i). The second one can be proved in a similar way using the result of [26].
3. APPLICATIONS

3.1. Quantitative Korovkin Theorems

The classical Korovkin theorem asserts that a sequence of bounded linear operators on $C[0,1]$ converges uniformly to the identity operator whenever it converges uniformly for three test functions $x \to 1$, $x \to x$, $x \to x^2$. There are several quantitative forms of this theorem. We refer to [8] for more information. The inequality (iii) in Theorem 2.4 above can be used to prove the following quantitative Korovkin-type theorem.

**Proposition 3.1.** Let $(X, d)$ be a Peano continuum and denote by $C \geq 1$ the same constant as in Theorem 2.4. Let $L$ be a positive linear operator from $C(X)$ into itself such that $L(1) = 1$, where $1$ denotes the constant function equal to 1 on $X$. Then,

$$|L(f)(x) - f(x)| \leq \left\{ 1 + C^{-1} |L(d^2(\cdot,x))(x)|^{1/2} \right\} \omega(f, \varepsilon)$$

holds for all $f \in C(X)$, $x \in X$, $\varepsilon > 0$.

**Proof.** The proof is similar to the proof of [27, Lemma 4], using the inequality (iii) of Theorem 2.4.

We refer to the recent book by Anastassiou [28] for some improved/optimal Korovkin inequalities for $X = [a, b]$.

3.2. Jackson-Type Theorems for Blending Approximation

The classical theorem of Jackson on the approximation of functions on $C[a, b]$ by polynomials says that $E_n(f) \leq C \omega(f, 1/n)$, where $C$ is a universal constant and

$$E_n(f) = \inf\{||f - p_n|| : p_n \text{ a polynomial of degree at most } n\}$$

denotes the best approximation functional. This was recently generalized for compact metric spaces by Stephani [29] who proved

$$a_n(f) \leq C \omega(f, \varepsilon_n(X)),$$

where $\omega$ is the first modulus of continuity for functions on the metric space $X$, $\varepsilon_n(X)$ denotes the $n$th entropy number of $X$ and $a_n(f)$ denotes the least deviation of $f$ to a certain class $\Phi_n$ in a nested sequence

$$\Phi_1 \subseteq \Phi_2 \subseteq \cdots \subseteq \Phi_n \subseteq \cdots,$$

having their union dense in $C(X)$. The $n$th entropy number $\varepsilon_n(X)$ of $X$ is defined as the infimum of all $\varepsilon > 0$ such that there are open balls $B(t, \varepsilon) = \{t \in X : d(t, t_i) < \varepsilon\}$, $1 \leq i \leq n$, of radius $\varepsilon$ which cover $X$. For $X = [a, b]$, we have $\varepsilon_n([a, b]) = (b - a)/2n$ and this was the starting point for Stephani's generalization. The class $\Phi_n$ replacing, in the case of metric spaces, the class of polynomials of degree at most $n$, is defined as the class of all continuous functions on $X$ which belong to a controllable partition subspace of dimension $k \leq n$. We refer to [24,29], for the definitions and more details.

Now, let $(X, d_X)$ and $(Y, d_Y)$ be two compact metric spaces with corresponding ascending chains $(\Phi_n)_{n \geq 1}$ and $(\Psi_n)_{n \geq 1}$, respectively. We are interested in proving some abstract Jackson-type theorems for blending approximation. These are expressed by replacing $\omega$ in the Jackson-type estimates with the mixed modulus of smoothness $\omega_{1,1}$. Since the functions from $B(M(X), M(Y)) \varsubseteq C(X) \otimes M(Y) + M(X) \otimes C(Y)$ are [24] dense in the class of functions verifying $\lim_{\delta_1, \delta_2 \to 0} \delta_{1,1}^2(f; \delta_1, \delta_2) = 0$, it is natural to define

$$a_{m,n}(f) = \inf \{||f - b_{m,n}|| : b_{m,n} \in B(M(X), M(Y); \Phi_m, \Psi_n)\}$$
as the least deviation of \( f \) from the space of abstract pseudopolynomials \( \Phi_m \otimes M(Y) + M(X) \otimes \Psi_n \) of order \((m, n)\) (in the sense of [24]). Several Jackson-type results due to Yu. A. Brudnyi, H. H. Gonska and K. Jetter, and B. H. Sendov and M. D. Takev are known for the usual space of pseudopolynomials in the case when \( X \) and \( Y \) are two compact intervals of the real axis (cf., the references cited in [24]). It is an open problem mentioned in [24] to find the most general class of compact metric spaces for which we have

\[
a_{m,n}(f) \leq C \omega_{1,1}(f; \varepsilon_m(X), \varepsilon_n(Y))
\]

for all bounded functions \( f \) on \( X \times Y \) with a constant \( C > 0 \) depending only on \( X \) and \( Y \). It was proved in [24] that (3.2) holds with \( C = 1 \) for all continuous functions \( f \) and for all metric spaces \( X \) and \( Y \). Moreover, if both of the spaces have finite coefficients of convex deformation, the inequality (3.2) holds for all bounded functions \( f \) with a suitable \( C \). The above inequalities for the mixed modulus for Peano continua and the results of [24] lead to an inequality like (3.2) for a more general class of metric spaces.

**THEOREM 3.2.** Let \( X \) and \( Y \) be Peano continua. Then, inequality (3.2) holds for all bounded functions on \( X \times Y \).

**PROOF.** It was proved in [24, Theorem 2.1] that one has

\[
a_{m,n}(f) \leq \inf \{ \omega_{1,1}(f; \varepsilon, \delta) : \varepsilon > \varepsilon_m(X), \delta > \varepsilon_n(Y) \}.
\]

Then,

\[
a_{m,n}(f) \leq \inf \{ \omega_{1,1}(f; \varepsilon, \delta) : \varepsilon > \varepsilon_m(X), \delta > \varepsilon_n(Y) \}
\]

\[
= \inf \{ \omega_{1,1}(f; \varepsilon, \delta) : 2\varepsilon_m(X) > \varepsilon > \varepsilon_m(X), 2\varepsilon_n(Y) > \delta > \varepsilon_n(Y) \}
\]

\[
\leq (1+C_1^2)(1+C_2^2) \{ \omega_{1,1}(f; \varepsilon - \varepsilon_m(X), \delta - \varepsilon_n(Y)) + \omega_{1,1}(f; \varepsilon - \varepsilon_m(X), \varepsilon_n(Y)) + \omega_{1,1}(f; \varepsilon_m(X), \delta - \varepsilon_n(Y)) + \omega_{1,1}(f; \varepsilon_m(X), \varepsilon_n(Y)) \}
\]

(\text{using (ii) of Theorem 2.6})

\[
\leq 4(1+C_1^2)(1+C_2^2) \omega_{1,1}(f; \varepsilon_m(X), \varepsilon_n(Y))
\]

(\text{using } 2\varepsilon_m(X) > \varepsilon, 2\varepsilon_n(Y) > \delta),

which completes the proof. \( \square \)

**REFERENCES**