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NOTE

CORRECTION TO A CONSTRUCTIVE DECOMPOSITION AND FULKERSON'S CHARACTERIZATION OF PERMUTATION MATRICES

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The purpose of this note is to point out an error in the paper [2], named in the title above, published under my authorship in this Journal in 1971.

Consider the set P of $n \times n$ matrices $X = (x_{ij})$ for which

$$\sum_{i \in I} \sum_{j \in J} x_{ij} \geqslant |I| + |J| - n ,$$

for all $I, J \subseteq \{1, 2, ..., n\}$, with $x_{ij} \geqslant 0$ for all $i, j \in \{1, 2, ..., n\}$.

In [2], I presented what I thought was an elementary constructive proof of the theorem:

Theorem 1. *Any matrix $X \in P$ may be decomposed as $X = S + N$, where S is a doubly stochastic matrix and N is non-negative in all entries.*

The result is true and had previously been proven by Fulkerson [1]. The proof in [2], however, is invalid.

In [2], the following lemma was stated with adequate proof:

Lemma 1. *If X is a matrix in P with at least one row sum or column sum greater than* 1, *then there exists a row i with row sum greater than* 1 *and a column j with column sum greater than* 1, *such that* $x_{ij} > 0$.

The invalid proof that if $X \in P$ then $X = S + N$ runs as follows. If no row sum of S is greater than 1, then X is doubly stochastic. Otherwise, X has a row i and column j as specified in Lemma 1. Lower the value of x_{ij} until either (a) row sum $i = 1$, or (b) column sum $j = 1$, or (c) $x_{ij} = 0$.

This gives a new matrix $X^{(1)}$. Continuing in this manner, we note that in finite time a matrix $X^{(k)}$ will result, where no row sums are greater than one. Thus $X = X^{(k)} + N$, where $X^{(k)}$ is doubly stochastic and N has entries corresponding to the non-negative entries subtracted from X.

In 1973 I received a letter from Allan B. Cruse, University of San Francisco, pointing out that the induction hypothesis required that each successive $X^{(i)}$ be in P, but that this was not proved. In fact it is not true, as the following counterexample shows. Consider the matrix

$$M = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

It is easily seen that the matrix M lies in P since it contains the permutation matrix

$$\pi = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Note too that: row sum $1 = 2 > 1$ and column sum $3 = 2 > 1$. But if the entry $M(1, 3)$ is reduced to zero (as the proof above would have it), the result is not in P and contains no doubly stochastic matrix; to see this, let $I = \{1,3\}$ and $J = \{1,3\}$.

I have had no success in finding a correct elementary constructive proof. A.B. Cruse has produced a generalization of this theorem which I find admirable. I hope it will soon be published. My own paper retains some value since it contains a proof that a subset of the inequalities defining P are "essential", as asserted by Fulkerson in [1] without proof.

References

[1] D.R. Fulkerson, Blocking Polyhedra, Graph Theory and its Applications (Academic Press, New York, 1970).

[2] P.E. O'Neil, A constructive decomposition and Fulkerson's characterization of permutation matrices, Discrete Math. 1 (2) (1971) 197–201.