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MV-Observables and MV-Algebras

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We introduce MV-observables, an analogue of observables for MV-algebras, as σ-homomorphisms from the Borel tribe generated by the Borel sets of $\mathbb{R}$ and constant functions from [0, 1] into an MV-algebra M. We show that it is possible to define such observables only for weakly divisible MV-algebras. We present a representation as well as a so-called calculus of MV-observables, which enables us to construct, e.g., the sum or product of MV-observables.

Key Words: MV-algebra; MV-observable; weakly divisible MV-algebra; σ-complete MV-algebra; joint MV-observable.

1. INTRODUCTION

In the classical Kolmogorov probability model [10], a random variable on a probability space $(\Omega, \mathscr{S}, P)$, where $\mathscr{S}$ is a σ-algebra of subsets of Ω, is any mapping $\xi : \Omega \to \mathbb{R}$ which is measurable with respect to the σ-algebra $\mathscr{S}$, i.e., $\xi^{-1}(E) \in \mathscr{S}$ for any Borel set $E \in \mathscr{B}(\mathbb{R})$. The measurability means that the mapping $x : \mathscr{B}(\mathbb{R}) \to \mathscr{S}$ defined by $x(E) = \xi^{-1}(E)$, $E \in \mathscr{B}(\mathbb{R})$, defines a σ-homomorphism from $\mathscr{B}(\mathbb{R})$ into $\mathscr{S}$. Conversely, it is well known [9] that any σ-homomorphism x from $\mathscr{B}(\mathbb{R})$ into $\mathscr{S}$ is determined by a unique random variable ξ via $x = \xi^{-1}$.

This approach is adopted in frames of quantum logic, a generalized probability structure which appears with foundations of quantum mechanics and where the classical Kolmogorov scheme can fail [15], where observables are analogues of random variables. In general, in Hilbert space quantum mechanics, observables due to spectral theorem correspond to Hermitian or, more general, to self-adjoint operators.

In both classical and non-classical probability models, the sum of two observables x and y can be defined as $x + y := (\xi + \eta)^{-1}$, where ξ and η

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are random variables corresponding to x and y, respectively. This is a usual way which enables us to construct a so-called calculus of observables.

In the last years, MV-algebras, introduced by Chang [2], also entered the theory of quantum structures, see, e.g., [7], due to their algebraic and fuzzy set ideas. Therefore, it seems to be reasonable to develop a theory of observables for MV-algebras, and in the present paper we introduce their analogue, MV-observables. We show that such observables can be defined only for weakly divisible MV-algebras. Using the Loomis–Sikorski theorem for σ-complete MV-algebras [5, 13], we present the representation for MV-algebras and develop a so-called calculus for MV-algebras which will enable us to construct the sum or product of MV-observables.

2. TRIBES AND MV-OBSERVABLES

MV-algebras are many-valued analogues of a two-valued logic, and they were introduced by Chang [2]. We recall that according to Mundici [12], or [4], they can be characterized as follows. An MV-*algebra* is a non-empty set M with two special elements 0 and 1 ($0 \neq 1$), with a binary operation $\oplus : M \times M \to M$ and with a unary operation $^* : M \to M$ such that, for all $a, b, c \in M$, we have

(MVi) $a \oplus b = b \oplus a$ (commutativity);

(MVii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ (associativity);

(MViii) $a \oplus 0 = a$;

(MViv) $a \oplus 1 = 1$;

(MVv) $(a^*)^* = a$;

(MVvi) $a \oplus a^* = 1$;

(MVvii) $0^* = 1$;

(MVviii) $(a^* \oplus b)^* \oplus b = (a \oplus b^*)^* \oplus a$.

We define binary operations $\odot$, $\vee$, $\wedge$ as

$$a \odot b := (a^* \oplus b^*)^*, \qquad a, b \in M,$$

$$a \vee b := (a^* \oplus b)^* \oplus b, \qquad a \wedge b := (a^* \vee b^*)^*, a, b \in M.$$

If, for $a, b \in M$, we define

$$a \leq b \Leftrightarrow a = a \wedge b,$$

then $\leq$ is a partial order on M, and $(M; \vee, \wedge, 0, 1)$ is a distributive lattice with the least and greatest elements 0 and 1, respectively [2]. We recall that $a \leq b$ iff $b \oplus a^* = 1$. An MV-algebra M is said to be σ-*complete* if M is in addition a σ-complete lattice.

A non-void subset I of M is said to be an *ideal* of M if

(i) $x, y \in I$ imply $x \oplus y \in I$,

(ii) $x \in I$, $y \leq x$ imply $y \in I$.

A proper ideal A of M is said to be *maximal* if there is no proper ideal of M containing A as a proper subset. Let $\mathscr{M}(M)$ denote the set of all maximal ideals of M. Then $\mathscr{M}(M) \neq \emptyset$. Denote by

$$Rad(M) := \bigcap \{A : A \in \mathscr{M}(M)\},$$

and we call $Rad(M)$ the *radical* of M. An MV-algebra M is said to be *semisimple* if $Rad(M) = \{0\}$. We recall that any σ-complete MV-algebra is semisimple.

A *state* on an MV-algebra M is a mapping $m : M \to [0,1]$ such that $m(1) = 1$, and $m(a \oplus b) = m(a) + m(b)$ whenever $a \leq b^*$. Denote by $\mathscr{S}(M)$ the set of all states on M. Then $\mathscr{S}(M)$ is a convex non-empty set. We denote by $Ext(\mathscr{S}(M))$ the set of extremal points of $\mathscr{S}(M)$. In addition, $\mathscr{S}(M)$ is a Hausdorff compact topological space in the weak topology of states, i.e., a net $\{m_\alpha\}$ converges weakly to m iff $m_\alpha(a) \to m(a)$ for each $a \in M$.

According to Belluce [1], we say that a subset $\mathscr{F} \subseteq [0,1]^\Omega$, where $\Omega \neq \emptyset$, is a *Bold algebra* if

(i) $0_\Omega \in \mathscr{F}$;

(ii) $f \in \mathscr{F}$ entails $1_\Omega - f \in \mathscr{F}$;

(iii) $f, g \in \mathscr{F}$ imply $f \oplus g \in \mathscr{F}$, where

$$(f \oplus g)(\omega) = \min\{f(\omega) + g(\omega), 1\}, \qquad \omega \in \Omega. \tag{2.1}$$

Then $\mathscr{F}$ with $\oplus$ defined by (2.1), with $f^* := 1_\Omega - f$ and with 0_Ω and 1_Ω is an MV-algebra which is semisimple. Conversely, any semisimple MV-algebra is MV-isomorphic to some Bold algebra.

The following notion is a direct generalization of a σ-algebra of crisp subsets.

A *tribe* is a non-void system $\mathscr{T} \subseteq [0,1]^\Omega$ of fuzzy sets on a set $\Omega \neq \emptyset$ such that

(i) $1_\Omega \in \mathscr{T}$;

(ii) if $a \in \mathscr{T}$, then $1 - a \in \mathscr{T}$;

(iii) if $\{a_n\}_{n=1}^\infty$ is a sequence of elements of $\mathscr{T}$, then

$$\min\left\{\sum_{n=1}^{\infty} a_n, 1\right\} \in \mathscr{T}.$$

(We note that all above operations with fuzzy sets are defined pointwisely on Ω.)

By [14, Proposition 3.13], if $\mathscr{T}$ is a tribe and if $a, b \in \mathscr{T}$, then (i) $a \vee b = \max\{a, b\} \in \mathscr{T}$, $a \wedge b = \min\{a, b\} \in \mathscr{T}$, (ii) $b - a \in \mathscr{T}$ if $a \le b$, i.e., if $a(\omega) \le b(\omega)$ for all $\omega \in \Omega$, (iii) if $a_n \in \mathscr{T}$, and $a_n \nearrow a$ (pointwisely), then $a = \lim_n a_n \in \mathscr{T}$. It is simple to verify that $\mathscr{T}$ is a Bold algebra and a σ-complete MV-algebra of fuzzy sets, where the partial order is determined by the set-theoretical ordering, with the least and greatest elements 0_Ω and 1_Ω, respectively.

Denote by

$$\mathscr{S}_0(\mathscr{T}) := \{A \subseteq \Omega : \chi_A \in \mathscr{T}\}.$$

The following result can be found, e.g., in [14, Theorem 8.1.4]:

PROPOSITION 2.1. *Let $\mathscr{T}$ be a tribe. Then*

(1) *$\mathscr{S}_0(\mathscr{T})$ is a σ-algebra of crisp subsets of Ω.*

(2) *If $f \in \mathscr{T}$, then f is $\mathscr{S}_0(\mathscr{T})$-measurable.*

(3) *$\mathscr{T}$ contains all $\mathscr{S}_0(\mathscr{T})$-measurable fuzzy functions on Ω if and only if $\mathscr{T}$ contains all constant functions with values in $[0, 1]$.*

It is evident that given a family T of fuzzy subsets of Ω there exists a tribe generated by T. Let $\mathscr{B}(\mathbb{R})$ be the algebra of Borel subsets of the real line $\mathbb{R}$. We denote by $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$ the tribe generated by $\{\chi_E : E \in \mathscr{B}(\mathbb{R})\}$ and by all constant fuzzy sets taking values in the interval $[0, 1]$. We called it the *Borel tribe*, and the elements of $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$ *Borel fuzzy subsets* of $\mathbb{R}$. In the same manner we define $\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$ and Borel fuzzy subsets of $\mathbb{R}^n$.

PROPOSITION 2.2. *The tribe $\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$ of Borel fuzzy subsets of $\mathbb{R}^n$ consists of all Borel measurable fuzzy sets on $\mathbb{R}^n$.*

Proof. It is evident that $\mathscr{B}(\mathbb{R}^n) \subseteq \mathscr{S}_0(\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n))$, and according to (3) of Proposition 2.1, $\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$ consists of all $\mathscr{S}_0(\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n))$-measurable fuzzy sets on $\mathbb{R}$. Denote by $\mathscr{T}$ the set of all elements from $\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$ which are Borel measurable. It is clear that the characteristic functions of all Borel sets and all constant functions from $[0, 1]$ belong to $\mathscr{T}$, and in addition, $\mathscr{T}$ is a tribe, consequently $\mathscr{T} = \mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$. ∎

An MV-*observable* on a σ-complete MV-algebra M is any MV-σ-homomorphism x from $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$ into M. It is clear that the *range*, $R(x)$, of an MV-observable x is an MV-subalgebra of M. In addition, if a is a crisp element in $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$, then $x(a)$ is a Boolean element in M, i.e., $x(a) \oplus x(a) = x(a)$.

THEOREM 2.3. *Let $\mathscr{T}$ be a tribe of fuzzy sets on a crisp set $\Omega \neq \emptyset$ containing all constant fuzzy sets on Ω, and let $f : \Omega \to \mathbb{R}$ be an $\mathscr{S}_0(\mathscr{T})$-measurable function. Then the mapping x_f defined by*

$$x_f(a) := a \circ f, \qquad a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R}), \tag{2.2}$$

is an MV-observable on the tribe $\mathscr{T}$. Conversely, let x be an MV-observable on the tribe $\mathscr{T}$. Then there exists a unique real-valued function f on Ω which is $\mathscr{S}_0(\mathscr{T})$-measurable such that $x = x_f$.

Proof. Let $E \in \mathscr{B}(\mathbb{R})$; by assumptions we have $f^{-1}(E) \in \mathscr{S}_0(\mathscr{T})$, which entails $x_f(\chi_E) = \chi_E \circ f = \chi_{f^{-1}(E)} \in \mathscr{T}$. Similarly, $c \circ f \in \mathscr{T}$ for any constant $c \in [0,1]$. Since the set of all Borel fuzzy sets a on $\mathbb{R}$ such that $a \circ f \in \mathscr{T}$ is a tribe containing the generators of $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$, x_f is a well-defined MV-observable on $\mathscr{T}$.

Conversely, let x be an MV-observable on $\mathscr{T}$. Then x maps all crisp sets from $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$ onto crisp sets from $\mathscr{T}$. Let $\mathbb{Q}$ be the set of all rational numbers in $\mathbb{R}$, and let $r_1, r_2, \dots$ be any enumeration of $\mathbb{Q}$. We set $\chi_{A_i} = x(\chi_{(-\infty, r_i)})$ for any $i \geq 1$. We define a mapping $f : \Omega \to \mathbb{R}$ as

$$f(\omega) = \inf\{r_j : \omega \in A_j\}.$$

Then f is a well-defined function such that $x(\chi_{(-\infty, r_j)}) = \chi_{f^{-1}(A_j)}$, and f is $\mathscr{S}_0(\mathscr{T})$-measurable. In addition, $x(\chi_E) = \chi_{f^{-1}(E)}$ for any $E \in \mathscr{B}(\mathbb{R})$.

For any integer $n \geq 1$, we have $1_\Omega = x(1_{\mathbb{R}}) = x(n\frac{1}{n}) = x(\frac{1}{n} \oplus \cdots \oplus \frac{1}{n}) = nx(\frac{1}{n})$ which proves that $x(\frac{1}{n}) = \frac{1}{n}$. Consequently, $x(\frac{m}{n}) = \frac{m}{n}$ for any $0 \leq m \leq n$. Using the density of rational numbers, we have that $x(c) = c$ for any constant $c \in [0,1]$. Hence, the set of all $a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R})$ such that $x(a) = x_f(a)$ is a tribe containing the generator of $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$; consequently, $x(a) = x_f(a)$ for any $a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R})$.

The uniqueness of f can be proved as follows. Let g be a real-valued function on Ω such that $x_f = x_g$. Then $\{\omega : f(\omega) \neq g(\omega)\} = \bigcup_{r \in \mathbb{Q}} (\{\omega : f(\omega) < r < g(\omega)\} \cup \{\omega : g(\omega) < r < f(\omega)\}) = \emptyset$. ∎

THEOREM 2.4. *Let x be an MV-observable on a σ-complete MV-algebra M. Then the range $R(x)$ of x is a weakly divisible σ-complete MV-subalgebra of M with a countable generator. Conversely, if M_1 is a weakly divisible σ-complete MV-subalgebra of M with a countable generator, then there exists an MV-observable x on M such that $M_1 = R(x)$.*

Proof. It is clear that $R(x)$ is a σ-complete MV-subalgebra of M with the countable generator $\{x(s\chi_{(-\infty, r)}) : r \in \mathbb{R}, s \in [0,1] \cap \mathbb{Q}\}$.

Suppose conversely that M_1 is a weakly divisible σ-complete MV-subalgebra of M. By the Loomis–Sikorski theorem for σ-complete MV-algebras [5, 13], there exists a tribe $\mathscr{T}$ of fuzzy sets on a non-void subset Ω and there exists an MV-σ-homomorphism h from $\mathcal{T}$ onto M_1. Let $\{a_n : n \geq 1\}$ be a generator of M_1, and let $\{b_n : n \geq 1\}$ be the set of such elements from $\mathcal{T}$ that $h(b_n) = a_n$ for any $n \geq 1$. We denote by $\mathcal{T}_0$ the smallest tribe containing all b_n; it contains all constants. The function $f: \Omega \to [0,1]^{\mathbb{N}}$ defined by

$$f(\omega) := (b_1(\omega), b_2(\omega), \dots)$$

is an $\mathcal{S}_0(\mathcal{T}_0)$-measurable mapping from Ω into the compact metric space $Y = [0,1]^{\mathbb{N}}$. We remark that $\mathcal{S}_0(\mathcal{T}_0)$ is equal to the σ-algebra generated by $\{b_i^{-1}(E) : E \in \mathcal{B}([0,1]), i \geq 1\}$. In addition, for any $E \in \mathcal{B}([0,1])$ we have $f^{-1}(\pi_i^{-1}(E)) = b_i^{-1}(E)$, where π_i is the ith projection from Y onto $[0,1]$. Consequently, $\mathcal{S}_0(\mathcal{T}_0) = \{f^{-1}(E) : E \in \mathcal{B}(Y)\}$. Now, by a classical theorem of Kuratowski [11], there exists a Borel isomorphism d of Y onto $\mathbb{R}$, so that the function $f_1(\omega) := d(f(\omega))$ is $\mathcal{S}_0(\mathcal{T}_0)$-measurable, and $\mathcal{S}_0(\mathcal{T}_0) = \{f_1^{-1}(E) : E \in \mathcal{B}(\mathbb{R})\}$. If we now define a mapping x by

$$x(a) := h(a \circ f_1), \qquad a \in \mathcal{T}_{\mathcal{B}}(\mathbb{R}),$$

then x is an MV-observable on M whose range is M_1. ∎

An element $a \in M$ is said to be *Boolean* or *idempotent* if $a \vee a^* = 1$. It is possible to show that a is Boolean iff $a \oplus a = a$ iff $a \odot a = a$ iff $a \odot a^* = 0$. Denote by $B(M)$ the set of all Boolean elements of M. It is easy to verify that (i) $0, 1 \in B(M)$; (ii) $\bigvee_i a_i \in B(M)$ whenever $a_i \in B(M)$ for any i; (iii) $B(M)$ is a Boolean σ-algebra whenever M is σ-complete.

We now present a finer version of the Loomis–Sikorski theorem for σ-complete MV-algebras.

THEOREM 2.5. *Let M be a σ-complete MV-algebra. Then there exists a tribe $\mathcal{T}$ of fuzzy sets on a compact Hausdorff space $\Omega \neq \emptyset$ and an MV-σ-homomorphism h from $\mathcal{T}$ onto M such that h maps $\mathcal{S}_0(\mathcal{T})$ onto $B(M)$.*

Proof. We define $\Omega := Ext(\mathcal{S}(M))$ and let $\hat{M} = \{\hat{a} \in [0,1]^{\Omega} : a \in M\}$, where $\hat{a}(m) := m(a)$, $m \in Ext(\mathcal{S}(M))$. According to the proof of the Loomis–Sikorski theorem, if we define $\mathcal{T}$ as the tribe of fuzzy subsets of Ω generated by $\hat{M}$, a mapping $h: \mathcal{T} \to M$ can be defined by $h(f) = a$ for $f \in \mathcal{T}$, $a \in M$, whenever $\{m \in \Omega : f(m) \neq m(a)\}$ is a meager set, h is a well-defined MV-σ-homomorphism from $\mathcal{T}$ onto M (see the proof of [5, Theorem 4.10]). It is evident that an element a of a semisimple MV-algebra M is Boolean iff $\hat{a}$ is a two-valued function on Ω, so that $\hat{a} = \chi_A$ for some $A \in \mathcal{S}_0(\mathcal{T})$, and in addition, A is a clopen subset of Ω. We define by $\mathcal{S}_0$ the set of $A \in \mathcal{S}_0(\mathcal{T})$ all such that $\chi_A = \hat{a}$ for some $a \in M$.

According to [8, Theorem 8.14], the space $Ext(\mathcal{S}(M))$ is homeomorphic with the set $Ext(\mathcal{S}(B(M)))$ of all extremal states on the Boolean σ-algebra $B(M)$. In addition, any restriction of $m \in Ext(\mathcal{S}(M))$ to $B(M)$ gives an element of $Ext(\mathcal{S}(B(M)))$, and conversely, any element of $Ext(\mathcal{S}(B(M)))$ can be uniquely extended to an extremal state on M, and this correspondence defines the mentioned homeomorphism.

Consequently, by the proof of the classical Loomis–Sikorski theorem, $\mathcal{S}_0$ is a σ-algebra of crisp subsets of Ω, and due to the definition of h, its restriction onto $\mathcal{S}_0$ defines a σ-homomorphism from $\mathcal{S}_0$ onto $B(M)$.

It is evident that $\mathcal{S}_0 \subseteq \mathcal{S}_0(\mathcal{T})$. On the other hand, $A \in \mathcal{S}_0(\mathcal{T})$ iff $\chi_A \in \mathcal{T}$, i.e., $h(\chi_A) = a$ for some $a \in M$. Since χ_A is an Boolean element of $\mathcal{T}$, so is a in M. Consequently, $\hat{a} = \chi_A$ and $A \in \mathcal{S}_0$. ∎

Let M be a σ-complete MV-algebra. The triplet $(\Omega, \mathcal{T}, h)$, where $\Omega = Ext(\mathcal{S}(M))$, $\mathcal{T}$ is the tribe, and h is the MV-σ-homomorphism from $\mathcal{T}$ onto M described in the proof of Theorem 2.5, is said to be a *canonical representation* of M.

Given an integer $n \geq 1$ and $a \in M$, we define

$$n \odot a := a_1 \oplus \cdots \oplus a_n,$$

where $a_i = a$, $i = 1, \ldots, n$.

Another version of the Loomis–Sikorski theorem is the following statement.

THEOREM 2.6. *Let $\mathcal{T}$ be a tribe of fuzzy sets on $\Omega \neq \emptyset$ containing all constant fuzzy sets on Ω. Let h be an MV-σ-homomorphism from $\mathcal{T}$ onto a σ-complete MV-algebra M. Then h maps $\mathcal{S}_0(\mathcal{T})$ onto $B(M)$.*

Proof. Let $a \in B(M)$. Then there exists an element $g \in \mathcal{T}$ with $h(g) = a$. According to (3) of Proposition 2.1, there are two cases.

Case 1. Let $g = \sum_{i=1}^{k} \alpha_i \chi_{A_i}$, where $\alpha_i \in [0,1]$, $A_i \in \mathcal{S}_0(\mathcal{T})$, and $A_i \cap A_j = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, k$. There exists an integer s such that $s\alpha_j \geq 1$ for $j = 1, \ldots, k$. Then $s \odot g = \chi_A$, where $A = \bigcup_{j=1}^{k} A_j$. Hence, $h(\chi_A) = h(s \odot g) = s \odot a = a$.

Case 2. There exists a non-decreasing sequence of simple functions $\{g_n\}_n$ of elements of $\mathcal{T}$ such that $\lim_n g_n = g$. Let $g_n = \sum_{i=1}^{k_n} \alpha_i^n \chi_{A_i^n}$, where $\alpha_i^n \in [0,1]$ and $A_i^n \cap A_j^n = \emptyset$ for $i \neq j$. We define $A_n = \bigcup_{i=1}^{k_n} A_i^n$ and $A = \bigcup_{n=1}^{\infty} A_n$. As in Case 1, there exists an integer $s \geq 1$ such that $s\alpha_i^1 \geq 1$ for $i = 1, \ldots, k_1$. Hence, $s\alpha_i^n \geq 1$ for any $i = 1, \ldots, k_n$ and any $n \geq 1$. Therefore, $s \odot g_n \leq s \odot g_{n+1} \leq s \odot g$, i.e., $\chi_{A_n} \leq \chi_{A_{n+1}} \leq s \odot g$, which entails $\chi_{A_n} \nearrow s \odot g$ and $h(\chi_A) = h(\lim_n \chi_{A_n}) = h(\lim_n s \odot g_n) = h(s \odot g) = s \odot a = a$. ∎

Let $c \in [0,1]$. An element a in a semisimple MV-algebra M is said to be a *c-constant* if $m(a) = c$ for any state $m \in M$. If M is a tribe, then the c-constant is equal to $c1_{\Omega}$.

According to [6], we say that a semisimple MV-algebra M is *weakly divisible* if, for any integer $n \geq 1$, there is an element $v \in M$ such that nv is defined in M and $nv = 1$. It is easy to see that v is uniquely determined. Indeed, $m(nv) = 1$ for any state m on M. Then $m(v) = 1/n$. We denote $v = \frac{1}{n}$.

For example, $\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$ is weakly divisible. In addition, if M is a σ-complete, weakly divisible MV-algebra, then any c-constant, where $c \in [0,1]$, belongs to M.

In the proof of [6, Theorem 3.10], it was shown that a weakly divisible σ-complete MV-algebra M is isomorphic as MV-algebras with the space $C(Ext(\mathscr{S}(M)))$ of all continuous functions on the basically disconnected space[1] $Ext(\mathscr{S}(M))$ with values in the interval $[0,1]$. In this case, the tribe generated by $C(Ext(\mathscr{S}(M)))$ is equal to the set of all Baire measurable fuzzy sets on $Ext(\mathscr{S}(M))$ [5, Proposition 3.4].

PROPOSITION 2.7. *Let h be an MV-σ-homomorphism from a tribe $\mathscr{T}$ of fuzzy sets on $\Omega \neq \emptyset$ containing all constant fuzzy sets into a σ-complete MV-algebra M. Then M is weakly divisible, and $h(c1_\Omega)$ is a c-constant for any $c \in [0,1]$.*

Proof. Let $n \geq 1$ be given. Then $1 = h(1_\Omega) = h(n\frac{1}{n}1_\Omega)$, which proves that M is weakly divisible. Hence, $h(\frac{m}{n}1_\Omega)$ is an $\frac{m}{n}$-constant in M. Consequently, $h(c1_\Omega)$ is a c-constant for any $c \in [0,1]$. ∎

The following statement is an easy consequence of the above proposition.

COROLLARY 2.8. *Let x be an MV-observable of a σ-complete MV-algebra M. Then M is weakly divisible, and $x(c)$ is a c-constant for any $c \in [0,1]$.*

For example, let $L_n := \{0, 1/n, 2/n, \dots, n/n\}$ be the so-called basic MV-algebra. If $n \geq 2$, then on L_n there exists no MV-observable. Similarly, since any finite MV-algebra M is a direct product of finitely many basic MV-algebras [3], if M is a finite non-Boolean MV-algebra, then it does not admit any MV-observable.

THEOREM 2.9. *Let $\mathscr{T}$ be a tribe of fuzzy sets on a non-void set Ω containing all constant fuzzy sets and let h be an MV-σ-homomorphism from $\mathscr{T}$ onto a σ-complete MV-algebra M. Then, for any MV-observable x on M, there exists an $\mathscr{S}_0(\mathscr{T})$-measurable real-valued function f defined on Ω such that*

$$x(a) = h(a \circ f), \qquad a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R}). \tag{2.3}$$

[1] A topological space X is said to be *basically disconnected* provided the closure of every open F_σ subset of X is open.

If g is any $\mathcal{S}_0(\mathcal{T})$*-measurable real-valued function defined on* Ω *such that* $x(a) = h(a \circ g)$, $a \in \mathcal{T}_{\mathcal{B}}(\mathbb{R})$, *then* $h(\chi_E) = 0$, *where* $E = \{\omega : f(\omega) \neq g(\omega)\}$.

Conversely, if f is a real-valued $\mathcal{S}_0(\mathcal{T})$*-measurable mapping on* Ω, *then the mapping* x_f *defined by*

$$x_f(a) := h(a \circ f), \qquad a \in \mathcal{T}_{\mathcal{B}}(\mathbb{R}), \tag{2.4}$$

defines an MV-observable on M.

Proof. Step 1. Suppose that $A, B \in \mathcal{S}_0(\mathcal{T})$, $A \subseteq B$, and $c \in B(M)$ are such that $h(\chi_A) \leq c \leq h(\chi_B)$. Then there exists an element $C \in \mathcal{S}_0(\mathcal{T})$ such that $A \subseteq C \subseteq B$, and $h(\chi_C) = c$. In fact, since h maps $\mathcal{S}_0(\mathcal{T})$ onto $B(M)$, there exists $C_1 \in \mathcal{T}$ such that $h(C_1) = c$. If we define $C = (C_1 \cap B) \cup A$, then $A \subseteq C \subseteq B$ and $h(C) = c$.

Step 2. Similarly as in the proof of Theorem 2.3, let $r_1, r_2, \dots$ be any enumeration of the rational numbers of $\mathbb{R}$. It is clear that $x(\chi_{(-\infty, r_i)}) \leq x(\chi_{(-\infty, r_j)})$ whenever $r_i < r_j$ and any $x(\chi_{(-\infty, r_i)})$ is a Boolean element of M. We shall now construct crisp sets $A_1, A_2, \dots$ from $\mathcal{S}_0(\mathcal{T})$ such that (a) $h(\chi_{A_i}) = x(\chi_{(-\infty, r_i)})$ for any i; (b) $A_i \subseteq A_j$ whenever $r_i < r_j$. According to Theorem 2.6, we can find a set A_1 in $\mathcal{S}_0(\mathcal{T})$ such that $h(A_1) = x(\chi_{(-\infty, r_1)})$. Suppose that $A_1, \dots, A_n$ in $\mathcal{S}_0(\mathcal{T})$ have been constructed such that (i) $h(\chi_{A_i}) = x(\chi_{(-\infty, r_i)})$ for $i = 1, \dots, n$; (ii) $A_i \subseteq A_j$ whenever $r_i < r_j$, $1 \leq i, j, \leq n$. We shall construct A_{n+1} as follows. Let $(i_1, \dots, i_n)$ be the permutation of $(1, \dots, n)$ such that $r_{i_1} < r_{i_2} < \cdots < r_{i_n}$. Then there exists a unique k such that $r_{i_k} < r_{n+1} < r_{i_{k+1}}$ (we define $r_{i_0} = -\infty$ and $r_{i_{n+1}} = \infty$), and by Step 1, we can select $A_{n+1} \in \mathcal{S}_0(\mathcal{T})$ such that $A_{i_k} \subseteq A_{n+1} \subseteq A_{i_{k+1}}$ (we define $A_{i_0} = \emptyset$ and $A_{i_{n+1}} = \Omega$). By induction, we have proved that there exists a sequence $A_1, A_2, \dots$ in $\mathcal{S}_0(\mathcal{T})$ with the properties (a) and (b).

As

$$h(\chi_{\cap_j A_j}) = \bigwedge_j h(\chi_{A_j}) = \bigwedge_j x(\chi_{(-\infty, r_j)}) = 0,$$

we may, by replacing A_k by $A_k \setminus \bigcap_j A_j$, if necessary, assume that $\bigcap_j A_j = \emptyset$. In addition, we have $h(\chi_{\cup_j A_j}) = \bigvee_j x(\chi_{(-\infty, r_i)}) = 1$.

We now define a function f on Ω as

$$f(\omega) = \begin{cases} \inf\{r_j : \omega \in A_j\} & \text{if } \omega \in \bigcup_j A_j, \\ 0 & \text{otherwise}. \end{cases}$$

Then f is a finite real-valued function on Ω, and, for any k

$$f^{-1}((-\infty, r_k)) \cap \bigcup_j A_j = \bigcup_{j:\ r_j < r_k} A_j,$$

so that f is $\mathscr{S}_0(\mathscr{T})$-measurable. Further,

$$f(\chi_{f^{-1}((-\infty, r_k))}) = h(\chi_{\bigcup_{j:r_j<r_k} A_j}) = \bigvee_{j:r_j<r_k} x(\chi_{(-\infty, r_j)}) = x(\chi_{(-\infty, r_k)}).$$

Therefore, $h(\chi_E \circ f) = x(\chi_E)$ whenever $E = (-\infty, r_k)$ for some k. Similarly, by Proposition 2.7, $h(c \circ f) = h(c1_\Omega) = x(c)$ for any $c \in [0,1]$. Hence, the system of all $a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R})$ such that $x(a) = h(a \circ f)$ is a tribe containing the generator of $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$, consequently, it coincides with $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$.

The uniqueness of a function f in the given sense can be proved in a similar way as in the proof of Theorem 2.3.

The second statement is now evident. ∎

It is worth recalling that according to Theorem 2.9, a σ-complete MV-algebra M admits an MV-observable iff M is weakly divisible:

THEOREM 2.10. *A σ-complete MV-algebra M admits an MV-observable if and only if M is weakly divisible.*

Proof. Let x be an MV-observable of M. Due to Corollary 2.8, M is weakly divisible.

Suppose now M is a weakly divisible σ-complete MV-algebra. By [6, Theorem 3.10], M is isomorphic to the system $C(\Omega)$ of all continuous fuzzy sets on $\Omega := Ext(\mathscr{S}(M))$. The tribe $\mathscr{T}(M)$ generated by $C(\Omega)$ is equal to the system of all Baire measurable functions on Ω [6, Proposition 3.4]. In addition, $\mathscr{T}(M)$ contains all constant functions taking values in the interval $[0,1]$. Take now a Baire function $f: \Omega \to \mathbb{R}$, and define x_f via (2.4). According to Theorem 2.9, x_f is an MV-observable on M, where h is defined in the same way as in the proof of Theorem 2.5. ∎

3. CALCULUS OF OBSERVABLES

Let f be a Borel measurable function from $\mathbb{R}$ into $\mathbb{R}$. Then $f \circ x$, where x is an MV-observable of M,

$$(f \circ x)(a) := x(a \circ f), \qquad a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R}),$$

and $a \circ f(\omega) := a(f(\omega))$, $\omega \in \mathbb{R}$, is an MV-observable of M. In this manner we can define x^2, x^3, or e^x, etc.

THEOREM 3.1. *Let x and y be two MV-observables of a σ-complete MV-algebra M. Then $R(x) \subseteq R(y)$ if and only if there exists a Borel measurable function $f: \mathbb{R} \to \mathbb{R}$ such that $x = f \circ y$.*

Proof. One direction is evident. Suppose now that $R(x) \subseteq R(y)$. We define $\mathscr{T} = \mathscr{T}_{\mathscr{B}}(\mathbb{R})$, $h = y$, $M = R(y)$, and apply Theorem 2.9. ∎

THEOREM 3.2. *Let $\{x_n\}$ be a sequence of MV-observables on a σ-complete MV-algebra M. Then there exists an MV-observable x and a sequence of Borel measurable functions $\{f_n\}$ from $\mathbb{R}$ into $\mathbb{R}$ such that $x_n = f_n \circ x$ for any n.*

Proof. Let M_0 be the MV-subalgebra of M generated by $\bigcup_n R(x_n)$. According to Theorem 2.4, there exists an MV-observable x such that $R(x) = M_0$. Then we apply Theorem 3.1 and we obtain the statement in question. ∎

The following result can be of a particular interest for many valued analysis and for a so-called calculus of MV-observables.

THEOREM 3.3. *Let $x_1, \dots, x_n$ be MV-observables of a σ-complete MV-algebra M. Then there exists a unique MV-σ-homomorphism x from $\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$ into M such that, for any Borel fuzzy set $a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R})$, $x_i(a) = x(a \circ \pi_i)$, where π_i is the ith projection of $\mathbb{R}^n$ onto $\mathbb{R}$, $i = 1, \dots, n$.*

Proof. Let M_0 be the MV-subalgebra of M generated by all ranges $R(x_1), \dots, R(x_n)$. Since M_0 has a countable generator, according to Theorem 2.4, there exists an MV-observable y of M such that $R(y) = M_0$. Consequently, by Theorem 2.9, there exist Borel measurable functions $f_1, \dots, f_n$ from $\mathbb{R}$ into $\mathbb{R}$ such that $x_i(a) = (f_i \circ y)(a)$, $a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R})$. Define the Borel measurable mapping $f: \mathbb{R} \to \mathbb{R}^n$ by $f(t) := (f_1(t), \dots, f_n(t))$, $t \in \mathbb{R}$, and set $x: \mathscr{T}_{\mathscr{B}}(\mathbb{R}^n) \to M$ via

$$x(b) := y(b \circ f), b \in \mathscr{T}_{\mathscr{B}}(\mathbb{R}^n). \tag{3.1}$$

Then x is an MV-σ-homomorphism from $\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$ into M. Moreover, let $a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R})$. Then

$$x(a \circ \pi_i) = y(a \circ \pi_i \circ f) = y(a \circ f_i) = x_i(a).$$

For the uniqueness, let z be any MV-σ-homomorphism from $\mathscr{T}_{\mathscr{B}}(\mathbb{R})$ into M such that $z(a \circ \pi_i) = x_i(a)$, $a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R})$, $i = 1, \dots, n$. Then, for $E_i \in \mathscr{B}(\mathbb{R})$ $(i = 1, \dots, n)$, we have

$$\begin{aligned} z(\chi_{E_1 \times \cdots \times E_n}) &= z\Big(\min_{1 \le i \le n} (\chi_{E_i} \circ \pi_i)\Big) = \bigwedge_{i=1}^n z(\chi_{E_i} \circ \pi_i) = \bigwedge_{i=1}^n x(\chi_{E_i} \circ \pi_i) \\ &= x\Big(\min_{1 \le i \le n} (\chi_{E_i} \circ \pi_i)\Big) = x(\chi_{E_1 \times \cdots \times E_n}). \end{aligned}$$

On the other hand, let $c \in [0, 1]$. Then $x(c\chi_{\mathbb{R}^n}) = x(c\chi_{\mathbb{R}} \circ \pi_i) = x_i(c\chi_{\mathbb{R}}) = z(c\chi_{\mathbb{R}} \circ \pi_i) = z(c\chi_{\mathbb{R}^n})$.

Hence the set of all Borel fuzzy sets b from $\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$ such that $x(b) = z(b)$ is a tribe containing the generator of $\mathscr{T}_{\mathscr{B}}(\mathbb{R}^n)$, so that $x = z$. ∎

The unique MV-σ-homomorphism x from Theorem 3.3 is said to be a *joint MV-observable* of $x_1, \ldots, x_n$.

THEOREM 3.4. *Let x be a joint MV-observable of the MV-observables $x_1, \ldots, x_n$ of a σ-complete MV-algebra M. Let y be an MV-observable of M and $f_1, \ldots, f_n$ let be Borel measurable functions from $\mathbb{R}$ into $\mathbb{R}$ such that $x_i = f_i \circ y$ for $i = 1, \ldots, n$. Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be a Borel function. Define $\phi \circ x : \mathscr{T}_{\mathscr{B}}(\mathbb{R}) \to M$ and $\phi(f_1, \ldots, f_n) \circ y : \mathscr{T}_{\mathscr{B}}(\mathbb{R}) \to M$, where $\phi(f_1, \ldots, f_n)(t) := \phi(f_1(t), \ldots, f_n(t))$, $t \in \mathbb{R}$, by*

$$(\phi \circ x)(a) := x(a \circ \phi), \qquad a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R}), \tag{3.2}$$

$$(\phi(f_1, \ldots, f_n) \circ y)(a) := y(a \circ \phi(f_1, \ldots, f_n)), \qquad a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R}). \tag{3.3}$$

Then $\phi \circ x$ and $\phi(f_1, \ldots, f_n) \circ y$ are identical MV-observables of M.

Proof. According to the proof of Theorem 2.9 and Theorems 3.1–3.2, it is evident that the construction of the joint MV-observable x of $x_1, \ldots, x_n$ does not depend on the used MV-observable y and Borel functions $f_1, \ldots, f_n$. It is clear that $\phi \circ x$ and $\phi(f_1, \ldots, f_n) \circ y$ are MV-observables of M. Consequently, by (3.1), for any $a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R})$, we have

$$\begin{aligned}(\phi \circ x)(a) &= x(a \circ \phi) = y(a \circ \phi \circ f) \\ &= y(a \circ \phi(f_1, \ldots, f_n)) \\ &= (\phi(f_1, \ldots, f_n) \circ y)(a),\end{aligned}$$

where $f(t) = (f_1(t), \ldots, f_n(t))$, $t \in \mathbb{R}$. ∎

The MV-observable $\phi \circ x$ from Theorem 3.4 will be denoted by $\phi \circ (x_1, \ldots, x_n)$, i.e., as a function of $x_1, \ldots, x_n$. We have seen that it can be defined by equivalent ways by (3.2) or by (3.3).

For example, if $\phi(t_1, t_2) = t_1 + t_2$, we can define addition of two MV-observables x_1 and x_2 by

$$x_1 + x_2 := \phi \circ (x_1, x_2). \tag{3.4}$$

In a similar way we can define product, difference, etc., of MV-observables.

THEOREM 3.5. *Let $x_1, \ldots, x_n$ be MV-observables of a σ-complete MV-algebra M, $g_1, \ldots, g_k$ be real-valued Borel functions on $\mathbb{R}^n$, and h be a real-valued functions on $\mathbb{R}^k$. Then $y_i = g_i \circ (x_1, \ldots, x_n)$, $i = 1, \ldots, k$, are MV-observables of M, and*

$$h \circ (y_1, \ldots, y_k) = (h(g_1, \ldots, g_k)) \circ (x_1, \ldots, x_n), \tag{3.5}$$

where $h(g_1, \dots, g_k)$ is the function from $\mathbb{R}^n$ into $\mathbb{R}$ defined by $(t_1, \dots, t_n) \mapsto h(g_1(t_1, \dots, t_n), \dots, g_k(t_1, \dots, t_n))$.

Proof. Suppose $y_i = g_i \circ (x_1, \dots, x_n)$ and let g be the mapping from $\mathbb{R}^n$ into $\mathbb{R}^k$ defined by $g(t_1, \dots, t_n) = (g_1(t_1, \dots, t_n), \dots, g_k(t_1, \dots, t_n))$. Then the mapping $y : \mathscr{T}_{\mathscr{B}}(\mathbb{R}^k) \to M$ given by

$$y(b) := x(b \circ g), \qquad b \in \mathscr{T}_{\mathscr{B}}(\mathbb{R}^k),$$

is an MV-σ-homomorphism such that, for $a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R})$,

$$y\left(a \circ \pi_i^k\right) = x\left(a \circ \pi_i^k \circ g\right) = x(a \circ g_i) = y_i(a),$$

where π_i^k is the ith projection from $\mathbb{R}^k$ on $\mathbb{R}$. Therefore, y is the unique joint MV-observable of $y_1, \dots, y_k$. Consequently, for $h \circ (y_1, \dots, y_k)$, we have

$$(h \circ (y_1, \dots, y_k))(a) = y(a \circ h) = x(a \circ h \circ g),$$

which proves the theorem. ∎

If we apply Theorems 3.3–3.4, we see that, for example, the sum $x_1 + \cdots + x_n$ of MV-observables $x_1, \dots, x_n$ is well-defined, and, in addition, $x_1 + \cdots + x_n = x_{i_1} + \cdots + x_{i_n}$, where $(i_1, \dots, i_n)$ is any permutation of $(1, \dots, n)$.

We now present a representation theorem for MV-observables. According to Theorem 2.10, we can limit ourselves to weakly divisible σ-complete MV-algebras.

THEOREM 3.6. *Let M be a weakly divisible σ-complete MV-algebra with the canonical representation $(\Omega, \mathscr{T}, h)$. If f is a Baire measurable real-valued function on Ω, then*

$$x_f(a) := h(a \circ f), \qquad a \in \mathscr{T}_{\mathscr{B}}(\mathbb{R}),$$

defines an MV-observable of M. Conversely, given an MV-observable x on M, there exists a real-valued Baire-function f on Ω such that $x = x_f$. If g is another real-valued Baire measurable function on Ω with $x_f = x_g$, then $\{\omega : f(\omega) \neq g(\omega)\}$ is a meager set.

Proof. The statement follows directly from Theorems 2.9–2.10. ∎

THEOREM 3.7. *Let the conditions of Theorem 3.6 be satisfied. Let $x_1, \dots, x_n$ be MV-observables with the joint MV-observable x. Then there exist real-valued Baire measurable functions $f_1, \dots, f_n$ on Ω such that $x_i = x_{f_i}$, $i = 1, \dots, n$, and $x = x_f$, where $f(\omega) = (f_1(\omega), \dots, f_n(\omega))$, and*

$$x_f(b) = h(b \circ f), \qquad b \in \mathscr{T}_{\mathscr{B}}(\mathbb{R}^n). \tag{3.6}$$

If $g : \Omega \to \mathbb{R}^n$ *is Baire measurable such that* $x_f = x_g$, *then* $\{\omega : f(\omega) \neq g(\omega)\}$ *is a meager set.*

Proof. According to Theorem 3.6, there exist Baire measurable functions $f_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, such that $x_i = x_{f_i}$ $(i = 1, \ldots, n)$. Due to Theorem 3.3, the joint MV-observable $x : \mathcal{T}_{\mathcal{B}}(\mathbb{R}^n) \to M$ is uniquely determined by $x(a \circ \pi_i) = x_i(a)$, $a \in \mathcal{T}_{\mathcal{B}}(\mathbb{R})$. Define x_f by (3.6). Then x_f is an MV-σ-homomorphism from $\mathcal{T}_{\mathcal{B}}(\mathbb{R}^n)$ into M with $x_f(a \circ \pi_i) = h(a \circ \pi_i \circ f) = h(a \circ f_i) = x_{f_i}(a) = x_i(a)$ for $a \in \mathcal{T}_{\mathcal{B}}(\mathbb{R})$. The uniqueness of a joint MV-observable entails $x = x_f$.

If $x = x_g$ for some Baire measurable function $g : \Omega \to \mathbb{R}^n$, then $\{\omega : f(\omega) \neq g(\omega)\} = \bigcup_{i=1}^n \{\omega : f_i(\omega) \neq g_i(\omega)\}$, where $g = (g_1, \ldots, g_n)$ with $g_i : \Omega \to \mathbb{R}$, so that the set under question is meager. ∎

Let x be an MV-observable of M. The *spectrum* of x is the set $\sigma(x) := \bigcap\{C : C$ is a closed crisp set of $\mathbb{R}$ with $x(\chi_C) = 1\}$. Since the natural topology of $\mathbb{R}$ satisfies the second countability axiom, there exists a sequence of closed sets $C_1, C_2, \ldots$ such that $x(\chi_{C_n}) = 1$ for all n, and $\sigma(x) = \bigcap_n C_n$. x is said to be (i) *bounded* if $\sigma(x)$ is a bounded set of $\mathbb{R}$; (ii) *discrete* if $\sigma(x) = \{c_1, c_2, \ldots\}$; (iii) *simple* if $\sigma(x) = \{c_1, \ldots, c_n\}$; (iv) *question* if $\sigma(x) \subseteq \{0, 1\}$, and (v) *constant* if $\sigma(x) = \{c\}$.

PROPOSITION 3.8. *Let the conditions of Theorem* 3.6 *be satisfied. Let* $f : \Omega \to \mathbb{R}$ *be a Baire measurable function. The MV-observable* x_f *is*

(i) *bounded if and only if f is a bounded function up to a meager set*;

(ii) *discrete if and only if* $f = \sum_i c_i \chi_{A_i}$, $A_i \cap A_j = \emptyset$ *for* $i \neq j$, $A_i \in \mathcal{S}_0(\mathcal{T})$;

(iii) *simple if and only if* $f = \sum_{i=1}^n c_i \chi_{A_i}$, $A_i \cap A_j = \emptyset$ *for* $i \neq j$, $A_i \in \mathcal{S}_0(\mathcal{T})$;

(iv) *question if and only if* $f = \chi_A$ *for some* $A \in \mathcal{S}_0(\mathcal{T})$;

(v) *constant if and only if* $f = c\chi_\Omega$.

(*All statements* (ii)–(v) *are valid up to a meager set.*)

Proof. (i) Let x be bounded. Then there exists a constant c such that $\sigma(x) \subseteq [-c, c]$. Hence

$$1 = x_f(\chi_{[-c,c]}) = h(\chi_{[-c,c]} \circ f) = h(\chi_{f^{-1}([-c,c])}),$$

which proves that $\{\omega : f(\omega) \notin [-c, c]\}$ is a meager set. The converse is clear.

From the rest we prove only (ii). Put $A_i = f^{-1}(\{c_i\})$ and calculate $x_f(\chi_{\{c_i\}}) = h(\chi_{\{c_i\}} \circ f) = h(\chi_{f^{-1}(\{c_i\})}) = h(\chi_{A_i})$. Therefore, $f = \sum_i c_i \chi_{A_i}$. ∎

Let m be a σ-additive state on a σ-complete MV-algebra M. If x is an MV-observable on M, then the composition m_x of m with x, $m_x := m \circ x$, is a σ-additive state on $\mathcal{T}_{\mathcal{B}}(\mathbb{R})$. In addition, the restriction $\widehat{m_x}$ of m_x onto $\mathcal{B}(\mathbb{R})$, defined via $\widehat{m_x}(E) := m_x(\chi_E)$, $E \in \mathcal{B}(\mathbb{R})$, is a usual probability measure on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$.

In a similar way, if h is an MV-σ-homomorphism of a tribe $\mathcal{T}$ onto M, then $m_h := m \circ h$ is a σ-additive state, and $\widehat{m_h} : \mathcal{S}_0(\mathcal{T}) \to [0, 1]$ defined by $\widehat{m_h}(A) := m_h(\chi_A)$, $A \in \mathcal{S}_0(\mathcal{T})$, is a usual probability measure on $\mathcal{S}_0(\mathcal{T})$.

Let x be an MV-observable; for the *mean value* of x in a σ-additive state m we understand the expression

$$m(x) := \int_{\mathbb{R}} t \, d\widehat{m_x}(t) \tag{3.7}$$

whenever the expression on the right-hand side of (3.7) exists and is finite.

The following result gives a way how to prove many of the known results from classical probability theory (convergence of MV-observables, limit theorems) for probability theory on MV-algebras.

PROPOSITION 3.9. *Let h be an MV-σ-homomorphism of a tribe containing all constants onto a σ-complete MV-algebra M. Let x_f be defined by* (2.2) *for some $\mathcal{S}_0(\mathcal{T})$-measurable function $f : \Omega \to \mathbb{R}$, and let m be a σ-additive state on M. Then $m(x_f)$ exists and is finite if and only if the integral $\int_\Omega f(\omega) \, d\widehat{m_h}(\omega)$ exists and is finite. In such a case*

$$m(x_f) = \int_\Omega f(\omega) \, d\widehat{m_h}(\omega).$$

Proof. It follows from the equality (3.7) and from transforms of integrals. ∎

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