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Periodic solutions of integro-differential equations in vector-valued function spaces ☆

Valentin Keyantuo [a], Carlos Lizama [b,*], Verónica Poblete [c]

[a] University of Puerto Rico, Department of Mathematics, Faculty of Natural Sciences, PO Box 23355, PR 00931, USA
[b] Universidad de Santiago de Chile, Departamento de Matemática, Facultad de Ciencias, Casilla 307-Correo 2, Santiago, Chile
[c] Universidad de Chile, Departamento de Matemática, Facultad de Ciencias, Las Palmeras 3425 Ñuñoa, Santiago, Chile

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A B S T R A C T

Operator-valued Fourier multipliers are used to study well-posedness of integro-differential equations in Banach spaces. Both strong and mild periodic solutions are considered. Strong well-posedness corresponds to maximal regularity which has proved very efficient in the handling of nonlinear problems. We are concerned with a large array of vector-valued function spaces: Lebesgue–Bochner spaces L^p, the Besov spaces $B^s_{p,q}$ (and related spaces such as the Hölder–Zygmund spaces C^s) and the Triebel–Lizorkin spaces $F^s_{p,q}$. We note that the multiplier results in these last two scales of spaces involve only boundedness conditions on the resolvents and are therefore applicable to arbitrary Banach spaces. The results are applied to various classes of nonlinear integral and integro-differential equations.

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* Corresponding author.
E-mail addresses: keyantuo@uprrp.edu (V. Keyantuo), carlos.lizama@usach.cl (C. Lizama), vpoblete@uchile.cl (V. Poblete).

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1. Introduction

Ever since completion of Fourier's ground-breaking work on the propagation of heat in solid bodies in 1807, followed by the monograph "Théorie Analytique de la Chaleur" in 1822, Fourier analysis has become an indispensable tool in analysis. It is not only essential in the analysis of differential equations, but is also a very important tool in most areas of pure and applied mathematics, science and technology. It was discovered recently that when dealing with operator equations in abstract spaces, the theory of Fourier multipliers can be used effectively. New challenges arise in this setting—the operator case—that are not present in the scalar or even the vector-valued case. Although the fundamental problem of characterizing bounded multiplier transformations in L^p remains open (that is, for $p \notin \{1,2,\infty\}$) even in the scalar case, in the case of resolvent operators, many advances have come to the fore in the last few years. The abstract results developed have concrete applications involving partial differential operators and integral equations arising in mathematical physics.

The aim of this paper is to study the integro-differential equation

$$\mu * u' + \nu * u - \eta * Au = f, \tag{1.1}$$

where A is a closed operator in a Banach space X; μ, ν, and η are finite scalar-valued measures on $\mathbb{R}$, u' stands for the time derivative of u and f is a 2π-periodic function with values in X. Here $\mu * u'$ represents the convolution product i.e. $(\mu * u')(t) = \int_{\mathbb{R}} u'(t-s)\,\mu(ds)$, and $\nu * u$ and $\eta * Au$ are defined analogously. The function u is extended to $\mathbb{R}$ by periodicity without change of notation. We are concerned with strong and mild solutions of (1.1) in various spaces of vector-valued functions. Specifically, we consider the Lebesgue–Bochner spaces $L^p((0,2\pi);X)$, $1 \leqslant p < \infty$, the Besov spaces $B^s_{p,q}((0,2\pi);X)$, $1 \leqslant p,q \leqslant \infty$, and in particular the Hölder–Zygmund spaces $\mathcal{C}^s$, $s > 0$ (these are identified with the Besov spaces $B^s_{\infty,\infty}((0,2\pi);X)$ and correspond to the familiar Hölder spaces C^s, if $0 < s < 1$). Also considered later in the paper are the Triebel–Lizorkin spaces $F^s_{p,q}((0,2\pi);X)$, $1 \leqslant p,q < \infty$. In the scalar case, the famous Littlewood–Paley inequalities show that $F^0_{p,2} = L^p$, $1 < p < \infty$, with equivalent norms. This is no longer true in the vector-valued case. In fact, the equality $F^0_{p,2}((0,2\pi);X) = L^p((0,2\pi);X)$ holds if and only if X is isomorphic to a Hilbert space. See [33] and [12].

Eq. (1.1) was studied by Staffans [36]. He considered the case where X is a Hilbert space and gave conditions for strong and mild well-posedness for L^2 solutions. The main tool he used was Plancherel's theorem. As is well known, this theorem is valid in $L^p((0,2\pi);X)$ if and only if $p = 2$ and X is (isomorphic to) a Hilbert space (see e.g. [6]). The more general situation we consider here therefore calls for other methods. In our study of Eq. (1.1), we employ the method of operator-valued Fourier multipliers which enables us to provide explicit conditions on the measures and on the operator A ensuring well-posedness. In recent years, the theory of operator-valued Fourier multipliers has been extensively developed and applied to well-posedness of abstract differential equations. We note for example the papers [2,4–6,11,12,15,20,25,27,29,31,39] and the references cited therein.

Strong well-posedness for special cases of (1.1) has been studied earlier (see [12,28,29]) using Fourier multipliers. There are earlier papers dealing with the special equations treated in [29] and [28] which make the assumption that the operator A generates an analytic semigroup (not necessarily strongly continuous) and use resolvent families to construct the solution (see e.g. [16,19] and the references given there). Among the equations not previously considered with the new methods, we mention the renewal equation and the delay equation

$$u'(t) = Au(t) - \gamma u(t-\tau) + f(t), \quad t \in \mathbb{R}, \tag{1.2}$$

where τ and γ are given real numbers. Of course, the differential equation

$$P_{\text{per}}(f)\begin{cases} u'(t) = Au(t) + f(t), & t \in [0,2\pi], \\ u(0) = u(2\pi) \end{cases} \tag{1.3}$$

is also a special case of (1.1). This is obtained when $\mu = \eta$ is the Dirac measure concentrated at the origin and $\nu = 0$. This equation is treated in [5,6] and [12] (see also the survey paper [4]).

In the present paper, we make a complete study of well-posedness of (1.1) in the above mentioned function spaces. We consider mild and strong well-posedness. In the case of mild well-posedness, it turns out that one can consider a one-parameter family of such notions depending on the index $\alpha \geqslant 0$ of the intermediate spaces $H_p^\alpha((0,2\pi);X)$ for $1 < p < \infty$. Both strong and mild well-posedness are important in the study of nonlinear problems. There are three important notions that are needed in the study, namely n-regularity of scalar sequences, M-boundedness and MR-boundedness of order n for operator sequences. The concept of n-regular sequences was introduced in [29] as a discrete version of k-regularity used in [32] and was subsequently used in [28] and [12]. On the other hand, M stands for Marcinkiewicz. Define the differences $\Delta^k M_n$ by $\Delta^0 M_n = M_n$, $\Delta^1 M_n = \Delta M_n = M_{n+1} - M_n$, and $\Delta^{k+1} M_n = \Delta(\Delta^k M_n)$, for $k \geqslant 1$. If $\{M_n\}$ is the operator family under consideration, M-boundedness (resp. MR-boundedness) of order m ($m \in \mathbb{N} \cup \{0\}$) means that the sequences $\{n^j \Delta^j M_n\}$ are bounded (resp. R-bounded) for $0 \leqslant j \leqslant m$.

Under appropriate assumptions on the Fourier coefficients of the measures involved, we give necessary and sufficient conditions for strong well-posedness of (1.1). These conditions are in terms of the resolvent. In the L^p case, as is shown in [6] (see also [20] and [39]), R-boundedness is a necessary condition for an operator family to be a multiplier. In the $B_{p,q}^s((0,2\pi);X)$ case, R-boundedness is not necessary but in general, one has to require a Marcinkiewicz condition of order two. A Marcinkiewicz condition of order one is enough if the Banach space X has nontrivial Fourier type (see [6] and [22]). Likewise, for the Triebel–Lizorkin spaces $F_{pq}^s((0,2\pi);X)$ the R-boundedness condition is not necessary. Sufficient conditions for multipliers involve M-boundedness of order 3 in general, and order 2 if $1 < p < \infty$, $1 < q \leqslant \infty$, $s \in \mathbb{R}$.

Compared to the previous papers [29] and [28] (see also [12]), we simplify our assumptions. They are now more symmetric and depend solely on the differences $\Delta^k M_n$. Compared to the paper of Staffans even in the L^2 context for Hilbert spaces, we give more specific conditions ensuring well-posedness. For example, we give conditions under which assumption (i) in [36, Theorem 3.2] already implies assumption (iii) of the same theorem. Moreover, for nonlinear problems, L^2 results are sometime not enough and one needs L^p estimates (see [1]). One surprising feature of our results is that in some cases, it is possible to characterize mild well-posedness directly in terms of boundedness or R-boundedness conditions on the resolvent.

The study of nonlinear equations is one of the main areas of application for maximal regularity. For example, quasilinear equations of convolution type on the real line have been studied in Amann [3] in the parabolic case. Maximal regularity is used by Chill and Srivastava [15] for the treatment of second order equations, both semi-linear and quasilinear. Other references include [4,16,28] and [36]. We take up nonlinear equations in Section 9. There, we illustrate through various examples the applicability of the results obtained for linear problems to nonlinear integral and integro-differential equations in Banach and Hilbert spaces. We now describe the content of the various sections. In Section 2, we give some preliminary definitions on Fourier multipliers and R-boundedness. In Section 3, we consider the Marcinkiewicz conditions and their behavior with respect to sums and products. Section 4 is devoted to n-regularity of scalar sequences and the behavior under sums, products and quotients. Strong L^p solutions are studied in Section 5. In Section 6, we deal with mild L^p solutions. Solutions in Besov spaces are the subject matter of Section 7 while the corresponding results in the Triebel–Lizorkin spaces are established in Section 8. We apply the results to semi-linear problems in Section 9.

2. Preliminaries

Let X, Y be complex Banach spaces. We denote by $\mathcal{B}(X,Y)$ the Banach space of all bounded linear operators from X to Y. When $X = Y$ we write simply $\mathcal{B}(X)$ and denote by I the identity operator in $\mathcal{B}(X)$. For a closed linear operator A with domain and range in X we write $\rho(A)$ for the resolvent set of A. When $\lambda \in \rho(A)$ we denote by $R(\lambda, A) = (\lambda I - A)^{-1}$ the resolvent operator. When we consider $D(A)$ as a Banach space we always understand that it is equipped with the graph norm.

For a function $f \in L^1((0,2\pi);X)$, we denote by

$$\hat{f}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)\,dt$$

the kth Fourier coefficient of f, where $k \in \mathbb{Z}$. The Fourier coefficients determine the function f; i.e., $\hat{f}(k) = 0$ for all $k \in \mathbb{Z}$ if and only if $f(t) = 0$ a.e. For $\mu \in M(\mathbb{R},\mathbb{C})$ (the space of bounded measures) we denote by $\tilde{\mu}$ the Fourier transform of μ, that is,

$$\tilde{\mu}(\omega) = \int_{-\infty}^{\infty} e^{-i\omega t}\,\mu(dt), \quad \omega \in \mathbb{R}.$$

If μ has a density $a \in L^1(\mathbb{R})$, then $\tilde{\mu}$ is the Fourier transform of the function a and we will continue to denote it by $\tilde{a}$.

Let $\mu \in M(\mathbb{R},\mathbb{C})$. Let $v \in L^1((0,2\pi);X)$ extended by periodicity to $\mathbb{R}$. Using Fubini's theorem we obtain, for $k \in \mathbb{Z}$,

$$\widehat{\mu * v}(k) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ik(t-s)} v(t-s)\,dt \int_{-\infty}^{\infty} e^{-iks}\,\mu(ds)$$

and hence

$$\widehat{\mu * v}(k) = \tilde{\mu}(k)\hat{v}(k), \quad k \in \mathbb{Z}. \tag{2.1}$$

This is a very important identity in our investigations.

As usual, we identify the spaces of (vector- or operator-valued) functions defined on $[0,2\pi]$ to their periodic extensions to $\mathbb{R}$. Thus, in this section, we consider the space $L^p((0,2\pi);X)$ (denoted also $L^p_{2\pi}(\mathbb{R};X)$), $1 \leqslant p \leqslant \infty$, of all 2π-periodic Bochner measurable X-valued functions f such that the restriction of f to $[0,2\pi]$ is p-integrable (usual modification in case $p = \infty$).

We recall the notion of operator-valued Fourier multiplier in L^p spaces (see [6]). Corresponding definitions for Besov and Triebel–Lizorkin spaces will appear in Sections 7 and 8, respectively.

Definition 2.1. Let X, Y be Banach spaces and $1 \leqslant p \leqslant \infty$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is an L^p-multiplier if for each $f \in L^p((0,2\pi);X)$ there exists a function $g \in L^p((0,2\pi);Y)$ such that

$$M_k\hat{f}(k) = \hat{g}(k), \quad k \in \mathbb{Z}.$$

A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is an L^p-multiplier if and only if there exists a bounded operator $\mathcal{M} : L^p((0,2\pi);X) \to L^p((0,2\pi);Y)$ such that

$$\widehat{(\mathcal{M}f)}(k) = M_k\hat{f}(k)$$

for all $k \in \mathbb{Z}$ and all $f \in L^p((0,2\pi);X)$.

Remark 2.2. (i) The set of L^p-multipliers is a vector space. Moreover, it is clear from the definition that if X, Y, Z are Banach spaces and $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ and $\{N_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(Y,Z)$ are L^p-multipliers then $\{N_kM_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Z)$ is an L^p-multiplier as well. When $X = Y$, the space of L^p-multipliers is an operator algebra.

(ii) We note that if for some fixed $N \in \mathbb{N}$, $M_k = 0$ for $|k| > N$ then $\{M_k\}_{k\in\mathbb{Z}}$ is a Fourier multiplier. This way, when we check conditions ensuring that a sequence $\{M_k\}_{k\in\mathbb{Z}}$ is a Fourier multiplier, what really matters is when $|k|$ is large. This observation applies to operator- and scalar-valued Fourier multipliers in various contexts considered throughout the paper.

Example 2.3. If $\mu \in M(\mathbb{R}, \mathbb{C})$ then the sequence $\{M_k = \tilde{\mu}(k)I\}$ is an L^p-multiplier for every $1 \leqslant p \leqslant \infty$. This follows directly from the definition, Eq. (2.1) and Young's inequality.

Definition 2.4. For $k \in \mathbb{Z}$, let

$$M_k = \begin{cases} I & \text{if } k \geqslant 0, \\ 0 & \text{if } k < 0. \end{cases}$$

We say that X is a *UMD* space if the sequence $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for all (equivalently one) $p \in (1, \infty)$.

Equivalently, X is a *UMD* space if and only if the sequence $\{N_k\}$ defined by

$$N_k = \begin{cases} I & \text{if } k \geqslant 0, \\ -I & \text{if } k < 0 \end{cases}$$

is an L^p-multiplier. Note that $\{M_k\}$ corresponds to the Riesz projection while $\{N_k\}$ is the representation of the Hilbert transform in the periodic case. For more on *UMD* spaces we refer to [1, Chapter IV], [8,13,18,20] and [32] where examples, properties and several equivalent definitions, notably the one involving martingales in Banach spaces can be found.

We introduce the means

$$\left\|(x_1, \ldots, x_n)\right\|_R := \frac{1}{2^n} \sum_{\epsilon_j \in \{-1,1\}^n} \left\| \sum_{j=1}^{n} \epsilon_j x_j \right\|$$

for $x_1, \ldots, x_n \in X$.

Definition 2.5. Let X, Y be Banach spaces. A subset $\mathcal{T}$ of $\mathcal{B}(X, Y)$ is called R-bounded if there exists a constant $c \geqslant 0$ such that

$$\left\|(T_1 x_1, \ldots, T_n x_n)\right\|_R \leqslant c \left\|(x_1, \ldots, x_n)\right\|_R \tag{2.2}$$

for all $T_1, \ldots, T_n \in \mathcal{T}$, $x_1, \ldots, x_n \in X$, $n \in \mathbb{N}$. The least c such that (2.2) is satisfied is called the R-bound of $\mathcal{T}$ and is denoted $R(\mathcal{T})$.

An equivalent definition using the Rademacher functions can be found in the references cited below.

The notion of R-boundedness was implicitly introduced and used by Bourgain [9] and later on also by Zimmermann [40]. Explicitly it is due to Berkson and Gillespie [8] and to Clément, de Pagter, Sukochev and Witvliet [18]. Its importance for operator-valued Fourier multipliers was realized by Weis [39], Clément and Prüss [17], Arendt and Bu [6], Kunstmann and Weis [30], among others. For abstract multipliers, it is also of great importance (see [18]).

R-boundedness clearly implies boundedness. If $X = Y$, the notion of R-boundedness is strictly stronger than boundedness unless the underlying space is isomorphic to a Hilbert space [6, Proposition 1.17]. Some useful criteria for R-boundedness are provided in [6,20] and [22].

Remark 2.6. (a) Let $\mathcal{S}, \mathcal{T} \subset \mathcal{B}(X, Y)$ be R-bounded sets, then $\mathcal{S} + \mathcal{T} := \{S + T\colon\ S \in \mathcal{S},\ T \in \mathcal{T}\}$ is R-bounded.

(b) Let $\mathcal{T} \subset \mathcal{B}(X,Y)$ and $\mathcal{S} \subset \mathcal{B}(Y,Z)$ be R-bounded sets, then $\mathcal{S} \cdot \mathcal{T} := \{S \cdot T\colon\ S \in \mathcal{S},\ T \in \mathcal{T}\} \subset \mathcal{B}(X,Z)$ is R-bounded and

$$R(\mathcal{S} \cdot \mathcal{T}) \leqslant R(\mathcal{S}) \cdot R(\mathcal{T}).$$

(c) Also, each subset $M \subset \mathcal{B}(X)$ of the form $M = \{\lambda I\colon\ \lambda \in \Omega\}$ is R-bounded whenever $\Omega \subset \mathbb{C}$ is bounded. This follows from Kahane's contraction principle (see [6,18] or [20]).

3. Marcinkiewicz conditions

Sufficient conditions for operator-valued Fourier multipliers in the L^p context have been derived recently and used by many authors in the study of maximal regularity for differential equations. We mention Weis [39], Arendt [4], Arendt and Bu [6], Denk, Hieber and Prüss [20] and the paper by Hytönen [25]. In order to present the conditions that we will need later we introduce some notation. Let $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ be a sequence of operators. We set

$$\Delta^0 M_k = M_k, \qquad \Delta M_k = \Delta^1 M_k = M_{k+1} - M_k$$

and for $n = 2, 3, \ldots$

$$\Delta^n M_k = \Delta\big(\Delta^{n-1} M_k\big).$$

Definition 3.1. We say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is M-bounded of order n ($n \in \mathbb{N} \cup \{0\}$), if

$$\sup_{0\leqslant l\leqslant n} \sup_{k\in\mathbb{Z}} \big\| k^l \Delta^l M_k \big\| < \infty. \tag{3.1}$$

Observe that for $j \in \mathbb{Z}$ fixed, we have $\sup_{0\leqslant l\leqslant n} \sup_{k\in\mathbb{Z}} \|k^l \Delta^l M_k\| < \infty$ if and only if $\sup_{0\leqslant l\leqslant n} \sup_{k\in\mathbb{Z}} \|k^l \Delta^l M_{k+j}\| < \infty$. This follows directly from the binomial formula.

To be more explicit when $n = 0$, M-boundedness of order n for $\{M_k\}$ means simply that $\{M_k\}$ is bounded. For $n = 1$ this is equivalent to

$$\sup_{k\in\mathbb{Z}} \|M_k\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}} \big\| k(M_{k+1} - M_k) \big\| < \infty. \tag{3.2}$$

When $n = 2$ we require in addition to (3.2) that

$$\sup_{k\in\mathbb{Z}} \big\| k^2 (M_{k+2} - 2M_{k+1} + M_k) \big\| < \infty \tag{3.3}$$

and when $n = 3$, we require in addition to (3.3) and (3.2)

$$\sup_{k\in\mathbb{Z}} \big\| k^3 (M_{k+3} - 3M_{k+2} + 3M_{k+1} - M_k) \big\| < \infty. \tag{3.4}$$

Remark 3.2. (i) The definition of M-boundedness, where M stands for Marcinkiewicz, was introduced in [29] but was already implicit in [5]. Here we reformulate the definition to make precise the order n.

(ii) Analogously, we define M-boundedness of order n in case of sequences $\{a_k\}_{k\in\mathbb{Z}}$ of real or complex numbers (this amounts to taking $M_k = a_k I$ in $\mathcal{B}(X)$).

(iii) Note that if $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are M-bounded of order n then $\{M_k \pm N_k\}_{k\in\mathbb{Z}}$ is M-bounded of order n. In fact, the set of n-bounded sequences is a vector space. This is obvious from the definition.

The following result establishes a useful property of sequences satisfying the M-boundedness condition of order n. In this paper we shall only need these conditions for $n \leqslant 3$. This is enough for the various characterizations of well-posedness in the sequel.

Theorem 3.3. *If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are sequences in $\mathcal{B}(Y,Z)$ and $\mathcal{B}(X,Y)$ that are M-bounded of order n ($n \leqslant 3$) then $\{M_kN_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Z)$ is also M-bounded of the same order.*

Proof. From the hypotheses, it is clear that $\sup_{k\in\mathbb{Z}} \|M_kN_k\| < \infty$. To verify M-boundedness of order n ($n = 1, 2, 3$), we have the following identities:

(i) Order 1:

$$\Delta(M_kN_k) = \Delta(M_k)N_{k+1} + \Delta(N_k)M_k.$$

(ii) Order 2:

$$\Delta^2(M_kN_k) = \Delta^2(M_k)N_{k+2} + M_{k+1}\Delta^2(N_k) + \Delta(M_k)\Delta(N_{k+1}) + \Delta(M_k)\Delta(N_k).$$

(iii) Order 3:

$$\begin{aligned}\Delta^3(M_kN_k) = {} & \Delta^3(M_k)N_{k+3} + \Delta^2(M_k)\big(\Delta(N_{k+2}) + \Delta(N_{k+1}) + \Delta(N_k)\big) \\ & + \Delta^3(N_k)M_{k+2} + 2\Delta(M_{k+1})\Delta^2(N_k) + \Delta(M_{k+1})\Delta^2(N_{k+1}).\end{aligned}$$

Since $\{M_k\}$ and $\{N_k\}$ are M-bounded of order n ($n = 1, 2, 3$) we obtain from the above identities that $\{M_kN_k\}$ verifies (3.2)–(3.4). $\square$

Remark 3.4. The result is also true in the case of sequences $\{a_k\}_{k\in\mathbb{Z}}$ of real or complex numbers satisfying M-boundedness of order n ($n = 1, 2, 3$). In this case we identify a_k with a_kI as already indicated.

Definition 3.5. We say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is MR-bounded of order n, if for each $0 \leqslant l \leqslant n$ the set

$$\big\{k^l\Delta^lM_k:\ k \in \mathbb{Z}\big\} \tag{3.5}$$

is R-bounded.

Remark 3.6. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X,Y)$ is MR-bounded of order 1 if the sets

$$\{M_k:\ k \in \mathbb{Z}\} \quad \text{and} \quad \big\{k(M_{k+1} - M_k):\ k \in \mathbb{Z}\big\} \tag{3.6}$$

are R-bounded.

If in addition we have that the set

$$\big\{k^2(M_{k+1} - 2M_k + M_{k-1}):\ k \in \mathbb{Z}\big\} \tag{3.7}$$

is R-bounded then $\{M_k\}_{k\in\mathbb{Z}}$ is MR-bounded of order 2.

If (3.6) and (3.7) are satisfied and

$$\big\{k^3(M_{k+1} - 3M_k + 3M_{k-1} - M_{k-2}):\ k \in \mathbb{Z}\big\} \tag{3.8}$$

is R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is MR-bounded of order 3.

Remark 3.7. According to the second section, in Hilbert spaces MR-bounded and M-bounded are identical concepts. In general, MR-bounded implies R-bounded which in turn implies boundedness. Moreover, note that MR-boundedness implies M-boundedness.

Using the same identities as in the proof of Theorem 3.3 one proves the following result.

Theorem 3.8. *If $\{M_k\}_{k\in\mathbb{Z}}$ and $\{N_k\}_{k\in\mathbb{Z}}$ are MR-bounded sequences of order n $(n \leqslant 3)$ then $\{M_kN_k\}_{k\in\mathbb{Z}}$ is MR-bounded of order n.*

The following theorem is the discrete analogue of the operator-valued version of Mikhlin's theorem due to Arendt and Bu in [6]. The continuous version was proved earlier by Weis [39] using different methods. They used the multiplier theorems to study maximal regularity for the first order Cauchy problem. In [6] maximal regularity for (1.3) is treated as well as boundary value problems for second order differential equations.

Theorem 3.9. *Let X, Y be UMD spaces. If the sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X, Y)$ is MR-bounded of order 1 then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.*

We observe that the condition of MR-boundedness of order 0 (that is, R-boundedness) for $\{M_k\}_{k\in\mathbb{Z}}$ is necessary.

The following corollary due to Zimmermann [40] is the vector-valued version of the Marcinkiewicz multiplier theorem. It shows the importance of the concept of *UMD* spaces. It is an extension of earlier multiplier results known for $L^p(l^q)$.

Corollary 3.10. *Let X be a UMD space. If $\{m_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ is M-bounded of order 1 then $\{m_kI\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.*

We note that if $\{M_k\}$ is the sequence considered in Definition 2.4, then $\{M_k\}$ is R-bounded (Kahane's inequality) of order n for any n but is not a Fourier multiplier unless X is a *UMD* space.

4. *n*-regular sequences

The notion of 1-regular and 2-regular scalar sequences was introduced in [29] to study maximal regularity of integro-differential equations on periodic Lebesgue and Besov spaces. This concept is the discrete analogue for the notion of n-regularity related to Volterra integral equations (see [32, Chapter I, Section 3.2]). Recently, Bu and Fang in [10] introduced the notion of 3-regular sequences to study maximal regularity of integro-differential equations on the scale $F^s_{p,q}$ of Triebel–Lizorkin spaces.

Definition 4.1. A sequence $\{a_k\}_{k\in\mathbb{Z}} \subseteq \mathbb{C} \setminus \{0\}$ is called n-regular $(n \in \mathbb{N})$ if

$$\sup_{1\leqslant l\leqslant n}\sup_{k\in\mathbb{Z}}\left\|k^l\left(\Delta^l a_k\right)/a_k\right\| < \infty. \tag{4.1}$$

Note that if $\{a_k\}_{k\in\mathbb{Z}}$ is 1-regular then $\lim_{|k|\to\infty} a_{k+1}/a_k = 1$. Observe that an n-regular sequence need not be bounded.

As an immediate consequence of the definition, we have the following result showing the interplay between n-regularity and M-bounded sequences.

Proposition 4.2. *If $\{a_k\}_{k\in\mathbb{Z}}$ is a bounded and n-regular sequence, then it is M-bounded of order n.*

Remark 4.3. The converse is false in general. For example, the sequence $a_k = e^{-k^2}$ is M-bounded of order n for every n but is not even 1-regular.

However, we have the following useful observation which follows at once from the definition of n-regular sequence.

Proposition 4.4. *Let $n \in \mathbb{N}$. If $\{a_k\}_{k\in\mathbb{Z}}$ is M-bounded of order n and $\{\frac{1}{a_k}\}$ is bounded, then $\{a_k\}_{k\in\mathbb{Z}}$ is an n-regular sequence.*

Remark 4.5. The boundedness of $\{\frac{1}{a_k}\}$ is not a necessary condition in order to have the conclusion of the above proposition. For example the sequence $a_k = \frac{1}{ik+1}$ is M-bounded of order n and n-regular for all $n \in \mathbb{N}$.

In the next theorem, we give some useful properties of n-regular sequences for $n \leqslant 3$.

Theorem 4.6. *Let $(a_k)_{k\in\mathbb{Z}}, (b_k)_{k\in\mathbb{Z}}$ be given sequences and let $n \leqslant 3$.*

(i) *If $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are n-regular sequences such that $\sup_k |\frac{a_k}{a_k+b_k}| < \infty$, then the sequence $\{a_k+b_k\}_{k\in\mathbb{Z}}$ is n-regular.*
(ii) *If the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are n-regular, then the sequence $\{a_kb_k\}_{k\in\mathbb{Z}}$ is n-regular.*
(iii) *The sequence $\{a_k\}_{k\in\mathbb{Z}}$ is n-regular if and only if the sequence $\{\frac{1}{a_k}\}_{k\in\mathbb{Z}}$ is n-regular.*
(iv) *If the sequences $\{a_k\}_{k\in\mathbb{Z}}$ and $\{b_k\}_{k\in\mathbb{Z}}$ are n-regular, then the sequence $\{a_k/b_k\}_{k\in\mathbb{Z}}$ is n-regular.*

Proof. First we prove (i). For 1-regularity observe that

$$\frac{k(\Delta[a_k+b_k])}{a_k+b_k} = \frac{k(\Delta a_k)}{a_k}\frac{a_k}{a_k+b_k} + \frac{k(\Delta b_k)}{b_k} - \frac{k(\Delta b_k)}{b_k}\frac{a_k}{a_k+b_k}.$$

In view of the hypothesis, 1-regularity of $\{a_k+b_k\}$ follows. To verify 2-regularity, we observe that

$$\frac{k^2(\Delta^2[a_{k-1}+b_{k-1}])}{a_k+b_k} = \frac{k^2(\Delta^2 a_{k-1})}{a_k}\frac{a_k}{a_k+b_k} + \frac{k^2(\Delta^2 b_{k-1})}{b_k} - \frac{k^2(\Delta^2 b_{k-1})}{b_k}\frac{a_k}{a_k+b_k}.$$

Finally, to verify 3-regularity, this time we note that

$$\frac{k^3(\Delta^3[a_{k-2}+b_{k-2}])}{a_k+b_k} = \frac{k^3(\Delta^3 a_{k-2})}{a_k}\frac{a_k}{a_k+b_k} + \frac{k^3(\Delta^3 b_{k-2})}{b_k} - \frac{k^3(\Delta^3 b_{k-2})}{b_k}\frac{a_k}{a_k+b_k}.$$

This completes the proof of (i). As for the proof of (ii), is suffices to note that

$$\frac{k(\Delta[a_kb_k])}{a_kb_k} = \frac{k(\Delta a_k)}{a_k}\frac{b_{k+1}}{b_k} + \frac{k(\Delta b_k)}{b_k}.$$

Since $\{a_k\}$ and $\{b_k\}$ are 1-regular sequences, it follows that $\{a_kb_k\}$ is 1-regular. In order to show that $\{a_kb_k\}$ is 2-regular, we take advantage of the following identity

$$\frac{k^2(\Delta^2[a_{k-1}b_{k-1}])}{a_kb_k} = \frac{k^2(\Delta^2 a_{k-1})}{a_k}\frac{b_{k+1}}{b_k} + \frac{k^2(\Delta^2 b_{k-1})}{b_k} + \frac{k(\Delta a_{k-1})}{a_k}\frac{k[(\Delta b_k)+(\Delta b_{k-1})]}{b_k}.$$

Since $\{a_k\}$ and $\{b_k\}$ are 2-regular sequences, it follows that $\{a_kb_k\}$ is 2-regular. Finally, using the relation

$$\frac{k^3(\Delta^3[a_{k-2}b_{k-2}])}{a_kb_k} = \frac{k^3(\Delta^3 a_{k-2})}{a_k}\frac{b_{k+1}}{b_k} + \frac{k^2(\Delta^2 a_{k-2})}{a_{k-1}}\frac{k[(\Delta b_k)+(\Delta b_{k-1})+(\Delta b_{k-2})]}{b_k}\frac{a_{k-1}}{a_k}$$
$$+\frac{k^3(\Delta^3 b_{k-2})}{b_k} + 2\frac{k^2(\Delta^2 b_{k-2})}{b_{k-1}}\frac{k(\Delta a_{k-1})}{a_k}\frac{b_{k-1}}{b_k} + \frac{k(\Delta a_{k-1})}{a_k}\frac{k^2(\Delta^2 b_{k-1})}{b_k},$$

we see that $\{a_kb_k\}$ is 3-regular.

Now we note that (iv) is a consequence of (ii) and (iii). Therefore to complete the proof of the theorem it remains to verify (iii). To this end, observe that $\frac{k(\Delta 1/a_k)}{1/a_k} = -\frac{k(\Delta a_k)}{a_k}\frac{a_k}{a_{k+1}}$. Since $\{a_k\}$ is a 1-regular sequence, it follows that $|\frac{a_{k+1}}{a_k} - 1| \leqslant M/|k|$, $k \neq 0$, for some $M > 0$, and hence $a_k/a_{k+1} \to 1$ as $|k| \to \infty$. It follows that $\{1/a_k\}$ is 1-regular.

To show 2-regularity, we write

$$\frac{k^2(\Delta^2 1/a_{k-1})}{1/a_k} = \frac{k[(\Delta a_k)+(\Delta a_{k-1})]}{a_{k-1}}\frac{k(\Delta a_k)}{a_{k+1}} - \frac{k^2(\Delta^2 a_{k-1})}{a_{k-1}}.$$

Finally, to verify 3-regularity, we write

$$\frac{k^3(\Delta^3 1/a_{k-2})}{1/a_k} = -\frac{a_{k-1}}{a_{k-2}}\frac{a_k}{a_{k-1}}\frac{a_k}{a_{k+1}}\frac{k^3(\Delta^3 a_{k-2})}{a_k} + 3\frac{a_{k-1}}{a_{k-2}}\frac{a_k}{a_{k+1}}\frac{k(\Delta a_{k-1})}{a_{k-1}}\frac{k^2(\Delta^2 a_{k-1})}{a_k}$$
$$-3\frac{a_k}{a_{k+1}}\frac{k(\Delta a_{k-1})}{a_{k-1}}\frac{k(\Delta a_{k-2})}{a_{k-2}}\frac{k(\Delta a_k)}{a_k} - 3\frac{a_{k-2}}{a_{k+1}}\frac{k(\Delta a_{k-1})}{a_{k-1}}\frac{k(\Delta a_{k-2})}{a_{k-2}}\frac{k(\Delta a_{k-2})}{a_{k-2}}$$
$$+3\frac{a_{k-1}}{a_{k+1}}\frac{k(\Delta a_{k-1})}{a_{k-1}}\frac{k^2(\Delta^2 a_{k-2})}{a_{k-1}}.$$

The result follows immediately. This completes the proof of the theorem. □

Remark 4.7. (i) In general, it is not enough to assume that the sequences $\{a_k\}$ and $\{b_k\}$ are n-regular in order for the sum $\{a_k + b_k\}$ to enjoy the same property. For example, a direct computation shows that $a_k = ik + e^{-ik}$ and $b_k = -ik$ are 1-regular sequences, whereas the sequence $a_k + b_k = e^{-ik}$ is not 1-regular.

(ii) The condition $\sup_k |\frac{a_k}{a_k+b_k}| < \infty$ in Theorem 4.6 is equivalent to $\sup_k |\frac{b_k}{a_k+b_k}| < \infty$.

(iii) We also note that the condition $\sup_k |\frac{a_k}{a_k+b_k}| < \infty$ in Theorem 4.6 is not necessary. This is evidenced by the following example. Take $a_k = k$, $b_k = 1 - k$, $k \in \mathbb{Z}$. Both sequences are n-regular for all $n \in \mathbb{N}$. Also, $a_k + b_k = 1$, $k \in \mathbb{Z}$, is n-regular for all $n \in \mathbb{N}$. Yet, $\{\frac{a_k}{a_k+b_k}\}$ is unbounded.

We now present a series of examples which correspond to various classes of equations that are subsumed under our main results.

Example 4.8. Let $c_k \neq -1$ for all $k \in \mathbb{Z}$ and define $b_k = \frac{ik}{1+c_k}$. Suppose c_k is M-bounded of order n ($n \leqslant 3$) and $\frac{1}{1+c_k}$ is bounded. Since $1 + c_k$ is also M-bounded, it follows from Proposition 4.4 that $1 + c_k$ is n-regular and then, using Theorem 4.6(iv) we conclude that b_k is an n-regular sequence (compare [29, p. 741]).

Example 4.9. Let $c_0, \gamma_0, \gamma_\infty \in \mathbb{R}$ be given and suppose $\{a_k\}$, $\{b_k\}$ are M-bounded sequences of order n ($n \leqslant 3$) with $\{\frac{1}{c_0-a_k}\}, \{\frac{1}{ik(\gamma_0+b_k)+\gamma_\infty}\}$ well defined and bounded. Define $d_k = \frac{ik(\gamma_0+b_k)+\gamma_\infty}{c_0-a_k}$. It follows by Theorem 3.3 and Remark 3.2(iii) that $\{c_0 - a_k\}$ and $\{ik(\gamma_0 + b_k) + \gamma_\infty\}$ are M-bounded. Hence from Proposition 4.4 we obtain that the same sequences are also n-regular. Finally, using Theorem 4.6(iv) we deduce that $\{d_k\}$ is an n-regular sequence (compare [28, p. 30]).

Example 4.10. Suppose $\{a_k\}$ is an M-bounded sequence of order n and such that $\{\frac{1}{a_k}\}$ is bounded. Then we obtain from Proposition 4.4 and Theorem 4.6(iv) that $d_k = \frac{-ik}{a_k}$ is an n-regular sequence. This example is important in the scalar case, i.e. with $A = I$ and $X = \mathbb{C}^n$, as we will see later (cf. [24, Theorem 3.11, p. 87]).

5. Well-posedness in L^p spaces

Having presented in the previous sections preliminary material on M-boundedness and Fourier multipliers we will now show how these tools can be used to handle the integro-differential equation (1.1).

In this section we proceed to study L^p well-posedness of the general integro-differential equation (1.1). Here we do not assume that A is densely defined but merely that A is a closed operator. The results give concrete conditions on the measures ν, μ, η as well as the operator A under which Eq. (1.1) is strongly well-posed. Special cases that have been studied before are incorporated into the new framework. In the next section we will study mild well-posedness in L^p spaces. Strong and mild well-posedness in other scales of function spaces will be taken up in the subsequent sections.

The definition of strong well-posedness which we investigate in this section is as follows.

Definition 5.1. We say that the problem (1.1) is strongly L^p well-posed ($1 \leqslant p < \infty$) if for each $f \in L^p((0, 2\pi); X)$ there exists a unique function $u \in H^1_p((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ such that (1.1) is satisfied (for almost every t).

The function u in Definition 5.1 will be called the *strong L^p solution* of Eq. (1.1). For a closed operator A in X with domain $D(A)$ and $1 \leqslant p < \infty$, we define the operator $\mathcal{A}$ on $L^p((0, 2\pi); X)$ by $D(\mathcal{A}) = H^1_p((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ and

$$\mathcal{A}u = \mu * u' + \nu * u - \eta * Au.$$

Here $H^1_p((0, 2\pi); X)$ is the vector-valued Sobolev space, which is denoted H^1 in case $p = 2$.

Remark 5.2. In terms of the operator $\mathcal{A}$ defined above, Definition 5.1 is equivalent to saying that it is one-to-one and surjective. By the closed graph theorem, it follows that $\mathcal{A}$ has a continuous inverse $\mathcal{B}$ that maps $L^p((0, 2\pi); X)$ into $H^1_p((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$.

We have the following preliminary result.

Proposition 5.3. *Let X be a UMD space and A a closed linear operator defined on X. Let $\{a_k\}_{k\in\mathbb{Z}}$, $\{b_k\}_{k\in\mathbb{Z}}$ be 1-regular sequences such that $\{\frac{b_k}{a_k}\}$ is bounded and $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. Then the following assertions are equivalent:*

(i) *$\{a_k(b_k I - A)^{-1}\}_{k\in\mathbb{Z}}$ is an L^p-multiplier, $1 < p < \infty$.*
(ii) *$\{a_k(b_k I - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. Let $M_k = a_k(b_k I - A)^{-1}$. By [6, Proposition 1.11], it follows that (i) implies (ii). Note that $\{\frac{1}{a_k}\}$ is 1-regular by Theorem 4.6(iii). Then the result is a consequence of the following identity

$$k(M_{k+1} - M_k) = M_{k+1}\frac{b_k}{a_{k+1}}k\frac{(b_k - b_{k+1})}{b_k}M_k - k\frac{\frac{1}{a_{k+1}} - \frac{1}{a_k}}{\frac{1}{a_k}}M_{k+1}. \qquad \square$$

When $a_k = b_k$ we obtain the following special case of Proposition 5.3 which will be used later (see also [29, Proposition 2.8]). Note that condition (ii) is independent of $p \in (1, \infty)$.

Corollary 5.4. *Let X be a UMD space and A a closed linear operator defined on X. Let $\{b_k\}_{k\in\mathbb{Z}}$ be a 1-regular sequence such that $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. Then the following assertions are equivalent*:

(i) $\{b_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ *is an L^p-multiplier,* $1 < p < \infty$.
(ii) $\{b_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ *is R-bounded.*

In the remaining part of this section we will assume that $\tilde{\eta}(k) \neq 0$ for all $k \in \mathbb{Z}$ and the sequence

$$\left\{\frac{1}{\tilde{\eta}(k)}\right\} \text{ is bounded.} \tag{5.1}$$

For example, we can take $\eta = a\delta_0 + \zeta$ where $a \neq 0$ and $\zeta \in M(\mathbb{R}, \mathbb{C})$ has a density $L^1(\mathbb{R})$ and $a + \tilde{\zeta}(k) \neq 0$ for all $k \in \mathbb{Z}$. We now address strong well-posedness of the integro-differential equation (1.1).

Theorem 5.5. *Assume that X is a UMD space and $1 < p < \infty$. Suppose that the sequences $\{ik\tilde{\mu}(k) + \tilde{\nu}(k)\}$ and $\{\tilde{\eta}(k)\}$ are 1-regular. Then the following assertions are equivalent*:

(i) *Problem* (1.1) *is strongly L^p well-posed*;
(ii) $\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} \subseteq \rho(A)$ *and* $\{\frac{ik}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is an L^p-multiplier*;
(iii) $\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} \subseteq \rho(A)$ *and* $\{\frac{ik}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is R-bounded.*

Proof. Set $M_k = \frac{ik}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}$.

(ii) $\Leftrightarrow$ (iii) Let $a_k = \frac{ik}{\tilde{\eta}(k)}$ and $b_k = \frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}$. From the hypotheses and Theorem 4.6 we have that $\{a_k\}$ and $\{b_k\}$ are 1-regular sequences. Since $\{\frac{b_k}{a_k}\} = \{\tilde{\mu}(k) + \frac{\tilde{\nu}(k)}{ik}\}_{k\in\mathbb{Z}\setminus\{0\}}$ it follows that $\{\frac{b_k}{a_k}\}$ is bounded and the assertion now follows from Proposition 5.3.

(i) $\Leftrightarrow$ (ii) Let $N_k = \frac{1}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}, k \in \mathbb{Z}$. Thus, $M_k = ikN_k$, $k \in \mathbb{Z}$. The solution u is constructed through

$$\hat{u}(k) = N_k\hat{f}(k), \quad k \in \mathbb{Z}. \tag{5.2}$$

Indeed, as is well known [6, Lemma 2.2], the assumption that $\{M_k\}$ is an L^p-multiplier implies that $\{N_k\}$ is an L^p-multiplier as well.

Except for the verification that the solution u constructed using multipliers belongs to $L^p((0, 2\pi); D(A))$, the proof follows the same lines as that of [28, Theorem 2.9] (see also [29]). In fact, by Theorem 4.6(iii) we have that $\{\tilde{\eta}(k)\}$ is 1-regular. It follows that $\{\frac{1}{\tilde{\eta}(k)}\}$ is 1-regular. Note that due to (5.1) the sequence $\{\frac{1}{\tilde{\eta}(k)}\}$ is also bounded, and so we obtain by Proposition 4.2 that the latter sequence is M-bounded of order 1. Hence it is an L^p-multiplier.

Let $\frac{b_k}{ik} = \frac{1}{\tilde{\eta}(k)}[\tilde{\mu}(k) + \frac{\tilde{\nu}(k)}{ik}]$. From hypothesis and Remark 4.7 we have that $\{b_k/ik\}$ is 1-regular and bounded, hence it is M-bounded of order 1 and therefore is an L^p-multiplier.

From the identity

$$AN_k = \frac{b_k}{ik}M_k - \frac{1}{\tilde{\eta}(k)}I \tag{5.3}$$

we conclude that $\{AN_k\}$ is an L^p-multiplier. The proof is complete. $\square$

From the proof of Theorem 5.5 we deduce the following result on maximal regularity.

Corollary 5.6. *The solution u of problem* (1.1) *given by Theorem* 5.5 *satisfies the following maximal regularity property*: $u, u', Au \in L^p((0,2\pi);X)$. *Moreover, there exists a constant $C > 0$ independent of $f \in L^p((0,2\pi);X)$ such that*

$$\|u\|_p + \|u'\|_p + \|Au\|_p \leqslant C\|f\|_p. \tag{5.4}$$

Note that by Definition 5.1 (cf. also Remark 5.2), u, u' as well as all the terms in the left-hand side of (5.4) belong to L^p with continuous dependence on f. Similarly $\mu * u'$, $\nu * u$, $\eta * Au$ belong to $L^p((0,2\pi);X)$ and for some positive constant K we have

$$\|\nu * u\|_p + \|\mu * u'\|_p + \|\eta * Au\|_p \leqslant K\|f\|_p.$$

Example 5.7. Consider the equation

$$u'(t) = Au(t) + \int_{-\infty}^{t} c(t-s)Au(s)\,ds + f(t) \tag{5.5}$$

with the boundary condition $u(0) = u(2\pi)$. This is a special case of Eq. (1.1) corresponding to $\mu = \delta_0$, $\nu = 0$, $\eta = \delta_0 - c(t)\chi_{[0,\infty)}(t)$ where we identify an L^1 function with the associated measure. By Example 4.8, it follows that if

$$c_k := \tilde{c}(k) \text{ is } M\text{-bounded of order 1,}$$

then by Theorem 5.5, Eq. (5.5) has a unique strong L^p solution for every $f \in L^p(0,2\pi;X)$ if and only if the equivalent conditions (ii) and (iii) of Theorem 5.5 hold. Hence we recover the results established in [29] in the L^p case (note incidentally that in that paper, we used $\tilde{c}$ to denote the Laplace transform of c).

Example 5.8. Let $\gamma \in \mathbb{R}$, $\tau > 0$ and consider the delay equation

$$u'(t) = Au(t) - \gamma u(t-\tau) + f(t), \quad t \in \mathbb{R}. \tag{5.6}$$

This problem is motivated by feedback-systems and control theory, see [7] and the references therein. In Eq. (5.6) the operator A corresponds to the system operator which is generally assumed to be the generator of a C_0-semigroup. The term $\gamma u(t-\tau)$ can be interpreted as the feedback. We note that usually the above equation is studied in the context of Hilbert spaces. Here we show that our theory applies and we obtain strong well-posedness.

Indeed, here we have $\tilde{\mu}(k) = 1$, $\tilde{\nu}(k) = \gamma e^{-ik\tau}$ and $\tilde{\eta}(k) = 1$. The hypotheses of the theorem are easily seen to be satisfied if $|\gamma| \notin \mathbb{N}$. More precisely, when X is a *UMD* space, problem (5.6) is strongly L^p well-posed ($1 < p < \infty$) if and only if $\{ik + \gamma e^{-ik\tau}\}_{k\in\mathbb{Z}} \subset \rho(A)$ and $\{ik(ik + \gamma e^{-ik\tau} - A)^{-1}\}_{k\in\mathbb{Z}}$ is R-bounded. When X is a Hilbert space, the last condition is equivalent to boundedness of $\{ik(ik + \gamma e^{-ik\tau} - A)^{-1}\}_{k\in\mathbb{Z}}$. For example, if A generates an analytic semigroup $\mathcal{T} = \{T(t)\}$ of type $\omega(\mathcal{T}) < -|\gamma|$ then it is easy to check that this condition is satisfied.

6. Mild well-posedness in L^p

In this section we study mild solutions of the integro-differential equation (1.1). The definition of mild solution we adopt here first appeared in Staffans [36] in the context of mild L^2 solutions on Hilbert spaces. For the special equation (1.3), another concept of mild solution is studied in [6] and its relationship to the present approach is considered in [26]. Later in this section we will relate the notion of mild solution to the strong solutions studied in Section 5. This will be done in a natural way by constructing a one-parameter family of concepts of mild solutions.

Definition 6.1. We say that problem (1.1) is (H_p^1, L^p) mildly well-posed if there exists a linear operator $\mathcal{B}$ that maps $L^p((0,2\pi);X)$ continuously into itself as well as $H_p^1((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into itself and which satisfies

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in H_p^1((0,2\pi);X) \cap L^p((0,2\pi);D(A))$. In this case the function $\mathcal{B}f$ is called the (H_p^1, L^p) mild solution of (1.1) and $\mathcal{B}$ the solution operator.

More specifically, we require that the following diagram be commutative:

$$\begin{tikzcd}
L^p((0,2\pi);X) \arrow[r, "\mathcal{B}"] & L^p((0,2\pi);X) \\
H_p^1((0,2\pi);X) \cap L^p((0,2\pi);D(A)) \arrow[u, "\mathcal{A}"] \arrow[ur, "\mathcal{I}"] \arrow[r, "\mathcal{B}"'] & H_p^1((0,2\pi);X) \cap L^p((0,2\pi);D(A)) \arrow[u, "\mathcal{A}"']
\end{tikzcd}$$

where $\mathcal{I}$ is the natural injection of $H_p^1((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into $L^p((0,2\pi);X)$.

Clearly, the solution operator $\mathcal{B}$ above is unique, if it exists. Recall that for two Banach spaces Z and X, the notation $Z \hookrightarrow X$ means that Z is continuously embedded into X.

Lemma 6.2. *Suppose $Z \hookrightarrow X$ and $B : D(B) \subset X \to X$ is a closed linear operator. Define B_Z, the part of B in Z by $D(B_Z) = \{x \in D(B) \cap Z, Bx \in Z\}$. Then B_Z is a closed operator.*

Proof. Suppose z_n is a sequence in $D(B_Z)$ which converges in Z to z (thus $z \in Z$) and $B_Z z_n$ converges in Z to w. Then by the continuity of the injection of Z into X, both sequences converge in X to the same limits. Since B is closed, $z \in D(B)$ and $Bz = w$. We then have that $z \in D(B) \cap Z$ and $Bz = w \in Z$. This concludes the proof. $\square$

The following corollary is immediate.

Corollary 6.3. *Suppose $Z \hookrightarrow X$ and $B \in \mathcal{L}(X)$. If Z is invariant under B, then B_Z which we continue to denote by B, satisfies $B \in \mathcal{L}(Z)$.*

Next, we characterize mild solutions using operator-valued Fourier multipliers.

Theorem 6.4. *Assume that $\overline{D(A)} = X$. Let $1 < p < \infty$. Assume that $\tilde{\eta}(k) \neq 0$, for all $k \in \mathbb{Z}$. Then the following assertions are equivalent:*

(i) *Problem (1.1) is (H_p^1, L^p) mildly well-posed;*
(ii) *$\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ and $\{\frac{1}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ is an L^p-multiplier.*

Proof. (ii) $\Rightarrow$ (i) Consider $d_k := \frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}$, $c_k = \frac{1}{\tilde{\eta}(k)}$ and let $\mathcal{B}$ be the operator which maps $f \in L^p((0,2\pi);X)$ into the function $u \in L^p((0,2\pi);X)$ whose kth Fourier coefficient is $c_k R(d_k, A)\hat{f}(k)$, i.e.

$$\widehat{(\mathcal{B}f)}(k) = c_k R(d_k, A)\hat{f}(k) = \hat{u}(k) \tag{6.1}$$

for all $k \in \mathbb{Z}$ and all $f \in L^p((0,2\pi);X)$. By the remark following Definition 2.1, $\mathcal{B}$ is a bounded linear operator on $L^p((0,2\pi);X)$. Let $g \in H^1_p((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and set $h = \mathcal{B}g$. Then,

$$ik\hat{h}(k) = c_k R(d_k, A) ik\hat{g}(k) = c_k R(d_k, A)\hat{g}'(k) \tag{6.2}$$

for all $k \in \mathbb{Z}$. Since $g' \in L^p((0,2\pi);X)$, by (i) there exists $w \in L^p((0,2\pi);X)$ such that

$$\hat{w}(k) = c_k R(d_k, A)\hat{g}'(k) \tag{6.3}$$

for all $k \in \mathbb{Z}$. Hence from (6.2), (6.3) and [6, Lemma 2.1] we obtain $h \in H^1_p((0,2\pi);X)$. Note that $\hat{h}(k) \in D(A)$, $k \in \mathbb{Z}$ since $\hat{h}(k) = c_k R(d_k, A)\hat{g}(k)$ and then $A\widehat{\mathcal{B}g}(k) = \widehat{\mathcal{B}Ag}(k)$. Since by assumption $Ag \in L^p((0,2\pi);X)$, the closedness of A implies that $A\mathcal{B}g \in L^p((0,2\pi);X)$, that is $\mathcal{B}g \in L^p((0,2\pi);D(A))$.

We have proved that $\mathcal{B}$ maps $H^1_p((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into itself. Continuity of $\mathcal{B}$ follows from Corollary 6.3 since the space $H^1_p((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ embeds continuously into $L^p((0,2\pi);X)$.

Finally, for $u \in H^1_p((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ we have

$$\widehat{(\mathcal{A}u)}(k) = \frac{1}{c_k}(d_k I - A)\hat{u}(k) \tag{6.4}$$

for all $k \in \mathbb{Z}$. Hence from (6.1) and [6, Lemma 3.1] we obtain $\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$.

(i) $\Rightarrow$ (ii) Let $x \in X$ and $x_n \in D(A)$ such that $x_n \to x$. Fix $k \in \mathbb{Z}$ and let $f_n(t) = e^{ikt}x_n$ for all $n \in \mathbb{N}$ and $f_0(t) = e^{ikt}x$. Note that $\hat{f}_n(k) = x_n$ and $\hat{f}_n(j) = 0$ for $j \neq k$. Clearly $f_n \to f_0$ in the L^p-norm as $n \to \infty$. Let $u_n = \mathcal{B}f_n$. Then we have

$$ik\tilde{\mu}(k)\hat{u}_n(k) + \tilde{\nu}(k)\hat{u}_n(k) - \tilde{\eta}(k)A\hat{u}_n(k) = \widehat{(\mathcal{A}u_n)}(k) = (\widehat{\mathcal{A}\mathcal{B}f_n})(k) = \hat{f}_n(k) = x_n.$$

Since $\mathcal{B}$ is bounded on $L^p((0,2\pi);X)$, $u_n \to u_0 := \mathcal{B}f_0$ in the L^p-norm, we conclude that $\hat{u}_n(k) \to \hat{u}_0(k)$, and

$$\big(ik\tilde{\mu}(k) + \tilde{\nu}(k) - \tilde{\eta}(k)A\big)\hat{u}_0(k) = x.$$

Hence, for all $k \in \mathbb{Z}$, $(ik\tilde{\mu}(k) + \tilde{\nu}(k) - \tilde{\eta}(k)A)$ is surjective.

Let $x \in D(A)$ be such that $(ik\tilde{\mu}(k) + \tilde{\nu}(k) - \tilde{\eta}(k)A)x = 0$, for $k \in \mathbb{Z}$ fixed. Define $u(t) = e^{ikt}x$. Then, clearly, $u \in W^{1,p}((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and $\mathcal{A}u = 0$. Hence

$$u = \mathcal{B}\mathcal{A}u = 0,$$

and therefore $x = 0$. Since A is closed, we have proved that $\{d_k\}_{k\in\mathbb{Z}} \subset \rho(A)$.

To verify that $(c_k R(d_k, A))_{k\in\mathbb{Z}}$ is an L^p-multiplier, let $f \in L^p((0,2\pi);X)$. We observe that since $\overline{D(A)} = X$ and $1 \leqslant p < \infty$, the space $H^1_p((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ is dense in $L^p((0,2\pi);X)$. Hence there exists a sequence $f_n \in H^1_p((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ such that $f_n \to f$ in the L^p-norm. Define

$$g_n = \mathcal{B}f_n, \quad n \in \mathbb{N}.$$

Then $g_n \in H^1_p((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ and

$$\mathcal{A}g_n = \mathcal{A}\mathcal{B}f_n = f_n, \quad n \in \mathbb{N}.$$

Taking Fourier coefficients and using the fact that $\{d_k\}_{k\in\mathbb{Z}} \subset \rho(A)$, we obtain from the above that

$$\hat{g}_n(k) = c_k(d_k I - A)^{-1}\hat{f}_n(k) \tag{6.5}$$

for all $k \in \mathbb{Z}$. By continuity of $\mathcal{B}$, $\{g_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p((0,2\pi);X)$. Hence there exists $g \in L^p((0,2\pi);X)$ such that $g_n \to g$ in the L^p-norm. From this and using Hölder's inequality we deduce that $\hat{g}_n(k) \to \hat{g}(k)$ and, analogously, $\hat{f}_n(k) \to \hat{f}(k)$. Therefore we conclude from (6.5) that $\hat{g}(k) = c_k(d_k I - A)^{-1}\hat{f}(k)$, for all $k \in \mathbb{Z}$. The claim is proved. $\square$

As a direct consequence of Proposition 5.3 and Theorem 4.6, we obtain the following result. It is remarkable that in some cases we can characterize mild well-posedness in terms of R-boundedness of resolvents. This phenomenon seems to be new.

Theorem 6.5. *Let $1 < p < \infty$. Let X be a UMD space and assume that $\overline{D(A)} = X$. Suppose that*

$$\tilde{\eta}(k) \text{ is 1-regular} \quad \text{and} \quad ik\tilde{\mu}(k) + \tilde{\nu}(k) \text{ is 1-regular and bounded.} \tag{6.6}$$

Then the following assertions are equivalent:

(i) *Problem* (1.1) *is (H^1_p, L^p) mildly well-posed*;
(ii) $\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} \subseteq \rho(A)$ *and* $\{\frac{1}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is an L^p-multiplier*;
(iii) $\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} \subseteq \rho(A)$ *and* $\{\frac{1}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is R-bounded.*

Remark 6.6. Condition (6.6) might seem strong. If we consider ϑ an arbitrary bounded measure and set $\mu = \frac{1}{2i}(\vartheta_\pi - \vartheta_{-\pi})$, where ϑ_a denotes the a-translate of ϑ, then we have $\tilde{\mu}(k) = 0$ for all $k \in \mathbb{Z}$. Another case is when μ has a density f with respect to the Lebesgue measure and $f \in W^{1,1}(\mathbb{R}) = \{g \in L^1(\mathbb{R}):\ g' \in L^1(\mathbb{R})\}$ where the derivative is taken in distributional sense.

Remark 6.7. Observe that when $\mu = \delta_0$, $\nu = 0$, $\eta = \delta_0$, condition (6.6) is not satisfied. In this case, which corresponds to the equation of the first order

$$u'(t) = Au(t) + f(t), \tag{6.7}$$

there is no regularization on the first derivative in Eq. (1.1). This case corresponds to mild solutions for (6.7) and was investigated in [6]. There it was observed that they cannot be characterized in terms of R-boundedness of the set $\{(ikI - A)^{-1}\}_{k\in\mathbb{Z}}$ solely.

Example 6.8 *(Renewal equation).* We take $\nu = \delta_0$, $\mu = 0$ and η is chosen such that $\tilde{\eta}(k)$ is 1-regular. Then, an application of Theorem 4.6(iii) shows that the assumptions in Theorem 6.5 are satisfied and we obtain that the integral equation

$$u = \eta * Au + f$$

is (H^1_p, L^p) mildly well-posed if and only if the equivalent conditions (ii), (iii) in Theorem 6.5 are verified. Note that maximal regularity to the above equation was characterized for periodic L^p spaces in the scalar case (cf. [24, Theorem 4.7, p. 48]). Our result extends such characterization to the infinite-dimensional setting.

We now introduce a one-parameter family of concepts of well-posedness for Eq. (1.1). The main idea is to embed the space $H^1_p((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into a scale of intermediate spaces of $L^p((0,2\pi);X)$. Related notions appear in [36]. In [6], the spaces $H^\alpha_p((0,2\pi);X)$ used below are also considered but just to obtain continuity and even Hölder continuity of mild solutions from a different definition.

For $1 < p < \infty$ and $\alpha \geqslant 0$, define the space $H_p^\alpha((0,2\pi);X)$ as

$$H_p^\alpha\big((0,2\pi);X\big) = \big\{f \in L^p\big((0,2\pi);X\big),\ \exists g \in L^p\big((0,2\pi);X\big) \text{ such that } \hat{g}(k) = |k|^\alpha \hat{f}(k),\ k \in \mathbb{Z}\big\}.$$

We note due to the *UMD* property (more precisely the continuity of the Hilbert transform on $L^p((0,2\pi);X)$), we have

$$W^{m,p}\big((0,2\pi);X\big) = H_p^m\big((0,2\pi);X\big), \quad \text{for } 1 < p < \infty \text{ and } m \in \mathbb{N} \cup \{0\} \tag{6.8}$$

(see for example [37, Chapter III], [1] and for the relationship with intermediate spaces, see [14, Chapter IV, especially Section 4.4, p. 272]). Now we give the definition of $(H_p^1, H_p^{1-\alpha})$ well-posedness.

Definition 6.9. Let $0 \leqslant \alpha \leqslant 1$. We say that the problem (1.1) is $(H_p^1, H_p^{1-\alpha})$ mildly well-posed if there exists a linear operator $\mathcal{B}$ that maps $L^p((0,2\pi);X)$ continuously into itself with range in $H_p^{1-\alpha}((0,2\pi);X)$, as well as $H_p^1((0,2\pi);X) \cap L^p((0,2\pi);D(A))$ into itself and which satisfies

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in H_p^1((0,2\pi);X) \cap L^p((0,2\pi);D(A))$.

This means that in the diagram following Definition 6.1, we replace $L^p((0,2\pi);X)$ in the upper right corner with $H_p^{1-\alpha}((0,2\pi);X)$. Thanks to the closed graph theorem, this means that $\mathcal{B}$ is continuous from $L^p((0,2\pi);X)$ into $H_p^{1-\alpha}((0,2\pi);X)$.

We have the following result.

Theorem 6.10. *Assume that $\overline{D(A)} = X$. Let $1 < p < \infty$ and $0 \leqslant \alpha \leqslant 1$. Assume that $\tilde{\eta}(k) \neq 0$, for all $k \in \mathbb{Z}$. Then the following assertions are equivalent:*

(i) *Problem (1.1) is $(H_p^1, H_p^{1-\alpha})$ mildly well-posed;*
(ii) $\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and* $\{\frac{(ik)^{1-\alpha}}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is an L^p-multiplier.*

Proof. The proof is a modification of the proof of Theorem 6.4 and we omit it. □

Remark 6.11. When $\alpha = 1$, Theorem 6.10 corresponds to Theorem 6.4 and when $\alpha = 0$ the concept of solution that appears in the new context differs from that of strong solution covered by Theorem 5.5. The main difference is that in the new context we do not require that the solution operator $\mathcal{B}$ maps into $L^p((0,2\pi),D(A))$. However, in some cases, the requirement that the range of $\mathcal{B}$ be in H_p^1 automatically implies that $\mathcal{B}$ also maps into $L^p((0,2\pi),D(A))$. See Proposition 6.15 below. A specific example is Eq. (6.7) for which the analysis was done in [26]. There, we also justified why it is reasonable to assume that $0 \leqslant \alpha \leqslant 1$.

The next theorem characterizes $(H_p^1, H_p^{1-\alpha})$ mild well-posedness under an additional assumption.

Theorem 6.12. *Let $1 < p < \infty$, $0 \leqslant \alpha \leqslant 1$ and X be a UMD space. Assume that $\overline{D(A)} = X$ and $\tilde{\eta}(k) \neq 0$, for all $k \in \mathbb{Z}$ and*

$$\tilde{\eta}(k) \text{ is 1-regular} \quad \text{and} \quad (ik)^\alpha \tilde{\mu}(k) + \frac{\tilde{\nu}(k)}{(ik)^{1-\alpha}} \text{ is 1-regular and bounded.} \tag{6.9}$$

Then the following assertions are equivalent:

(i) *Problem* (1.1) *is* $(H_p^1, H_p^{1-\alpha})$ *mildly well-posed*;

(ii) $\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and* $\{\frac{(ik)^{1-\alpha}}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is an* L^p*-multiplier*;

(iii) $\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and* $\{\frac{(ik)^{1-\alpha}}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is* R*-bounded.*

Proof. Thanks to (6.9) we can use Proposition 5.3 to prove the equivalence between (ii) and (iii). The equivalence between (i) and (ii) is Theorem 6.10. □

Example 6.13. Consider the equation

$$u'(t) = \int_{-\infty}^{\infty} Au(t-s)\,\eta(ds) + f(t). \tag{6.10}$$

This case corresponds to Eq. (1.1) with $\mu = \delta_0$, $\nu = 0$ and η a bounded measure. It follows from Theorem 6.12 with $\alpha = 0$ that if

$$a_k := \tilde{\eta}(k) \text{ is 1-regular,} \tag{6.11}$$

then Eq. (6.10) is (H_p^1, H_p^1) mildly well-posed if and only if the equivalent conditions (ii) and (iii) hold. Hence we extend the results established in [24, Theorem 3.11, p. 87] to the vector-valued L^p case.

Example 6.14. Let $a > 0$ and $\gamma > -1$. We take in (1.1) $\mu(dt) = \frac{1}{\Gamma(\gamma+1)} t^{\gamma} e^{-at}\,dt$ $(t > 0)$ and $\mu(t) = 0$ for $t < 0$; $\nu = 0$ and η a bounded measure. In this case Eq. (1.1), which reads $\mu * u' = \eta * Au + f$, is (H_p^1, L^p) mildly well-posed if $\gamma \geqslant \alpha - 1$, $\gamma > -1$ and one of the equivalent conditions (ii) or (iii) is satisfied. We note that in this case $\tilde{\mu}(k) = \frac{1}{(ik+a)^{\gamma+1}}$.

In case $\alpha = 0$ we have

Proposition 6.15. *Assume that either* A *is bounded or* $\{\frac{1}{\tilde{\eta}(k)}\}$ *is an* L^p*-multiplier. Then, problem* (1.1) *is* (H_p^1, H_p^1) *mildly well-posed if and only if problem* (1.1) *is strongly* L^p *well-posed.*

Proof. Suppose problem (1.1) is (H_p^1, H_p^1) mildly well-posed. In case A is bounded, we note that $D(\mathcal{A}) = H_p^1((0, 2\pi); X)$ and the assertion follows. On the other hand, let N_k and M_k be as in the proof of Theorem 5.5. By Theorem 6.10 we have that M_k is an L^p-multiplier. When $\frac{1}{\tilde{\eta}(k)}$ is an L^p-multiplier, the identity (5.3) and the fact that $\frac{d_k}{ik} = \frac{1}{\tilde{\eta}(k)}[\tilde{\mu}(k) + \frac{\tilde{\nu}(k)}{ik}]$ show that AN_k is an L^p-multiplier as well (see Example 2.3). Then the solution u, defined by (5.2), satisfies $Au \in L^p((0, 2\pi), X)$ and hence the range of $\mathcal{B}$ is contained in $H_p^1((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$, proving the proposition.

We point out that several concrete criteria for R-boundedness have been established (see e.g. [20,23] and [6]). □

7. Well-posedness on Besov spaces

In this section we consider solutions in Besov spaces. For the definition and main properties of these spaces we refer to [5] or [28]. For the scalar case, see [14,34]. Contrary to the L^p case the multiplier theorems established so far are valid for arbitrary Banach spaces; see [2,5] and [22]. Special cases here allow one to treat Hölder–Zygmund spaces. Specifically, we have $B^s_{\infty,\infty} = \mathcal{C}^s$ for $s > 0$. Moreover, if $0 < s < 1$ then $B^s_{\infty,\infty}$ is just the usual Hölder space C^s. We begin with the definition of operator-valued Fourier multipliers in the context of Besov spaces.

Definition 7.1. Let $1 \leqslant p \leqslant \infty$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$ is a $B^s_{p,q}$-multiplier if for each $f \in B^s_{p,q}((0,2\pi);X)$ there exists a function $g \in B^s_{p,q}((0,2\pi);X)$ such that

$$M_k \hat{f}(k) = \hat{g}(k), \quad k \in \mathbb{Z}.$$

The following general multiplier theorem for periodic vector-valued Besov spaces is due to Arendt and Bu [5, Theorem 4.5]. The continuous case (multipliers on the real line) was studied by Amann [2] and later by Girardi and Weis [22].

Theorem 7.2. (i) *Let X be a Banach space and suppose that $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$ is M-bounded of order 2. Then for $1 \leqslant p,q \leqslant \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$ is a B^s_{pq}-multiplier.*

(ii) *Let X be a Banach space with nontrivial Fourier type. Then any sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$ which is M-bounded of order 1 is a B^s_{pq}-multiplier for all $1 \leqslant p,q \leqslant \infty$, $s \in \mathbb{R}$.*

The analogue of Proposition 5.3 in the present context is:

Proposition 7.3. *Let A be a closed linear operator defined on the Banach space X. Let $\{a_k\}_{k\in\mathbb{Z}}$, $\{b_k\}_{k\in\mathbb{Z}}$ be 2-regular sequences such that $\{\frac{b_k}{a_k}\}$ is bounded and $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. Then the following assertions are equivalent:*

(i) *$\{a_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ is a $B^s_{p,q}$-multiplier, $1 \leqslant p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$;*
(ii) *$\{a_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded.*

Proof. Let $M_k = a_k(b_kI - A)^{-1}$. By [6, Proposition 1.11], it follows that (i) implies (ii). We turn to (ii) implies (i). The part corresponding to M-boundedness of order 1 is contained in Proposition 5.3. To complete the verification of M-boundedness of order 2, we use the following identity

$$\begin{aligned} k^2(M_{k+1} - 2M_k + M_{k-1}) = {} & k^2(a_{k+1} - 2a_k + a_{k-1})\frac{1}{a_{k+1}}M_{k+1} \\ & - 2k\left[\frac{a_k - a_{k-1}}{a_k}\right]k\frac{1}{a_{k-1}}(b_{k+1} - b_k)M_kM_{k-1} \\ & - k^2\frac{1}{a_k}(b_{k+1} - 2b_k + b_{k-1})M_kM_{k-1} \\ & + 2k\frac{1}{a_{k+1}}(b_{k+1} - b_k)k\frac{1}{a_{k-1}}(b_{k+1} - b_{k-1})M_{k+1}M_kM_{k-1} \\ & - k\frac{1}{a_k}(b_{k+1} - b_k)k\frac{1}{a_{k+1}}(b_{k+1} - b_{k-1})M_{k+1}M_kM_{k-1}. \qquad \square \end{aligned}$$

We remark that the case $a_k = b_k$ was proved in [29, Proposition 3.4].
Next, we consider strong well-posedness for Eq. (1.1).

Definition 7.4. We say that problem (1.1) is strongly $B^s_{p,q}$ well-posed if for each $f \in B^s_{p,q}((0,2\pi);X)$ there exists a unique function $u \in B^{s+1}_{p,q}((0,2\pi);X) \cap B^s_{p,q}((0,2\pi);D(A))$ and (1.1) is satisfied almost everywhere.

As above, we call u the strong solution of (1.1). As in Section 5, in what follows we will assume that (5.1) is valid. Strong well-posedness of (1.1) in the $B^s_{p,q}$ spaces is established in the following theorem.

Theorem 7.5. *Let $1 \leqslant p,q \leqslant \infty$, $s \in \mathbb{R}$. Suppose that the sequences $\{ik\tilde{\mu}(k) + \tilde{\nu}(k)\}$ and $\{\tilde{\eta}(k)\}$ are 2-regular. Then the following assertions are equivalent:*

(i) *Problem* (1.1) *is strongly* $B^s_{p,q}$ *well-posed*;

(ii) $\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\} \subseteq \rho(A)$ *and* $\{\frac{k}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is a* $B^s_{p,q}$*-multiplier*;

(iii) $\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\} \subseteq \rho(A)$ *and* $\{\frac{k}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is bounded.*

Proof. The proof follows the same lines as the proof of Theorem 5.5 using Proposition 7.3 with $a_k = b_k$ instead of Corollary 5.4 and making use of the properties on M-boundedness of order 2 and 2-regularity for sequences established in Sections 4 and 5. □

Example 7.6. In reference to Example 4.9 we consider the following integro-differential equation with infinite delay studied in [28]

$$\begin{cases} \gamma_0 u'(t) + \dfrac{d}{dt}\left(\displaystyle\int_{-\infty}^{t} b(t-s)u(s)\,ds\right) + \gamma_\infty u(t) \\ \quad = c_0 Au(t) - \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)\,ds + f(t), \quad 0 \leqslant t \in \mathbb{R}, \end{cases} \tag{7.1}$$

where $\gamma_0, \gamma_\infty, c_0$ are constants and $a(\cdot), b(\cdot) \in L^1(\mathbb{R}_+)$. In [28], strong well-posedness on periodic Besov spaces for Eq. (7.1) was characterized as in Theorem 7.5 (see [28, Theorem 3.12]) under a set of conditions which we reformulate as:

(IDE1) $\{\frac{1}{c_0 - \tilde{a}(k)}\}_{k\in\mathbb{Z}}$ is a bounded sequence.

(IDE2) $\{\tilde{a}(k)\}$ and $\{\tilde{b}(k)\}$ are M-bounded of order 2.

(IDE3) $\{k\tilde{a}(k)\}$ and $\{k\tilde{b}(k)\}$ are bounded sequences.

Eq. (7.1) is a special case of Eq. (1.1) corresponding to $\eta = c_0\delta_0 - a(t)\chi_{[0,\infty)}(t)$, $\mu = \gamma_0\delta_0 + b(t)\chi_{[0,\infty)}(t)$, $\nu = \gamma_\infty\delta_0$ when we identify an L^1 function with the associated measure. As a consequence of Example 4.9 with $a_k = \tilde{a}(k)$ and $b_k = \tilde{b}(k)$ one easily checks that Theorem 7.5 applies only under $c_0 \neq \tilde{a}(k)$ for all $k \in \mathbb{Z}$ and (IDE2).

We observe here that the removal of condition (IDE3) is due to Proposition 7.3. As a consequence the hypotheses are formulated entirely in terms of M-boundedness and n-regularity.

The particular case of Eq. (7.1) with $\gamma_\infty = 0$ and $c_0 = \gamma_\infty = 1$ and $b \equiv 0$ was considered in [29, Theorem 3.9]. From the above, we conclude that there, we only need the condition

$$\{a_k\} \text{ is } M\text{-bounded of order 2} \tag{7.2}$$

in order to have the characterization of strong well-posedness. It shows that the set of conditions imposed in Theorem 7.5 is in some sense more natural, giving an improvement over the results in the above mentioned papers.

In analogy to mild solutions in the L^p case we proceed to define mild $B^s_{p,q}$ solutions.

Definition 7.7. Let $1 \leqslant p, q \leqslant \infty$ and $s > 0$. We say that the problem (1.1) is $(B^{s+1}_{p,q}, B^s_{p,q})$ mildly well-posed if there exists a linear operator $\mathcal{B}$ that maps $B^s_{p,q}((0,2\pi); X)$ continuously into itself as well as $B^{s+1}_{p,q}((0,2\pi); X) \cap B^s_{p,q}((0,2\pi); D(A))$ into itself and which satisfies

$$\mathcal{A}\mathcal{B}u = \mathcal{B}\mathcal{A}u = u$$

for all $u \in B^{s+1}_{p,q}((0,2\pi);X) \cap B^s_{p,q}((0,2\pi);D(A))$. In this case the function $\mathcal{B}f$ is called the $(B^{s+1}_{p,q}, B^s_{p,q})$ mild solution of (1.1) and $\mathcal{B}$ the associated solution operator.

The following result follows directly from Proposition 7.3 and Theorem 4.6.

Theorem 7.8. *Let $1 \leqslant p,q \leqslant \infty$, $s>0$ and X be a Banach space. Assume that $\overline{D(A)} = X$ and*

$$\tilde{\eta}(k) \text{ is 2-regular} \quad \text{and} \quad ik\tilde{\mu}(k) + \tilde{\nu}(k) \text{ is 2-regular and bounded.} \tag{7.3}$$

Then the following assertions are equivalent:

(i) *Problem* (1.1) *is $(B^{s+1}_{p,q}, B^s_{p,q})$ mildly well-posed*;
(ii) $\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and* $\{\frac{1}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is a $B^s_{p,q}$-multiplier*;
(iii) $\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\}_{k\in\mathbb{Z}} \subseteq \rho(A)$ *and* $\{\frac{1}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is bounded.*

Remark 7.9. When the space X has nontrivial Fourier type, then due to Theorem 7.2(ii) the assumptions of M-boundedness of order 2 and 2-regularity in Theorems 7.5 and 7.8 can be replaced by M-boundedness of order 1 and 1-regularity respectively.

8. Well-posedness on Triebel–Lizorkin spaces

In this section we study strong and mild well-posedness of problem (1.1) on the scale of Triebel–Lizorkin spaces of vector-valued functions. The important feature in this case, as in the context of Besov spaces, is that the results do not use R-boundedness but merely boundedness conditions on resolvents. In concrete applications, one can therefore handle operators on familiar spaces X like $C(\overline{\Omega})$, the Schauder spaces $C^s(\overline{\Omega})$, $0<s<1$, and $L^1(\Omega)$ where Ω is a bounded open subset of $\mathbb{R}^n$. These spaces are not *UMD*, and are not even reflexive. The price to pay is that when $p=1$ or $q=1$ then one needs a Marcinkiewicz estimate of order 3 whereas, for the Besov scale, order 2 is enough.

We briefly recall the definition of periodic Triebel–Lizorkin spaces in the vector-valued case used in [12]. For the scalar case, these spaces have been studied for a long time, see Triebel [38, Chapter II, Section 9], Schmeisser and Triebel [34] and references therein. A vector-valued Fourier multiplier in the Triebel–Lizorkin scale appears in [37, Chapter 3, Section 15.6].

Let $\mathcal{S}$ be the Schwartz space on $\mathbb{R}$ and let $\mathcal{S}'$ be the space of all tempered distributions on $\mathbb{R}$. Let $\Phi(\mathbb{R})$ be the set of all systems $\phi = \{\phi_j\}_{j\geqslant 0} \subset \mathcal{S}$ satisfying

$$\begin{gathered}
\operatorname{supp}(\phi_0) \subset [-2,2], \\
\operatorname{supp}(\phi_j) \subset \left[-2^{j+1}, -2^{j-1}\right] \cup \left[2^{j-1}, 2^{j+1}\right], \quad j \geqslant 1, \\
\sum_{j\geqslant 0} \phi_j(t) = 1, \quad t \in \mathbb{R},
\end{gathered}$$

and for $\alpha \in \mathbb{N} \cup \{0\}$, there exists $C_\alpha > 0$ such that

$$\sup_{j\geqslant 0,\, x\in\mathbb{R}} 2^{\alpha j} \left\|\phi_j^{(\alpha)}(x)\right\| \leqslant C_\alpha. \tag{8.1}$$

Recall that such a system can be obtained by choosing $\phi \in \mathcal{S}(\mathbb{R})$ with

$$\operatorname{supp}(\phi_0) \subset [-2,2]$$

and $\phi_0(x) = 1$ if $\|x\| \leqslant 1$, then setting $\phi_1(x) = \phi_0(x/2) - \phi_0(x)$ and $\phi_j(x) = \phi_1(2^{-j}x)$, $j \geqslant 2$.

Let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$ and $\phi = (\phi_j)_{j\geqslant 0} \in \Phi(\mathbb{R})$. The X-valued periodic Triebel–Lizorkin spaces are defined by

$$F^{s,\phi}_{p,q} = \left\{ f \in \mathcal{D}'(\mathbb{T}; X)\colon\ \|f\|_{F^{s,\phi}_{p,q}} = \left\| \left(\sum_{j\geqslant 0} 2^{sjq} \left\| \sum_{k\in\mathbb{Z}} e_k \otimes \phi_j(k)\hat{f}(k) \right\|^q \right)^{1/q} \right\|_p < \infty \right\}.$$

The usual modification is adopted when $q = \infty$.

Here $(e_k \otimes x)(t) := e^{it}x, t \in [0, 2\pi]$. The space $F^{s,\phi}_{p,q}$ is independent of $\phi \in \Phi(\mathbb{R})$ and the norms $\|\cdot\|_{F^{s,\phi}_{p,q}}$ are equivalent. We will simply denote $\|\cdot\|_{F^{s,\phi}_{p,q}}$ by $\|\cdot\|_{F^s_{p,q}}$.

We remark that when X is a Banach space, the scale of Triebel–Lizorkin spaces does not in general contain the L^p scale. In fact, the Littlewood–Paley assertions $F^0_{p,2}((0, 2\pi); X) = L^p((0, 2\pi); X)$, $1 < p < \infty$ hold if and only if X can be renormed as a Hilbert space. This follows from [33]. In the scalar case, the well-known assertions may be found in [37, Chapter 3, Section 10]. For the nonvalidity of the Littlewood–Paley assertions in the vector-valued case, see also the introduction to [12].

Note that $F^s_{p,p}((0, 2\pi); X) = B^s_{p,p}((0, 2\pi); X)$ by a look at the definitions (for the scalar case see [32]). This relation is true when X is the scalar field $\mathbb{C}$ (see [34, Remark 4, p. 164]). Using the definitions of the spaces one easily sees that the relation remains true in the vector-valued case.

Definition 8.1. Let $1 \leqslant p < \infty$. A sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$ is an $F^s_{p,q}$-multiplier if, for each $f \in F^s_{p,q}((0, 2\pi); X)$ there exists a function $g \in F^s_{p,q}((0, 2\pi); X)$ such that

$$M_k\hat{f}(k) = \hat{g}(k), \quad k \in \mathbb{Z}.$$

The following multiplier theorem for periodic vector-valued Triebel–Lizorkin spaces is due to Bu and Kim [12, Theorem 3.2 and Remark 3.4].

Theorem 8.2. *Let X be a Banach space and suppose $\{M_k\}_{k\in\mathbb{Z}} \subset \mathcal{B}(X)$. Then the following assertions hold.*

(1) *Assume that $\{M_k\}_{k\in\mathbb{Z}}$ is M-bounded of order 3. Then for $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier.*
(2) *Assume that $\{M_k\}_{k\in\mathbb{Z}}$ is M-bounded of order 2. Then for $1 < p < \infty$, $1 < q \leqslant \infty$, $s \in \mathbb{R}$, $\{M_k\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier.*

Remark 8.3. When $p = q$ the assertion (1) of Theorem 8.2 holds true for $\{M_k\}_{k\in\mathbb{Z}}$ M-bounded of order 2. Moreover if X has nontrivial Fourier type, M-boundedness of order 1 suffices. This follows from the relation $F^s_{p,p}((0, 2\pi); X) = B^s_{p,p}((0, 2\pi); X)$ and [5, Theorems 4.2 and 4.5] (see also [22]) or simply Theorem 7.2.

The following result was proved in [10, Theorem 2.2].

Proposition 8.4. *Let X be a Banach space and A a closed linear operator defined on X. Let $\{b_k\}_{k\in\mathbb{Z}}$ be a 3-regular sequence such that $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. Then the following assertions are equivalent:*

(i) *$\{b_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier, $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$, $s \in \mathbb{R}$;*
(ii) *$\{b_k(b_kI - A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded.*

In case $p = q$ the same observations as in Remark 8.3 allow us to relax the hypotheses of Proposition 8.4. More specifically, in this case, it is enough to assume that $\{b_k\}_{k\in\mathbb{Z}}$ is 2-regular. Moreover if X has nontrivial Fourier type, the assumption that $\{b_k\}_{k\in\mathbb{Z}}$ is 1-regular suffices to ensure the conclusion. This again follows from Theorem 7.2 and the relation $F^s_{p,p}((0, 2\pi); X) = B^s_{p,p}((0, 2\pi); X)$. A similar remark holds for Proposition 8.5 and Theorem 8.7 below.

We have the following extension of this result (in analogy to Propositions 5.3 and 7.3).

Proposition 8.5. *Let X be a Banach space and A a closed linear operator defined on X. Let $\{a_k\}_{k\in\mathbb{Z}}$, $\{b_k\}_{k\in\mathbb{Z}}$ be 3-regular sequences such that $\{b_k\}_{k\in\mathbb{Z}} \subset \rho(A)$. Suppose that $\{b_k/a_k\}$ is bounded. Then the following assertions are equivalent*:

(i) *$\{a_k(b_kI-A)^{-1}\}_{k\in\mathbb{Z}}$ is an $F^s_{p,q}$-multiplier, $1 \leqslant p \leqslant \infty$, $1 \leqslant q < \infty$, $s \in \mathbb{R}$;*
(ii) *$\{a_k(b_kI-A)^{-1}\}_{k\in\mathbb{Z}}$ is bounded.*

Proof. Let $M_k = a_k(b_k-A)^{-1}$ and $N_k = b_k(b_k-A)^{-1}$. Note that $N_k = \frac{b_k}{a_k}M_k$. Since by hypothesis $\{M_k\}$ and $\{b_k/a_k\}$ are bounded, we obtain that $\{N_k\}_{k\in\mathbb{Z}}$ is bounded.

From the proofs of Propositions 5.3 and 7.3 we obtain that

$$\sup_{k\in\mathbb{Z}}\|M_k\| < \infty, \quad \sup_{k\in\mathbb{Z}}\bigl\|k\Delta(M_k)\bigr\| < \infty \quad \text{and} \quad \sup_{k\in\mathbb{Z}}\bigl\|k^2\Delta^2(M_k)\bigr\| < \infty.$$

Hence in order to prove that $\{M_k\}$ is M-bounded of order 3, we need only check that $\sup\{\|k^3\Delta^3(M_k)\|\colon\ k\in\mathbb{Z}\} < \infty$. In fact, from the proof of Theorem 3.3 part (iii) (writing $N_k = \frac{b_k}{a_k}M_k$, $k\in\mathbb{Z}$), we have that

$$\begin{aligned}
\Delta^3(M_k) &= \Delta^3\left(\frac{a_k}{b_k}N_k\right)\\
&= \Delta^3\left(\frac{a_k}{b_k}\right)N_{k+3} + \Delta^2\left(\frac{a_k}{b_k}\right)\bigl[\Delta(N_{k+2}) + \Delta(N_{k+1}) + \Delta(N_k)\bigr]\\
&\quad + \Delta^3(N_k)\frac{a_{k+2}}{b_{k+2}} + 2\Delta\left(\frac{a_{k+1}}{b_{k+1}}\right)\Delta^2(N_k) + \Delta\left(\frac{a_{k+1}}{b_{k+1}}\right)\Delta^2(N_{k+1}),
\end{aligned}$$

where each term in the right-hand side of the above identity can be handled separately as follows:

$$\begin{aligned}
\Delta^3\left(\frac{a_k}{b_k}\right)N_{k+3} &= \frac{\Delta^3(a_k/b_k)}{a_{k+2}/b_{k+2}}\frac{b_{k+3}}{b_{k+2}}\frac{a_{k+2}}{a_{k+3}}M_{k+3},\\
\Delta^2\left(\frac{a_k}{b_k}\right)\Delta(N_{k+2}) &= \frac{\Delta^2(a_k/b_k)}{a_{k+1}/b_{k+1}}\left[\frac{\Delta(b_{k+2})}{b_{k+1}}\frac{a_{k+1}}{a_{k+3}}M_{k+3} - \frac{b_{k+3}}{b_{k+1}}\frac{a_{k+1}}{a_{k+3}}\frac{b_{k+2}}{a_{k+2}}M_{k+3}\frac{\Delta(b_{k+2})}{b_{k+3}}M_{k+2}\right],\\
\Delta^2\left(\frac{a_k}{b_k}\right)\Delta(N_{k+1}) &= \frac{\Delta^2(a_k/b_k)}{a_{k+1}/b_{k+1}}\left[\frac{\Delta(b_{k+1})}{b_{k+1}}\frac{a_{k+1}}{a_{k+2}}M_{k+2} - \frac{b_{k+2}}{a_{k+2}}M_{k+2}\frac{\Delta(b_{k+1})}{b_{k+2}}M_{k+1}\right],\\
\Delta^2\left(\frac{a_k}{b_k}\right)\Delta(N_k) &= \frac{\Delta^2(a_k/b_k)}{a_{k+1}/b_{k+1}}\left[\frac{\Delta(b_k)}{b_{k+1}}M_{k+1} - \frac{b_k}{a_k}M_{k+1}\frac{\Delta(b_k)}{b_{k+1}}M_k\right],\\
\Delta^3(N_k)\frac{a_{k+2}}{b_{k+2}} &= -\frac{\Delta^3(b_k)}{b_{k+2}}M_{k+2}(N_{k+1}-I)\\
&\quad + \frac{\Delta^2(b_{k+1})}{b_{k+2}}\frac{\Delta(b_{k+2})+\Delta(b_{k+1})+\Delta(b_k)}{b_{k+3}}M_{k+2}N_{k+3}(N_{k+1}-I)\\
&\quad + \frac{\Delta^2(b_k)}{b_{k+2}}\frac{\Delta(b_{k+2})+\Delta(b_{k+1})+\Delta(b_k)}{b_{k+3}}M_{k+2}N_{k+3}(N_k-I)\\
&\quad - 2\frac{\Delta(b_k)}{b_{k+1}}\frac{\Delta(b_{k+1})}{b_{k+2}}\frac{\Delta(b_{k+2})+\Delta(b_{k+1})+\Delta(b_k)}{b_{k+3}}N_{k+1}M_{k+2}N_{k+3}(N_k-I),
\end{aligned}$$

$$\Delta\left(\frac{a_{k+1}}{b_{k+1}}\right)\Delta^2(N_k) = \frac{\Delta(a_{k+1}/b_{k+1})}{a_{k+1}/b_{k+1}}\frac{\Delta(b_k)}{b_k}\frac{\Delta(b_{k+1})+\Delta(b_k)}{b_{k+1}}M_{k+1}N_k(N_{k+2}-I) - \frac{\Delta(a_{k+1}/b_{k+1})}{a_{k+1}/b_{k+1}}\frac{\Delta^2(b_k)}{b_{k+1}}M_{k+1}(N_{k+2}-I),$$

$$\Delta\left(\frac{a_{k+1}}{b_{k+1}}\right)\Delta^2(N_{k+1}) = \frac{\Delta(a_{k+1}/b_{k+1})}{a_{k+1}/b_{k+1}}\frac{\Delta(b_{k+1})}{b_{k+1}}\frac{\Delta(b_{k+2})+\Delta(b_{k+1})}{b_{k+2}}N_{k+2}M_{k+1}(N_{k+3}-I) - \frac{\Delta(a_{k+1}/b_{k+1})}{a_{k+1}/b_{k+1}}\frac{b_{k+2}}{b_{k+1}}\frac{a_{k+1}}{a_{k+2}}\frac{\Delta^2(b_{k+1})}{b_{k+2}}M_{k+2}(N_{k+3}-I).$$

Then a moment of reflection shows that the assertion follows from the hypothesis and Theorem 4.6 (see also the observation after Definition 4.1). □

Definition 8.6. We say that the problem (1.1) is strongly $F^s_{p,q}$ well-posed if for each $f \in F^s_{p,q}((0,2\pi);X)$ there exists $u \in F^{s+1}_{p,q}((0,2\pi);X) \cap F^s_{p,q}((0,2\pi);D(A))$ such that (1.1) is satisfied.

We now discuss the conditions on the parameters appearing in Eq. (1.1) which ensure that the above theorem applies. As in Section 5, we assume that (5.1) is valid.

Theorem 8.7. *Let $1 \leqslant p < \infty$, $1 \leqslant q \leqslant \infty$, $s \in \mathbb{R}$. Suppose that the sequences $\{ik\tilde{\mu}(k) + \tilde{\nu}(k)\}$ and $\{\tilde{\eta}(k)\}$ are 3-regular. Then the following assertions are equivalent*:

(i) *Problem* (1.1) *is strongly $F^s_{p,q}$ well-posed*;
(ii) $\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\} \subseteq \rho(A)$ *and* $\{\frac{ik}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is an $F^s_{p,q}$-multiplier*;
(iii) $\{\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}\} \subseteq \rho(A)$ *and* $\{\frac{ik}{\tilde{\eta}(k)}(\frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)} - A)^{-1}\}$ *is bounded.*

Proof. (ii) ⇔ (iii) The assertion follows from hypothesis, Remark 4.7 and Proposition 8.5. The equivalence (ii) ⇔ (iii) is shown in the same way as the analogous parts of Theorems 7.5 and 5.5. □

Due to Theorem 8.2(2) we note that when $p > 1$, the requirement can be relaxed to 2-regularity for the sequences $\{ik\tilde{\mu}(k) + \tilde{\nu}(k)\}$ and $\{\tilde{\eta}(k)\}$.

Finally, we observe that one can study mild solutions in this context as well.

9. Application to nonlinear equations

In this section, we apply the above results to nonlinear equations in Banach and Hilbert spaces. We consider three situations where equations can be solved by the method of maximal regularity. One corresponds to Theorem 9.1 in which one deals with a semi-linear problem. Such problem was previously considered in [28] in Hölder spaces. We cover here the complete scale of Lebesgue, Besov and Triebel–Lizorkin spaces. The second application uses a method based on [16, Theorem 4.1] to solve a nonlinear integro-differential equation. The third application is concerned with semi-linear equations in Hilbert spaces (Theorem 9.6) corresponds to an extension of Staffans [36]. One of the main assumptions made in Theorem 9.6 below is that A has compact resolvent. Typically, this occurs in problems involving elliptic operators on bounded domains in $\mathbb{R}^n$ with appropriate boundary conditions. Such equations arise in heat conduction of materials with memory.

As already indicated, linear results on maximal regularity are very useful in dealing with nonlinear problems. For example the following problem was considered in [28,35]:

$$\begin{cases} \dfrac{d}{dt}\left(\gamma_0 u(t,x) + \displaystyle\int_{-\infty}^{t} b(t-s)u(s,x)\,ds\right) + \gamma_\infty u(t,x) \\ \quad = c_0 \Delta u(t,x) - \displaystyle\int_{-\infty}^{t} a(t-s)Au(s)\,ds + g\big(x,u(t,x)\big) + f(t,x), \quad x \in \Omega. \end{cases} \tag{9.1}$$

Here, $\Omega \subset \mathbb{R}^n$ is open and bounded, and $\Delta = \sum_{j=1}^{n} \frac{\partial}{\partial x_j^2}$ is the Laplace operator with Dirichlet boundary conditions on $X = C(\overline{\Omega})$. The positive constants γ_0 and c_0 represent the heat capacity and the thermal conductivity respectively, for the material under study (see e.g. [35] where Hölder continuous solutions on the real line are considered).

Let X be a Banach space and μ, ν, η be bounded measures. We shall say that a closed linear operator A belongs to the *class* $\mathcal{K}(X)$ if

$$\left\{\frac{ik\tilde{\mu}(k) + \tilde{\nu}(k)}{\tilde{\eta}(k)}\right\}_{k\in\mathbb{Z}} \subseteq \rho(A) \quad \text{and} \quad \sup_{k\in\mathbb{Z}}\left\|\frac{ik}{\tilde{\eta}(k)}\left(\frac{ik\tilde{\mu}(k) + \tilde{\nu}(k)}{\tilde{\eta}(k)} - A\right)^{-1}\right\| < \infty. \tag{9.2}$$

On the other hand, we say that A belongs to the *class* $\mathcal{K}_R(X)$ if

$$\left\{\frac{ik\tilde{\mu}(k) + \tilde{\nu}(k)}{\tilde{\eta}(k)}\right\}_{k\in\mathbb{Z}} \subseteq \rho(A) \quad \text{and} \quad \left\{\frac{ik}{\tilde{\eta}(k)}\left(\frac{ik\tilde{\mu}(k) + \tilde{\nu}(k)}{\tilde{\eta}(k)} - A\right)^{-1}\right\}_{k\in\mathbb{Z}} \text{ is } R\text{-bounded}. \tag{9.3}$$

Given $m \in \{1,2,3\}$, we will say that μ, ν, η are *m-admissible* if the sequences $\{ik\tilde{\mu}(k) + \tilde{\nu}(k)\}$ and $\{\tilde{\eta}(k)\}$ are m-regular, and η is a finite scalar-valued measure on $\mathbb{R}$ such that (5.1) is valid.

We consider the semi-linear problem:

$$(\mu * u')(t) + (\nu * u)(t) - (\eta * Au)(t) = G(u)(t) + \rho f(t), \quad 0 \leqslant t \leqslant 2\pi, \tag{9.4}$$

with periodic boundary conditions. Here $\rho > 0$ is a small parameter and G is a nonlinear mapping.

Suppose $a \in L^1(\mathbb{R})$ and $b \in W^{1,1}(\mathbb{R})$. Note that Eq. (9.1) with periodic boundary conditions corresponds to problem (9.4), where we have $\mu = \gamma_0\delta_0$, $\nu = (\gamma_\infty + b(0))\delta_0 + b'(t)\chi_{[0,\infty)}(t)$ and $\eta = -c_0\delta_0 + a(t)\chi_{[0,\infty)}(t)$. By Theorem 4.6 it follows that under the hypothesis of n-regularity of $\tilde{a}(k)$ and $\tilde{b}(k)$ we have that μ, η, ν are n-admissible ($n = 1,2,3$). In such case, for example condition (9.2) reads as

$$\{d_k\}_{k\in\mathbb{Z}} \subseteq \rho(\Delta) \quad \text{and} \quad \sup_{k\in\mathbb{Z}}\left\|\frac{ik}{\tilde{a}(k) - c_0}(d_k - \Delta)^{-1}\right\| < \infty, \tag{9.5}$$

where $d_k := \frac{ik(\gamma_0 + \tilde{b}(k)) + (\gamma_\infty + b(0))}{\tilde{a}(k) - c_0}$.

The following result deals with the general situation.

Theorem 9.1. *Let X be a UMD space and suppose $A \in \mathcal{K}_R(X)$; μ, η, ν are 1-admissible. Furthermore, assume that $1 < p < \infty$ and*

(i) *G maps $H_p^1((0,2\pi); X) \cap L^p((0,2\pi); D(A))$ into $L^p((0,2\pi); X)$ and $f \in L^p((0,2\pi); X)$;*
(ii) *$G(0) = 0$; G is continuously (Fréchet) differentiable at $u = 0$ and $G'(0) = 0$.*

Then there exists $\rho^* > 0$ such that Eq. (9.4) *is solvable for each $\rho \in [0, \rho^*)$, with solution $u = u_\rho \in L^p((0, 2\pi); X)$.*

Proof. Define the operator $L_0 : H_p^1((0, 2\pi); X) \cap L^p((0, 2\pi); D(A)) \to L^p((0, 2\pi); X)$ where as usual, $D(A)$ is endowed with the graph norm, by

$$(L_0 u)(t) = (\mu * u')(t) + (\nu * u)(t) - (\eta * Au)(t). \tag{9.6}$$

Since A is closed, the space $Z := H_p^1((0, 2\pi); X) \cap L^p((0, 2\pi); D(A))$ becomes a Banach space with the norm

$$\|u\|_Z = \|u\|_p + \|u'\|_p + \|Au\|_p. \tag{9.7}$$

By hypothesis and Theorem 5.5 it follows that L_0 is an isomorphism onto. We consider for $\rho \in (0, 1)$, the one-parameter family of problems:

$$\mathcal{H}[u, \rho] = -L_0 u + G(u) + \rho f = 0. \tag{9.8}$$

Keeping in mind that $G(0) = 0$, we see that $\mathcal{H}[0, 0] = 0$. Also, by hypothesis, $\mathcal{H}$ is continuously differentiable at $(0, 0)$. Since L_0 is an isomorphism, the partial Fréchet derivative $\mathcal{H}^1_{(0,0)} = L_0$ is invertible. The conclusion of the theorem now follows from the implicit function theorem (see [21, Theorem 17.6]). $\square$

Remark 9.2. When X is an arbitrary Banach space, an analogous result holds for the cases Besov or Triebel–Lizorkin spaces. In such cases we have to assume for the kernels μ, η, ν the hypothesis of 2 or 3 admissibility, respectively.

Specifically, for the Besov case, we have

Theorem 9.3. *Let $1 \leqslant p, q \leqslant \infty$ and set $s > 0$. Let X be a Banach space and suppose $A \in \mathcal{K}(X)$ and μ, η, ν are 2-admissible. Assume that*

(i) *G maps $B^s_{p,q}((0, 2\pi); X) \cap B^{s+1}_{p,q}((0, 2\pi); D(A))$ into $B^s_{p,q}((0, 2\pi); X)$ and $f \in B^s_{p,q}((0, 2\pi); X)$;*
(ii) *$G(0) = 0$; G is continuously (Fréchet) differentiable at $u = 0$ and $G'(0) = 0$.*

Then there exists $\rho^* > 0$ such that Eq. (9.4) *is solvable for each $\rho \in [0, \rho^*)$, with solution $u = u_\rho \in B^s_{p,q}((0, 2\pi); X)$.*

Let $a \in \mathbb{R} \setminus \{0\}$, $0 < \alpha < 1$ and $b \in L^1(\mathbb{R}, |t|^\alpha\, dt) \cap L^1_{\mathrm{loc}}(\mathbb{R})$. Let D be a Banach space continuously embedded in X and let $G : D \to X$ be a nonlinear mapping. Let $g \in C^\alpha((0, 2\pi); X)$. We consider next the following nonlinear integral equation:

$$u(t) = \int_{-\infty}^{t} b(t-s)\bigl(G\bigl(u(s)\bigr) + g(s)\bigr)\, ds + aG\bigl(u(t)\bigr) + ag(t), \quad t \geqslant 0, \tag{9.9}$$

with the boundary condition $u(0) = u(2\pi)$. In case $a = 0$, existence and regularity of solutions for Eq. (9.9) (on the line), in several vector-valued spaces, has been studied in [16] under the assumption that $A := G'(0)$ generates an analytic semigroup.

Define $T : C^\alpha((0,2\pi);X) \to C^\alpha((0,2\pi);X)$ by

$$T(v) = \eta * v, \tag{9.10}$$

where $\eta = b(t)\chi_{[0,\infty)}(t) + a\delta_0$ (we identify an L^1 function with the associated measure).

Proposition 9.4. *Suppose $\{\tilde{b}(k)\}_{k\in\mathbb{Z}}$ is 2-regular and $\tilde{b}(k) + a \neq 0$ for all $k \in \mathbb{Z}$. Then T defined as above is an isomorphism of $C^\alpha((0,2\pi);X)$.*

Proof. Suppose $T(v) = 0$. By (2.1) we have, for all $k \in \mathbb{Z}$

$$\widehat{T(v)} = \widehat{\eta * v}(k) = \tilde{\eta}(k)\hat{v}(k) = \big(\tilde{b}(k) + a\big)\hat{v}(k) = 0.$$

Then $\hat{v}(k) = 0$ for all $k \in \mathbb{Z}$, i.e. $v = 0$. Note that $b \in L^1(\mathbb{R})$ and thus, by the Riemann–Lebesgue lemma, we have $\lim_{|s|\to\infty} \tilde{b}(s) = 0$.

Define $M_k = \frac{1}{\tilde{b}(k)+a} I$. Let $k \in \mathbb{N}$. It is not difficult to see, using the results of Section 4 (specifically Proposition 4.2 and Theorem 4.6), that (M_k) is M-bounded of order 2 if $\{\tilde{b}(k)\}_{k\in\mathbb{Z}}$ is 2-regular. In particular, it follows from Theorem 7.2 that (M_k) is a C^α-multiplier. Let $f \in C^\alpha((0,2\pi);X)$. Then there exists $u \in C^\alpha((0,2\pi);X)$ such that $\hat{u}(k) = M_k \hat{f}(k) = \frac{1}{\tilde{b}(k)+a}\hat{f}(k)$, for all $k \in \mathbb{Z}$. This proves that $u \in C^\alpha((0,2\pi);X)$ satisfies $T(u) = f$. □

Other conditions under which T defined as above is an isomorphism (in case $a = 0$) have been studied in [16, Proposition 2.1].

Theorem 9.5. *Suppose that T is an isomorphism and $G : D \to X$ is continuously (Fréchet) differentiable with $G(0) = 0$. Let $A := G'(0)$ and assume that A is a closed operator with domain $D(A) = D$ dense in X. Suppose moreover that $A \in \mathcal{K}(X)$ and $\{\tilde{b}(k)\}_{k\in\mathbb{Z}}$ is 2-regular. Then there exist $r > 0$, $s > 0$ such that for each $g \in C^\alpha((0,2\pi);X)$ satisfying $\|g\|_\alpha < r$ Eq. (9.9) has a unique solution $u \in C^{\alpha+1}((0,2\pi);X) \cap C^\alpha((0,2\pi);D(A))$ verifying the estimate*

$$\|u\|_{C^{\alpha+1}((0,2\pi);X)} + \|u\|_{C^\alpha((0,2\pi);D(A))} \leqslant s.$$

Proof. Define the mapping $F : C^{\alpha+1}((0,2\pi);X) \cap C^\alpha((0,2\pi);D(A)) \to C^\alpha((0,2\pi);X)$ by

$$F(u)(t) = T^{-1}(u)(t) - G\big(u(t)\big),$$

where T is defined by (9.10).

Since T is an isomorphism, we see that Eq. (9.9) is equivalent to

$$F(u) = g. \tag{9.11}$$

Note that by hypothesis, F is continuously differentiable, $F(0) = 0$ and

$$F'(u)v = T^{-1}v - G'(u)v \tag{9.12}$$

for all $u, v \in C^{\alpha+1}((0,2\pi);X) \cap C^\alpha((0,2\pi);D(A))$. Then $F'(0)v = T^{-1}v - Av$. Consider the linear problem

$$v(t) = \int_{-\infty}^{t} c(t-s)Av(s)\,ds + aAu(t) + \int_{-\infty}^{t} c(t-s)g(s)\,ds + ag(t) = (\eta * Av)(t) + T(g)(t). \tag{9.13}$$

Note that (9.13) is of the form (1.1) with $\mu = 0$, $\nu = \delta_0$ and $\eta = b(t)\chi_{[0,\infty)}(t) + a\delta_0$ and $f(t) = T(g)(t)$. Since the sequence $(\tilde{b}(k))$ is 2-regular and $A \in \mathcal{K}(X)$, that is,

$$\left\{\frac{1}{\tilde{b}(k)+a}\right\}_{k\in\mathbb{Z}} \subseteq \rho(A) \quad \text{and} \quad \sup_{k\in\mathbb{Z}}\left\|\frac{ik}{\tilde{b}(k)+a}\left(\frac{1}{\tilde{b}(k)+a}I - A\right)^{-1}\right\| < \infty, \tag{9.14}$$

we conclude by Theorem 7.5 that Eq. (9.13) is C^α-well-posed. We next prove that $F'(0)$ is an isomorphism from $C^{\alpha+1}((0,2\pi);X) \cap C^\alpha((0,2\pi);D(A))$ to $C^\alpha((0,2\pi);X)$. In fact, for $f = T(g) \in C^\alpha((0,2\pi);X)$ there exists a unique $v \in C^{\alpha+1}((0,2\pi);X) \cap C^\alpha((0,2\pi);D(A))$ such that (9.13) is satisfied, that is $F'(0)v = T^{-1}v - Av = f$. This shows that $F'(0)$ is onto. Suppose $F'(0)v = 0$. Then $v(t) = (\eta * Av)(t)$. By uniqueness, $v = 0$. Hence, $F'(0)$ is injective, proving the claim. The conclusion of Theorem 9.5 is now a direct consequence of the implicit function theorem. □

In the next application, we consider semi-linear equations in Hilbert space associated with operators with compact resolvent. Let H be a Hilbert space. We consider the problem:

$$(\mu * u')(t) + (\nu * u)(t) - (\eta * Au)(t) = G(u)(t), \quad t \in [0,2\pi], \tag{9.15}$$

where G is a nonlinear function that maps $L^2((0,2\pi);H)$ into $L^2((0,2\pi);H)$.

We define the bounded linear operator

$$\mathcal{B} : L^2\big((0,2\pi);H\big) \to H_2^1\big((0,2\pi);H\big) \cap L^2\big((0,2\pi);D(A)\big)$$

by $\mathcal{B}(g) = u$ where u is the unique solution of the linear problem

$$(\mu * u')(t) + (\nu * u)(t) - (\eta * Au)(t) = g(t), \quad t \in [0,2\pi].$$

Observe that $\mathcal{B}$ is well defined due to Theorem 5.5. Also, $\mathcal{B}$ is a bounded operator regarded as an operator from $L^2((0,2\pi);H)$ into itself (cf. Corollary 5.6).

We assume that for some $M > 0$,

$$\sup_{\|u\|\leqslant M}\big\|G(u)\big\|_{L^2((0,2\pi);H)} \leqslant M/\|\mathcal{B}\|, \tag{9.16}$$

then one proves the following result.

Theorem 9.6. *Let H be a Hilbert space, and suppose $A \in \mathcal{K}(H)$ and μ, η, ν are* 1*-admissible measures. Assume that the unit ball of $D(A)$ is compact in H. Let G be a continuous mapping of $L^2((0,2\pi);H)$ into itself, and such that* (9.16) *holds. Then Eq.* (9.15) *has a solution $u \in H_2^1((0,2\pi);H) \cap L^2((0,2\pi);D(A))$ such that* (9.15) *is satisfied, with $\|u\|_{L^2((0,2\pi);H)} \leqslant M$.*

Proof. Define $d_k := \frac{ik\tilde{\mu}(k)+\tilde{\nu}(k)}{\tilde{\eta}(k)}$ and $c_k := \frac{1}{\tilde{\eta}(k)}$. Since $A \in \mathcal{K}(H)$, for each $K \in \mathbb{N}$ we can define operators $\mathcal{B}_K : L^2((0,2\pi);H) \to L^2((0,2\pi);H)$ by

$$(\mathcal{B}_K g)(t) = \sum_{k=-K}^{K} c_k R(d_k, A)\hat{g}(k)e^{ikt}, \quad 0 \leqslant t \leqslant 2\pi. \tag{9.17}$$

Since the unit ball of $D(A)$ is compact in H we have that $R(d_k, A)$ is compact for all $k \in \mathbb{Z}$. Hence for each K, the operator $\mathcal{B}_K$ is a finite sum of compact operators, hence compact. Now, because of (9.2), as $K \to \infty$, $\mathcal{B}_K$ converges in norm to $\mathcal{B}$, so $\mathcal{B}$ is compact.

Define $\mathcal{H}: L^2((0,2\pi);H) \to H_2^1((0,2\pi);H) \cap L^2((0,2\pi);D(A))$ by $\mathcal{H}(u) = \mathcal{B}(G(u))$. Let $E := \{u \in L^2((0,2\pi);H)\colon \|u\| \leqslant M\}$ be the closed ball of radius M centered at the origin in $L^2((0,2\pi);H)$. Owing to (9.16) we have $\mathcal{H}: E \to E$ and $\mathcal{H}$ is compact. Hence the conclusion of the theorem is achieved by applying Schauder's fixed point theorem to the operator $\mathcal{H}$ in E. $\square$

Of course, if H is finite-dimensional, then the assumption that the unit ball of $D(A)$ is compact in H is redundant.

We remark that in [36, Theorem 6.1] the additional condition $\sup_{k\in\mathbb{Z}} \| \frac{1}{\tilde\eta(k)} A(\frac{ik\tilde\mu(k)+\tilde\nu(k)}{\tilde\eta(k)} - A)^{-1}\| < \infty$ was required. Instead, we require admissibility of the kernels μ,ν and η only.

We end this paper with the following application of Theorem 9.6. Let us consider the equation

$$u'(t) - M(\eta * u)(t) = f(t), \quad t \in [0,2\pi], \tag{9.18}$$

where $f \in L^2((0,2\pi);\mathbb{C}^n)$ and η is a finite, scalar-valued measure and M is an $n\times n$ matrix.

Eq. (9.18) corresponds to a particular case of an integro-differential equation studied in [24, Theorem 3.11, p. 87]. There it was proved that Eq. (9.18) has a unique solution in the same space as f if and only if $\det[ikI - \tilde\eta(k)M] \neq 0$ for all $k \in \mathbb{Z}$. Here we are interested in solutions of the semi-linear version:

$$u'(t) - M(\eta * u)(t) = G(u)(t), \quad t \in [0,2\pi].$$

To recast (9.18) in the form of Eq. (1.1) we make $\mu = \delta_0$ and $\nu = 0$ and $A = M$. Then Theorem 5.5 gives that, provided $\{\tilde\eta(k)\}$ is 1-regular and $\{1/\tilde\eta(k)\}$ is bounded, there exists a unique solution $u \in H_2^1((0,2\pi);\mathbb{C}^n)$ if and only if $\det[ikI - \tilde\eta(k)M] \neq 0$ for all $k \in \mathbb{Z}$ and

$$\sup_{k\in\mathbb{Z}} \left\| \frac{ik}{\tilde\eta(k)} \left(\frac{ik}{\tilde\eta(k)} - M \right)^{-1} \right\| < \infty. \tag{9.19}$$

Observe that condition (9.19) is always satisfied (since $\frac{ik}{\tilde\eta(k)} \to \infty$ as $|k| \to \infty$). We note that the condition of 1-regularity of $\tilde\eta(k)$, or equivalently 1-regularity of $\tilde\zeta(k)$, is satisfied by the class of functions $\zeta \in W^{1,1}(\mathbb{R}_+)$. This follows from [28, Remark 3.5].

The foregoing comments, together with Theorem 9.6, lead to the following corollary.

Corollary 9.7. *Assume that*

(i) $\det[ikI - \tilde\eta(k)M] \neq 0$ *for all* $k \in \mathbb{Z}$,
(ii) $\{\tilde\eta(k)\}$ *is a* 1*-regular sequence, where* $\eta = a\delta_0 + \zeta$, $a \neq 0$,
(iii) G *is a continuous mapping of* $L^2((0,2\pi);\mathbb{C}^n)$ *into itself, and there exists* $\delta > 0$ *such that*

$$\int_0^{2\pi} \|G(\phi)(s)\|^2\, ds \leqslant \delta^2$$

whenever $\phi \in L^2((0,2\pi);\mathbb{C}^n)$ *satisfies*

$$\int_0^{2\pi} \|\phi(s)\|^2\, ds \leqslant \|\mathcal{B}\|^2 \delta^2.$$

Then there exists a solution $u \in H_2^1((0, 2\pi); \mathbb{C}^n)$ *of the equation*

$$u'(t) - K(\eta * u)(t) = G(u)(t), \quad t \in [0, 2\pi],$$

satisfying $\int_0^{2\pi} \|u(s)\|^2 \, ds \leqslant \|\mathcal{B}\|^2 \delta^2$.

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References

[1] H. Amann, Linear and Quasilinear Parabolic Problems, Monogr. Math., vol. 89, Birkhäuser, Basel, 1995.
[2] H. Amann, Operator-valued Fourier multipliers, vector-valued Besov spaces and applications, Math. Nachr. 186 (1997) 5–56.
[3] H. Amann, Quasilinear parabolic functional evolution equations, in: M. Chipot, H. Ninomiya (Eds.), Recent Advances in Elliptic and Parabolic Issues, Proc. of the 2004 Swiss–Japanese Seminar, World Scientific, 2006, pp. 19–44.
[4] W. Arendt, Semigroups and evolution equations: Functional calculus, regularity and kernel estimates, in: Evolutionary Equations, vol. I, in: Handb. Differ. Equ., North-Holland, Amsterdam, 2004, pp. 1–85.
[5] W. Arendt, S. Bu, Operator-valued Fourier multiplier on periodic Besov spaces and applications, Proc. Edinb. Math. Soc. (2) 47 (2004) 15–33.
[6] W. Arendt, S. Bu, The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, Math. Z. 240 (2002) 311–343.
[7] A. Bátkai, S. Piazzera, Semigroups for Delay Equations, Res. Notes Math., vol. 10, A.K. Peters, Boston, MA, 2005.
[8] E. Berkson, T.A. Gillespie, Spectral decompositions and harmonic analysis on *UMD*-spaces, Studia Math. 112 (1994) 13–49.
[9] J. Bourgain, Vector-valued singular integrals and the H^1–*BMO* duality, in: Probability Theory and Harmonic Analysis, Marcel Dekker, New York, 1986.
[10] S. Bu, Y. Fang, Maximal regularity for integro-differential equation on periodic Triebel–Lizorkin spaces, Taiwanese J. Math. 12 (2) (2008) 281–292.
[11] S. Bu, Y. Fang, Periodic solutions for second order integro-differential equations with infinite delay in Banach spaces, Studia Math. 184 (2) (2008) 103–119.
[12] S. Bu, J. Kim, Operator-valued Fourier multipliers on periodic Triebel spaces, Acta Math. Sin. (Engl. Ser.) 21 (2005) 1049–1056.
[13] D.L. Burkholder, Martingales and Fourier analysis in Banach spaces, in: Probability and Analysis, Varenna, 1985, in: Lecture Notes in Math., vol. 1206, Springer-Verlag, Berlin, 1986, pp. 61–108.
[14] P.L. Butzer, H. Berens, Semi-Groups of Operators and Approximation, Grundlehren Math. Wiss., vol. 145, Springer-Verlag, 1967.
[15] R. Chill, S. Srivastava, L^p-maximal regularity for second order Cauchy problems, Math. Z. 251 (2005) 751–781.
[16] Ph. Clément, G. Da Prato, Existence and regularity results for an integral equation with infinite delay in Banach space, Integral Equations Operator Theory 11 (1988) 480–500.
[17] Ph. Clément, J. Prüss, An operator-valued transference principle and maximal regularity on vector-valued L_p-space, in: G. Lumer, L. Weis (Eds.), Evolution Equations and Their Applications in Physical and Life Sciences. Proceeding of the Bad Herrenalb Conference, Germany, 1999, in: Lect. Notes Pure Appl. Math., vol. 215, Marcel Dekker, New York, NY, 2001, pp. 67–87.
[18] Ph. Clément, B. de Pagter, F.A. Sukochev, M. Witvliet, Schauder decomposition and multiplier theorems, Studia Math. 138 (2000) 135–163.
[19] G. Da Prato, A. Lunardi, Periodic solutions for linear integrodifferential equations with infinite delay in Banach spaces, Lecture Notes in Math. 1223 (1985) 49–60.
[20] R. Denk, M. Hieber, J. Prüss, R-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Amer. Math. Soc. 166 (788) (2003).
[21] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, Grundlehren Math. Wiss., vol. 224, Springer-Verlag, 1983.
[22] M. Girardi, L. Weis, Operator-valued Fourier multiplier theorems on Besov spaces, Math. Nachr. 251 (2003) 34–51.
[23] M. Girardi, L. Weis, Criteria for R-boundedness of operator families, in: Evolution Equations, in: Lect. Notes Pure Appl. Math., vol. 234, Dekker, New York, 2003, pp. 203–221.
[24] G. Gripenberg, S.-O. Londen, O. Staffans, Volterra Integral and Functional Equations, Encyclopedia Math. Appl., vol. 34, Cambridge Univ. Press, Cambridge, New York, 1990.
[25] T. Hytönen, Convolutions, multipliers and maximal regularity on vector-valued Hardy spaces, J. Evol. Equ. 5 (2005) 205–225.
[26] V. Keyantuo, C. Lizama, Mild well-posedness of abstract differential equations, in: H. Amann, W. Arendt, M. Hieber, F. Neubrander, S. Nicaise, J. von Below (Eds.), Functional Analysis and Evolution Equations, Birkhäuser, Basel, 2008, pp. 371–387.

[27] V. Keyantuo, C. Lizama, Hölder continuous solutions for integro-differential equations and maximal regularity, J. Differential Equations 230 (2006) 634–660.
[28] V. Keyantuo, C. Lizama, Maximal regularity for a class of integro-differential equations with infinite delay in Banach spaces, Studia Math. 168 (1) (2005) 25–49.
[29] V. Keyantuo, C. Lizama, Fourier multipliers and integro-differential equations in Banach spaces, J. London Math. Soc. 69 (3) (2004) 737–750.
[30] P. Kunstmann, L. Weis, Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, in: M. Iannelli, R. Nagel, S. Piazzera (Eds.), Functional Analytic Methods for Evolution Equations, in: Lecture Notes in Math., vol. 1855, Springer-Verlag, 2004, pp. 65–311.
[31] V. Poblete, Maximal regularity of second-order equations with delay, J. Differential Equations 246 (1) (2009) 261–276.
[32] J. Prüss, Evolutionary Integral Equations and Applications, Monogr. Math., vol. 87, Birkhäuser, Basel, 1993.
[33] J.L. Rubio de Francia, J.L. Torrea, Some Banach space techniques in vector-valued Fourier analysis, Colloq. Math. 54 (1987) 273–284.
[34] H.J. Schmeisser, H. Triebel, Topics in Fourier Analysis and Function Spaces, Wiley, Chichester, 1987.
[35] D. Sforza, Existence in the large for a semilinear integrodifferential equation with infinite delay, J. Differential Equations 120 (1995) 289–303.
[36] O.J. Staffans, Periodic L^2-solutions of an integrodifferential equation in Hilbert space, Proc. Amer. Math. Soc. 117 (3) (1993) 745–751.
[37] H. Triebel, Fractals and Spectra. Related to Fourier Analysis and Function Spaces, Monogr. Math., vol. 78, Birkhäuser, Basel, 1983.
[38] H. Triebel, Theory of Function Spaces, Monogr. Math., vol. 91, Birkhäuser, Basel, 1997.
[39] L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, Math. Ann. 319 (2001) 735–758.
[40] F. Zimmermann, On vector-valued Fourier multiplier theorems, Studia Math. 93 (1989) 201–222.