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Boundary Value Problems on the Half-Line with Impulses and Infinite Delay[1]

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This paper presents some results on the existence and uniqueness of solutions for the boundary value problems on the half-line with impulses and infinite delay. Moreover, an existence theorem of multiple solutions is obtained also. The problems may be singular at the boundary. © 2001 Academic Press

Key Words: boundary value problems; impulses; infinite delay; fixed point index.

1. INTRODUCTION

Boundary value problems on the half-line arise quite naturally in the study of radially symmetric solutions of nonlinear elliptic equations and there are many results in this area; see [1–5] for examples. On the other hand, the theory of impulsive differential equations has been emerging as an important area of investigations in recent years (see [14, 15]). Very recently, some authors considered the delay equation $x'(t) = f(t, x_t)$ together with impulses and established some existence and uniqueness results with initial value(see [16–21]). However, no results are available for the boundary value problems on the half-line with impulses and infinite delay. As a result the goal of this paper is to fill the gap in this area. We obtain some results on the existence and uniqueness of solutions and on the existence of multiple solutions. Moreover, the problems may be suplinear and singular. A more direct condition than that in [16–19] is obtained and our theorems improve some results in [8, 11]. Our main techniques

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are the Leray–Schauder theorem and fixed point index theory (see [9]). Finally, we give some examples in Section 5.

2. PRELIMINARIES

First we give a lemma.

LEMMA 2.1. *Assume that* $x : [a, b] \to R$ *is a bounded and measurable function. Then* $g(t) = \sup\{|x(s)|, s \in [a, t]\}$ *is a measurable function on* $[a, b]$.

Proof. Given $\alpha > 0, E = \{t \in [a, b], g(t) \geq \alpha\}$. If $t \in E$, then for all $t' \geq t$, $g(t') \geq \alpha$. So E is an interval. Thus E is measurable. Consequently, $g(t)$ is a measurable function. The proof is complete. ∎

Assume that $h : (-\infty, 0] \to R$ is a continuous function with $l = \int_{-\infty}^{0} h(t)dt < +\infty$, $h(t) > 0$, $t \in [0, +\infty)$, and for $a > 0$, define

$$BM([-a, 0]), R) = \Big\{\psi : [-a, 0] \to R | \psi(t) \text{ is a bounded and measurable function on } [-a, 0]\Big\}$$

with norm $\|\psi\|_{[-a,0]} = \sup_{s\in[-a,0]} |x(s)|$. By Lemma 2.1 and the definition of BM, we can define

$$BM_h((-\infty, 0], R) = \left\{\psi : (-\infty, 0] \to R | \text{ for any } c > 0, \psi\Big|_{[-c,0]} \in BM([-c, 0], R) \text{ and } \int_{-\infty}^{0} h(t)\|\psi\|_{[t,0]}dt < +\infty\right\}$$

with norm

$$\|\psi\|_h = \int_{-\infty}^{0} h(t)\|\psi\|_{[t,0]}dt < +\infty,$$

and

$$PC_l([0, +\infty), R) = \Big\{\psi : [0, +\infty) \to R | \psi(t) \text{ is continuous at each } t \neq t_k, \text{ left continuous at } t = t_k, \psi(t_k^+) \text{ exists } (k = 1, 2, \ldots, m) \text{ and } \lim_{t\to+\infty} x(t) \text{ exists}\Big\}$$

with norm

$$\|x\|_l = \sup_{t\in[0,+\infty)} |x(t)|,$$

$$PC([0,+\infty),R) = \{\psi : [0,+\infty) \to R | \psi(t) \text{ is continuous at each } t \neq t_k, \text{ left continuous at } t = t_k, \psi(t_k^+) \text{ exists } (k = 1,2,\ldots,m)\}.$$

It is easy to see that $BM_h((-\infty,0],R)$ and $PC_l([0,+\infty),R)$ are Banach spaces.

Specifically, consider the problem

$$\begin{cases} (Lx)(t) + f(t,x_t) = 0, & t \neq t_k; \\ \Delta x|_{t=t_k} = I_k(x_{t_k}), & k = 1,2,\ldots,m; \\ \lambda x(0) - \beta \lim_{t\to 0} p(t)x'(t) = a; \\ \gamma x(\infty) + \delta \lim_{t\to\infty} p(t)x'(t) = b; \\ x(t) \text{ is bounded on } [0,+\infty), \end{cases} \tag{2.1}$$

where $\Phi \in BM_h((-\infty,0],R)$ and x_t is defined by

$$x_t(s) = \begin{cases} x(t+s), & t \geq t+s \geq 0; \\ \Phi(t+s), & -\infty < t+s < 0, \end{cases} \tag{2.2}$$

and $f : (0,+\infty) \times BM_h((-\infty,0],R) \to R, I_k : BM_h((-\infty,0],R) \to R, k = 1,2,\ldots,m$. Denote $(Lx)(t) = \frac{1}{p(t)}(p(t)x'(t))'$, $p \in C([0,+\infty),R) \bigcap C^1(0,+\infty)$, $p(t) > 0$ for $t \in (0,\infty)$,

$$\Delta x|_{t_k} = \lim_{\varepsilon\to 0^+} [x(t_k+\varepsilon) - x(t_k-\varepsilon)].$$

Let $\lambda,\beta,\gamma,\delta \geq 0$ with $\beta\gamma + \lambda\delta + \lambda\gamma > 0$, and $a,b \geq 0$. The following conditions will be assumed throughout:

$$\int_0^{+\infty} \frac{dt}{p(t)} < +\infty. \tag{2.3}$$

Denote $\tau_0(t) = \int_0^t \frac{1}{p(s)}ds$, $\tau_\infty(t) = \int_t^\infty \frac{1}{p(s)}ds$, $\rho^2 = \beta\gamma + \lambda\delta + \lambda\gamma\int_0^\infty \frac{1}{p(t)}dt$, and $\rho > 0$. Define

$$u(t) = \frac{1}{\rho}[\delta + \gamma\tau_\infty(t)], \qquad v(t) = \frac{1}{\rho}[\beta + \lambda\tau_0(t)]. \tag{2.4}$$

Then $\gamma v + \lambda u \equiv \rho$. Let

$$G(t,s) = \begin{cases} u(t)v(s)p(s), & 0 \leq s \leq t < \infty \\ v(t)u(s)p(s), & 0 \leq t \leq s < \infty. \end{cases} \tag{2.5}$$

Then

$$\lim_{t\to+\infty} G(t,s) = u(+\infty)v(s)p(s),\ s\in[0,+\infty). \tag{2.6}$$

We denote $\overline{G}(s) = u(+\infty)v(s)p(s)$, $s\in[0,+\infty)$. Let

$$e(t) = \frac{1}{\rho^2}\big[b\lambda\tau_0(t) + a\gamma\tau_\infty(t)\big] + \frac{1}{\rho^2}(a\delta + b\beta). \tag{2.7}$$

From (2.4), (2.5), and (2.7), there exist $t_m < a^* < b^* < +\infty$ and $1 \ge c^* = c^*(a^*, b^*) > 0$ such that

$$\begin{aligned} G(t,s) \ge c^* G(u,s) \quad &\text{for } \forall t\in[a^*,b^*],\\ &u\in[0,+\infty),\ s\in[0,+\infty), \end{aligned} \tag{2.8}$$

$$e(t) \ge c^* e(s) \quad \text{for } \forall t\in[a^*,b^*],\ s\in[0,+\infty), \tag{2.9}$$

$$\delta + \gamma\tau_\infty(t) \ge c^*\big[\delta + \gamma\tau_\infty(s)\big] \quad \text{for } \forall t\in[a^*,b^*],\ s\in[0,+\infty). \tag{2.10}$$

Write $Q = \{x \in PC_l([0,+\infty),R),\ x(t)\ge 0 \text{ with } x(t)\ge c^*x(s) \text{ for } \forall t\in[a^*,b^*],\ s\in[0,+\infty)\}$.

Now we give following lemma.

LEMMA 2.2. *Suppose* $x\in PC_l([0,+\infty),R)(a>0)$ *and* x_t *is defined by* (2.2). *Then for* $t\in[0,+\infty)$, $x_t\in BM_h((-\infty,0],R)$. *Moreover,*

$$\|x_t\|_h \ge l|x(t)|,$$

where

$$l = \int_{-\infty}^0 h(s)ds.$$

Proof. For any $t\in[0,a]$, it is easy to see that for any $a>0$, x_t is bounded and measurable on $[-a,0]$ and

$$\begin{aligned} &\int_{-\infty}^0 h(s)\|x_t\|_{[s,0]}ds\\ &\quad = \int_{-\infty}^{-t} h(s)\|x_t\|_{[s,0]}ds + \int_{-t}^0 h(s)\|x_t\|_{[s,0]}ds\\ &\quad \le \int_{-\infty}^{-t} h(s)\max\{\|x_0\|_{[t+s,0]}, \|x_t\|_{[-t,0]}\}|ds + \int_{-t}^0 h(s)\|x\|_{[0,t]}ds\\ &\quad \le \int_{-\infty}^0 h(s)\|x_0\|_{[s,0]}ds + 2\int_{-\infty}^0 h(s)ds \sup_{s\in[0,t]}|x(s)|\\ &\quad = \|x_0\|_h + 2l\sup_{s\in[0,t]}|x(s)|\\ &\quad \le \|\Phi\|_h + 3l\sup_{s\in[0,t]}|x(s)|. \end{aligned} \tag{2.11}$$

Since $\Phi \in BM_h((-\infty, 0], R)$, then $x_t \in BM_h((-\infty, 0], R)$. Moreover,

$$\|x_t\|_h = \int_{-\infty}^{0} h(s)\|x_t\|_{[s,0]}ds \geq \int_{-\infty}^{0} h(s)ds|x(t)| = l|x(t)|.$$

The proof is complete. ∎

Remark 2.1. Generally, for $x \in PC_l([0, +\infty), R)$, x_t is not continuous at $t \in [0, +\infty)$. For example, let

$$x(t) = \begin{cases} 0, & [0, t_1], \\ 1, & (t_1, +\infty). \end{cases}$$

Assume that $t', t \in (t_1, +\infty)$ and $t > t'$. Then for any $s < -t$, $\sup_{r\in[s,0]} |x(t+r) - x(t'+r)| \geq 1$. Then

$$\begin{aligned} \|x_t - x_{t'}\|_h &= \int_{-\infty}^{0} h(s)\|x_t - x_{t'}\|_{[s,0]}ds \\ &\geq \int_{-\infty}^{-t} h(s)\|x_t - x_{t'}\|_{[s,0]}ds \\ &\geq \int_{-\infty}^{-t} h(s)ds, \end{aligned}$$

which implies that $\lim_{t\to t'} \|x_t - x_{t'}\| \geq \int_{-\infty}^{-t'} h(s)ds$. Consequently, x_t is not continuous at t'. Since t' is arbitrary in $(t_1, +\infty)$, x_t is not continuous everywhere on $(t_1, +\infty)$. Then it is possible that $f(t, x_t)$ is not continuous everywhere even if $f(t, \psi)$ is continuous at each $(t, \psi) \in (0, +\infty) \times BM_h((-\infty, 0], R)$. In order to assure the integrality of $f(t, x_t)$, it is necessary to advance a suitable condition.

DEFINITION 2.1. A function $G : BM_h((-\infty, 0], R) \to R$ is said to be weakly continuous at $\phi_0 \in BM_h((-\infty, 0], R)$ if for any $\{\phi_n\} \subseteq BM_h((-\infty, 0], R)$ with $\lim_{n\to+\infty} \phi_n(s) = \phi_0(s)$, a.e. $s \in (-\infty, 0]$, then

$$\lim_{n\to+\infty} G(\phi_n) = G(\phi_0).$$

G is said to be weakly continuous on $BM_h((-\infty, 0], R)$ if G is weakly continuous at ϕ for any $\phi \in BM_h((-\infty, 0], R)$.

Remark 2.2. This condition is more direct than that in [16], which requires that $f(t, x_t)$ be measurable. The definition includes the conditions in [17–21], where it is required that $f(t, \psi)$ be continuous at each $\psi \in L^1([-r, 0], R^n)$ for all $t \in (0, +\infty)$.

DEFINITION 2.2. A function $g : (0, +\infty) \times BM_h \to R$ is a $L\frac{w}{}$ Carathéodary function if the following conditions hold:

(a) the map $t \to g(t, \psi)$ is measurable for all $\psi \in BM_h$;

(b) the map $\psi \to g(t, \psi)$ is weakly continuous for almost all $t \in (0, +\infty)$;

(c) for any $r > 0$, there exists a $\mu_r \in L((0, +\infty), R^+)$ such that for all $\|\psi\|_h \le r$,

$$|g(s, \psi)| \le \mu_r(s)$$

with

$$\sup_{t \in [0, +\infty)} \int_0^{+\infty} G(t, s)\mu_r(s)ds < +\infty$$

and

$$\int_0^{+\infty} \overline{G}(s)\mu_r(s)ds < +\infty.$$

LEMMA 2.3. *Assume that $D \subseteq PC_l([0, +\infty), R)$ is bounded. Then there exists an $r > 0$ such that*

$$\|x_t\|_h \le r \qquad \text{for all } x \in D \text{ and all } t \in [0, +\infty).$$

Proof. Since D is bounded, there exists a $d > 0$ such that $\|x\|_l \le d$ for all $x \in D$. By Lemma 2.2, for all $t \in [0, +\infty)$,

$$\|x_t\|_h \le \|\Phi\|_h + 3l \sup_{s \in [0, t]} |x(s)| \le \|\Phi\|_h + 3ld.$$

Let $r = \|\Phi\|_h + 3ld$. Then r is what we need. The proof is complete. ∎

LEMMA 2.4. *Assume that a function $g : (0, +\infty) \times BM_h \to R$ is a $L\overset{w}{-}$Carathéodory function and $x \in PC_l([0, +\infty), R)$. Then $g(t, x_t)$ is measurable.*

Proof. We can define $x(t) = \Phi(t)$ for all $t \in (-\infty, 0)$. Then there exists a continuous sequence $\{x_n\}$ such that

$$\lim_{n \to +\infty} x_n(t) = x(t), \qquad \text{for all } t \in (-\infty, +\infty).$$

By Lemma 4 in [22], x_{n_t} is continuous at $t \in [0, +\infty)$. Since $g : (0, +\infty) \times BM_h \to R$ is a $L\overset{w}{-}$Carathéodory function, then $g(t, x_{nt})$ is measurable on $[0, +\infty)$. Since $\lim_{n \to +\infty} x_{n_t}(s) = x_t(s)$, for all $s \in (-\infty, 0]$, then

$$\lim_{n \to +\infty} g(t, x_{nt}) = g(t, x_t), \qquad \text{for all } t \in [0, +\infty).$$

So $g(t, x_t)$ is measurable on $[0, +\infty)$. ∎

By Lemma 2.4 and normal method as in [7, 8], we have following lemma.

LEMMA 2.5. *Suppose f is a $L\frac{w}{}$Carathéodary function. If x satisfies equation*

$$x(t) = e(t) + (Ax)(t) + (Bx)(t), \tag{2.12}$$

then x is a solution of Eq. (2.1), *where*

$$(Ax)(t) = \int_0^{+\infty} G(t,s) f(s, x_s) ds,$$

$$(Bx)(t) = [\delta + \gamma\tau_\infty(t)] \sum_{0<t_k<t} \frac{I_k(x_{t_k})}{\delta + \gamma\tau_\infty(t_k)}.$$

The following theorems are needed in this paper.

THEOREM 2.1 (Nonlinear Alternative). *Let C be a convex subset of a normed linear space E, and U be an open subset of C, with $p^* \in U$. Then every compact, continuous map $N : \overline{U} \to C$ has at least one of the following two properties*:

(a) *N has a fixed point*;

(b) *there is an $x \in \partial U$, with $x = (1-\bar{\lambda})p^* + \bar{\lambda}Nx$ for some $0 < \bar{\lambda} < 1$.*

THEOREM 2.2. *Let $M \subseteq PC_l([0,+\infty), R)$. Then M is compact in PC_l $\times([0,+\infty), R)$ if the following conditions hold*:

(a) *M is bounded in PC_l*;

(b) *the functions belonging to M are piecewise equicontinuous on any interval of* $[0,+\infty)$;

(c) *the functions from M are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|f(t) - f(+\infty)| < \varepsilon$ for any $t \geq T(\varepsilon)$ and $f \in M$.*

This theorem is a simple improvement of the Corduneanu Theorem in [10].

3. EXISTENCE AND UNIQUENESS OF SOLUTIONS

In this section, we list the following conditions for convenience:

(H_1) f is a $L\frac{w}{}$Carathéodary function;

(H_2) $I_k \in C(BM_h, R)$ is bounded, $k = 1, 2, \ldots, m$.

From the proof of Lemma 2.4, for $x \in PC_l([0,+\infty), R^+)$, we can define

$$(Jx)(t) = e(t) + (Ax)(t) + (Bx)(t), \qquad t \in [0,+\infty). \tag{3.1}$$

LEMMA 3.1. *Assume that* (H_1) *and* (H_2) *hold. Then* $J : PC_l([0,+\infty), R) \to PC_l([0,+\infty), R)$ *is completely continuous.*

Proof. First we need to prove that for all $x \in PC_l([0,+\infty), R)$, $Jx \in PC_l([0,+\infty), R)$. By $x \in PC_l([0,+\infty), R)$ and Lemma 2.3, there exists an $r > 0$ such $|f(t, x_t)| \le \mu_r(t)$. For any $t' \in [0,+\infty)$ and $t < t'$,

$$
\begin{aligned}
&|(Jx)(t) - (Jx)(t')| \\
&\quad = |e(t) - e(t') + (Ax)(t) - (Ax)(t') + (Bx)(t) - (Bx)(t')| \\
&\quad \le |e(t) - e(t')| + \int_0^{+\infty} |G(t,s) - G(t',s)|\,|f(s,x_s)|ds + |\gamma\tau_\infty(t') \\
&\qquad - \gamma\tau_\infty(t)| \sum_{0<t_k<t} \frac{|I_k(x_{t_k})|}{\delta + \gamma\tau_\infty(t_k)} + |\delta + \gamma\tau_\infty(t')| \sum_{t\le t_k<t'} \frac{|I_k(x_{t_k})|}{\delta + \gamma\tau_\infty(t_k)} \\
&\quad \le |e(t) - e(t')| + \int_0^{+\infty} |G(t,s) - G(t',s)|\mu_r(s)ds + |\gamma\tau_\infty(t') \\
&\qquad - \gamma\tau_\infty(t)| \sum_{0<t_k<t} \frac{|I_k(x_{t_k})|}{\delta + \gamma\tau_\infty(t_k)} + |\delta + \gamma\tau_\infty(t')| \sum_{t\le t_k<t'} \frac{|I_k(x_{t_k})|}{\delta + \gamma\tau_\infty(t_k)}.
\end{aligned}
$$

By virtue of (c) of Definition 2.2, one has

$$\int_0^{+\infty} |G(t,s) - G(t',s)|\mu_r(s)ds \le 2 \sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t,s)\mu_r(s)ds.$$

Then by the Lebesgue Dominated Convergence Theorem, we get

$$\lim_{t\to t'-} (Jx)(t) = (Jx)(t') \qquad \text{for } t' \in [0,+\infty). \tag{3.2}$$

By a similar proof, it is easy to see that

$$\lim_{t\to t'+} (Jx)(t) = (Jx)(t') \qquad \text{for all but } t' \ne t_k, k = 1, 2, \ldots, t_m,$$

and $(Jx)(t_k^+)$ exists, $k = 1, 2, \ldots, m$. Now we will prove

$$\lim_{t\to+\infty} (Jx)(t) = e(+\infty) + \int_0^{+\infty} \overline{G}(s) f(s,x_s)ds + \delta \sum_{k=1}^m \frac{I_k(x_{t_k})}{\delta + \gamma\tau_\infty(t_k)}.$$

For $t > t_m$, one has

$$
\begin{aligned}
&|(Jx)(t) - e(+\infty) + \int_0^{+\infty} \overline{G}(s) f(s,x_s)ds + \delta \sum_{k=1}^m \frac{I_k(x_{t_k})}{\delta + \gamma\tau_\infty(t_k)}| \\
&\quad \le |e(t) - e(+\infty)| + \int_0^{+\infty} |G(t,s) - \overline{G}(s)|\,|f(s,x_s)|ds \\
&\qquad + \gamma|\tau_\infty(t)| \sum_{k=1}^m \frac{|I_k(x_{t_k})|}{\delta + \gamma\tau_\infty(t_k)}.
\end{aligned}
$$

By virtue of (c) of Definition 2.2, we get

$$\int_0^{+\infty} |G(t,s) - \overline{G}(s)|\,|f(s,x_s)|ds \le \sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t,s)\mu_r(s)ds + \int_0^{+\infty} \overline{G}(s)\mu_r(s)ds.$$

Then by the Lebesgue Dominated Convergence Theorem, one has

$$|(Jx)(t) - e(+\infty) + \int_0^{+\infty} \overline{G}(s)f(s,x_s)ds + \delta \sum_{k=1}^{m} \frac{I_k(x_{t_k})}{\delta + \gamma\tau_\infty(t_k)}| \to 0, \qquad t \to +\infty.$$

Therefore, $Jx \in PC_l([0,+\infty), R)$.

Next we show that $J : PC_l([0,+\infty), R) \to PC_l([0,+\infty), R)$ is continuous. Let $y^{(n)} \to y^{(0)}$ in $PC_l([0,+\infty), R)$. Then we must show that $Jy^{(n)} \to Jy^{(0)}$ in $PC_l([0,+\infty), R)$. Since $y^{(n)} \to y^{(0)}$ in $PC_l([0,+\infty), R)$, $\{y^{(0)}, y^{(1)}, \ldots, y^{(n)}, \ldots\}$ is bounded in $PC_l([0,+\infty), R)$. By Lemma 2.3, there exists an $r > 0$ such that

$$\|y_t^{(n)}\|_h \le r, \ \|y_t^{(0)}\|_h \le r$$

for all $n \ge 1$ and all $t \ge 0$. Then condition (H_1) implies that there exists a $\mu_r : (0,+\infty)$ such that $|f(t,\psi)| \le \mu_r(t)$ for all $\|\psi\|_h \le r$ and

$$\sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t,s)\mu_r(s)ds < +\infty,$$

and

$$\int_0^{+\infty} \overline{G}(s)\mu_r(s) < +\infty.$$

First we note that

$$|(Jy^{(n)})(+\infty) - (Jy^{(0)})(+\infty)| \le \int_0^{+\infty} \overline{G}(s)|f(s,y_s^{(n)}) - f(s,y_s^{(0)})|ds + \delta \sum_{k=1}^{m} \frac{|I_k(y_{t_k}^{(n)}) - I_k(y_{t_k}^{(0)})|}{\delta + \gamma\tau_\infty(t_k)}.$$

Since $y^{(n)} \to y^{(0)}$ in $PC_l([0,+\infty), R)$, by the proof of Lemma 2.2, for any $t \in [0,+\infty)$, one has

$$\begin{aligned}
\lim_{n\to+\infty} \|y_t^{(n)} - y_t^{(0)}\| &= \lim_{n\to+\infty} \int_{-\infty}^{0} h(s)\|y_t^{(n)} - y_t^{(0)}\|_{[s,0]}ds \\
&\le \lim_{n\to+\infty} \int_{-\infty}^{0} h(s)ds \sup_{s\in[0,t]} |y^{(n)}(s) - y^{(0)}(s)| \\
&\le l\|y^{(n)} - y^{(0)}\|_l \\
&= 0.
\end{aligned}$$

The Lebesgue Dominated Covergence Theorem guarantees that

$$\lim_{n\to+\infty}\int_0^{+\infty}\overline{G}(s)|f(s,y_s^{(n)})-f(s,y_s^{(0)})|ds=0.$$

Since $I_k : BM_h \to R$ is continuous, $k=1,2,\dots,m$, we get

$$\lim_{n\to+\infty}\delta\sum_{k=1}^{m}\frac{|I_k(y_{t_k}^{(n)})-I_k(y_{t_k}^{(0)})|}{\delta+\gamma\tau_\infty(t_k)}=0.$$

So

$$\lim_{n\to+\infty}|(Jy^{(n)})(+\infty)-(Jy^{(0)})(+\infty)|=0.$$

Then for any $\varepsilon>0$, there exists an $N_0\geq 1$ such that for all $n\geq N_0$

$$|(Jy^{(n)})(+\infty)-(Jy^{(0)})(+\infty)|<\frac{\varepsilon}{3}. \tag{3.3}$$

Also note that for $t>t_m$

$$\begin{aligned}
&|(Jy^{(n)})(t)-(Jy^{(n)})(+\infty)|\\
&\quad\leq|e(t)-e(+\infty)|+\int_0^{+\infty}|G(t,s)-\overline{G}(s)|\,|f(s,y_s^{(n)})|ds\\
&\qquad+\gamma|\tau_\infty(t)|\sum_{k=1}^{m}\frac{|I_k(y_{t_k}^{(n)})|}{\delta+\gamma\tau_\infty(t_k)}\\
&\quad\leq|e(t)-e(+\infty)|+\int_0^{+\infty}|G(t,s)-\overline{G}(s)|\mu_r(s)ds\\
&\qquad+|\gamma\tau_\infty(t)|\sum_{k=1}^{m}\frac{|I_k(y_{t_k}^{(n)})|}{\delta+\gamma\tau_\infty(t_k)}\\
&\quad\to 0,\qquad \text{as } t\to+\infty.
\end{aligned}$$

Similarly, we get

$$|(Jy^{(0)}(t)-(Jy^{(0)})(+\infty)|\to 0,\qquad \text{as } t\to+\infty.$$

Therefore there exists $T_0\geq t_m$ such that for all $t\geq T_0$

$$|(Jy^{(n)})(t)-(Jy^{(n)})(+\infty)|<\frac{\varepsilon}{3},\quad |(Jy^{(0)})(t)-(Jy^{(0)})(+\infty)|<\frac{\varepsilon}{3} \tag{3.4}$$

for all $n\geq 1$. Combining (3.3) and (3.4) yields an $N_0\geq 1$ and $T_0>t_m$ such that for all $n\geq N_0$

$$|(Jy^{(n)})(t)-(Jy^{(0)})(t)|<\varepsilon \tag{3.5}$$

for all $t \geq T_0$. Let $\overline{I_1} = [0, t_1]$, $\overline{I_2} = (t_1, t_2], \ldots, \overline{I_m} = (t_{m-1}, t_m]$, $\overline{I}_{m+1} = (t_m, T_0]$. For $t \in \overline{I_1}$,

$$\left|(Jy^{(n)})(t') - (Jy^{(n)})(t'')\right| \leq \left|e(t') - e(t'')\right| + \int_0^{+\infty} \left|G(t', s) - G(t'', s)\right|\mu_r(s)ds. \quad (3.6)$$

So $\{Jy^{(n)}\}$ is equicontinuous on $\overline{I_1}$. By a similiar proof, we know that $\{Jy^{(n)}\}$ is equicontinuous on $\overline{I_k}$, $k = 1, 2, \ldots, m+1$. On the other hand, by the Lebesgue Dominated Convergence Theorem, for any $t \in [0, T_0]$

$$\left|(Jy^{(n)})(t) - (Jy^{(0)})(t)\right| \leq \int_0^{+\infty} G(t,s)\left|f(s, y_s^{(n)}) - f(s, y_s^{(0)})\right|ds + \left|\delta + \gamma\tau_\infty(t)\right| \sum_{k=1}^{m} \frac{|I_k(y_{t_k}^{(n)}) - I_k(y_{t_k}^{(0)})|}{\delta + \gamma\tau_\infty(t_k)} \to 0,$$
$$\text{as } n \to +\infty. \quad (3.7)$$

Combining (3.6) and (3.7), and using the compactness of $[0, T_0]$, yield an N_1 such that for all $n \geq N_1$,

$$|(Jy^{(n)})(t) - (Jy^{(0)})(t))| < \varepsilon$$

for all $t \in [0, T_0]$. Let $N = \max\{N_0, N_1\}$. Then for $n \geq N$, we get

$$|(Jy^{(n)})(t) - (Jy^{(0)})(t)| < \varepsilon \quad (3.8)$$

for all $t \in [0, +\infty)$, which implies that

$$\|Jy^{(n)} - Jy^{(0)}\|_l \leq \varepsilon.$$

So $Jy^{(n)} \to Jy^{(0)}$ as $n \to +\infty$.

Finally, we show that $J : PC_l([0, +\infty), R) \to PC_l([0, +\infty), R)$ is completely continuous. Assume that $\Omega \subseteq PC_l([0, +\infty), R)$ is bounded. By Lemma 2.3, there exists an $r > 0$ such that for all $x \in \Omega$, $t \in [0, +\infty)$, $\|x_t\| \leq r$. From condition (H_1), there exists an $\mu_r \in L$ such that $|f(t, \psi)| \leq \mu_r(t)$ for all $\|\psi\|_h \leq r$, $t \in (0, +\infty)$. Then for $x \in \Omega$,

$$|(Jx)(t)| \leq |e(t)| + \int_0^{+\infty} G(t,s)\mu_r(s)ds + [\delta + \gamma\tau_\infty(t)] \sum_{0<t_k<t} \frac{|I_k(x_{t_k})|}{\delta + \gamma\tau_\infty(t_k)}.$$

$$\leq \|e\|_l + \sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t,s)\mu_r(s)ds + \sup_{t\in[0,+\infty)} (Bx)(t) = M_0.$$

We can prove (b) and (c) of Theorem 2.2 in exactly the same way that we prove (3.6) and (3.3), respectively. Therefore, by Theorem 2.2, $J : PC_l \times([0, +\infty), R) \to PC_l([0, +\infty), R)$ is completely continuous. The proof is complete. ∎

THEOREM 3.1. *Suppose that* (H_1) *and* (H_2) *hold. In addition suppose there exists an* $M > 0$ *such that for any* $\bar{\lambda} \in (0, 1)$, *if* y *is a solution of equation*

$$\begin{cases} (Lx)(t) + \bar{\lambda} f(t, x_t) = 0, & t \neq t_k; \\ \Delta x|_{t=t_k} = \bar{\lambda} I_k(x_{t_k}), & k = 1, 2, \ldots, m, \\ \lambda x(0) - \beta \lim_{t\to 0} p(t)x'(t)(t) = a; \\ \gamma x(\infty) + \delta \lim_{t\to\infty} p(t)x'(t)(t) = b; \\ x(t) \text{ is bounded on } [0, +\infty), \end{cases} \tag{1.1$_{\bar{\lambda}}$}$$

then $\|y\|_l \neq M$, *where* $\Phi \in BM_h((-\infty, 0], R)$ *and* x_t *is defined by* (2.2). *Then Eq.* (2.1) *has at least one solution.*

Proof. For $x \in PC_l([0, +\infty), R)$, the operator $J : PC_l([0, +\infty), R) \to PC_l([0, +\infty), R)$ is defined by (3.1). By Lemma 3.1, J is completely continuous.

Let $\Omega = \{x \in PC_l([0, +\infty), R), \|x\|_l < M\}$. We can now apply the Nonliear Alternative with $N = [0, +\infty)$ and $C = PC_l([0, +\infty), R)$. Possibility (b) of Theorem 2.1 cannot occur. Consequently, J has a fixed point in Ω, that is, Eq. (2.1) has a solution. ∎

By virtue of Theorem 3.1, we have the following theorem.

THEOREM 3.2. *Suppose that* (H_1) *and* (H_2) *hold. In addition suppose the following conditions hold*:

(a) *there exists a continuous and nondecreasing function* $f_1 : [0, +\infty) \to [0, +\infty)$ *with* $f_1(u) > 0$ *for* $u > 0$ *and a* $q \in L$ *with* $q(t) > 0$ *for a.e.* $t \in [0, +\infty)$ *such that* $|f(t, \phi)| \leq q(t) f_1(\|\phi\|_h)$ *for all* $t \in [0, +\infty)$, $\phi \in BM_h$;

(b) *there exist continuous and nondicreasing functions* $P_k : [0, +\infty) \to [0, +\infty)$, $k = 1, 2, \ldots, m$ *such that*

$$|I_k(\psi)| \leq P_k(\|\psi\|_h), \qquad k = 1, 2, \ldots, m$$

for all $\psi \in BM_h$;

(c) *let*

$$\sup_{c\in(0,+\infty)} \frac{c}{\|e\|_l + q_0 f_1(\|\Phi\|_h + 3lc) + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^m (P_k(\|\Phi\|_h + 3lc)/(\delta + \gamma\tau_\infty(t_k)))} > 1,$$

where $q_0 = \sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t, s)q(s)ds < +\infty$.

Then Eq. (2.1) *has at least one solution in* $PC_l([0, +\infty), R)$.

Proof. Let $M_0 > 0$ satisfy

$$\frac{M_0}{\|e\|_l + q_0 f_1(\|\Phi\|_h + 3lM_0) + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^m (P_k(\|\Phi\|_h + 3lM_0)/(\delta + \gamma\tau_\infty(t_k)))} > 1.$$

For $\lambda_0 \in (0, 1)$, suppose y is a solution of Eq. $(3.9)_{\lambda_0}$ with $\|y\|_l = M_1$. Then for $t \in [0, +\infty)$

$$\begin{aligned}|y(t)| &\le |e(t)| + |(Ay)(t)| + |(By)(t)| \\ &\le |e(t)| + \int_0^{+\infty} G(t,s)q(s)f_1(\|y_s\|_h)ds \\ &\quad + |\delta + \gamma\tau_\infty(t)| \sum_{0<t_k<t} \frac{P_k(\|y_{t_k}|_h)}{\delta + \gamma\tau_\infty(t_k)} \\ &\le \|e\|_l + \int_0^{+\infty} G(t,s)q(s)f_1(\|\Phi\|_h + 3lM_1)ds \\ &\quad + [\delta + \gamma\tau_\infty(t)] \sum_{0<t_k<t} \frac{P_k(\|\Phi\|_h + 3lM_1)}{\delta + \gamma\tau_\infty(t_k)} \\ &\le \|e\|_l + q_0 f_1(\|\Phi\|_h + 3lM_1) + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^{m} \frac{P_k(\|\Phi\|_h + 3lM_1)}{\delta + \gamma\tau_\infty(t_k)}.\end{aligned}$$

So

$$\begin{aligned}M_1 = \sup_{t\in[0, +\infty)} |y(t)| &\le \|e\|_l + q_0 f_1(\|\Phi\|_h + 3lM_1) \\ &\quad + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^{m} \frac{P_k(\|\Phi\|_h + 3lM_1)}{\delta + \gamma\tau_\infty(t_k)},\end{aligned}$$

which implies that

$$\frac{M_1}{\|e\|_l + q_0 f_1(\|\Phi\|_h + 3lM_1) + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^{m} (P_k(\|\Phi\|_h + 3lM_1)/(\delta + \gamma\tau_\infty(t_k)))} \le 1.$$

Then $M_1 \neq M_0$, which yields that the condition of Theorem 3.1 is true. By Theorem 3.1, Eq. (2.1) has at least one solution. ∎

THEOREM 3.3. *Suppose that* (H_1) *and* (H_2) *hold. In addition suppose there exist* $q \in L$ *and* $c_1, \ldots, c_m$ *such that*

$$|f(t, \psi) - f(t, \phi)| \le q(t)\|\psi - \phi\|_h$$

and

$$|I_k(\psi) - I_k(\phi)| \le c_k\|\psi - \phi\|_h, \qquad k = 1, 2, \ldots, m$$

for all $\psi, \phi \in BM_h$ *with*

$$k = l\left(\sup_{t\in[0, +\infty)} \int_0^{+\infty} G(t, s)q(s)ds + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^{m} \frac{c_k}{\delta + \gamma\tau_\infty(t_k)}\right) < 1.$$

Then Eq. (2.1) *has a unique solution.*

Proof. For $x, y \in PC_l([0, +\infty), R)$, by the proof of Lemma 2.2, we get

$$\|x_t - y_t\|_h = \int_0^{+\infty} h(s)\|x_t - y_t\|_{[s,0]} ds \le l\|x - y\|_l$$

for all $t \ge 0$. Then

$$\begin{aligned}
&|(Jx)(t)-(Jy)(t)| \\
&\le \int_0^{+\infty} G(t,s)|f(s,x_s)-f(s,y_s)|ds \\
&\quad +[\delta+\gamma\tau_\infty(t)] \sum_{0<t_k<t} \frac{|I_k(x_{t_k})-I_k(y_{t_k})|}{\delta+\gamma\tau_\infty(t_k)} \\
&\le \int_0^{+\infty} G(t,s)q(s)\|x_s-y_s)\|_h ds+[\delta+\gamma\tau_\infty(t)] \sum_{0<t_k<t} \frac{c_k\|x_{t_k}-y_{t_k}\|_h}{\delta+\gamma\tau_\infty(t_k)} \\
&\le \int_0^{+\infty} G(t,s)q(s)dsl\|x-y\|_l+[\delta+\gamma\tau_\infty(t)] \sum_{0<t_k<t} \frac{lc_k\|x_{t_k}-y_{t_k}\|_h}{\delta+\gamma\tau_\infty(t_k)} \\
&\le k\|x-y\|_l.
\end{aligned}$$

Then

$$\|Jx - Jy\|_l = \sup_{t\in[0,+\infty)} |(Jx)(t) - (Jy)(t)| \le k\|x - y\|_l.$$

So $J : PC_l([0, +\infty), R) \to PC_l([0, +\infty), R)$ is a contraction map and there exists a unique fixed point in $PC_l([0, +\infty), R)$, that is, Eq. (2.1) has a unique solution. ∎

4. THE EXISTENCE OF MULTIPLE SOLUTIONS

Let $Q_h = \{x \in BM_h, x(t) \ge 0 \text{ for all } t \ge (-\infty, 0]\}$. By the fixed point index theory in cone, we can obtain an results on the existence of multiple solutions.

THEOREM 4.1. *Suppose that the conditions* (H_1) *and* (H_2) *hold in Section* 3. *In addition suppose* $f : (0, +\infty) \times Q_h \to [0, +\infty)$ *and* $I_k : Q_h \to [0, +\infty)$ *and*

(a) *there exist two continuous and nondecreasing functions* $f_1, f_2 : [0, +\infty) \to R^+$ *with* $f_1(u) > 0, f_2(u) > 0$ *for* $u > 0$ *and two nonnegative* $q, \bar{q} \in L$ *with* $q(t) > 0, \bar{q}(t) > 0$ *for a.e.* $t \in [0, +\infty)$ *such that* $\bar{q}(t)f_2(|\phi(0)|) \le f(t, \phi) \le q(t)f_1(\|\phi\|_h)$ *for all* $t \in (0, +\infty)$, $\phi \in BM_h$;

(b) *there exist continuous and nondecreasing functions* $P_k : [0, +\infty) \to [0, +\infty)$, $k = 1, 2, \ldots, m$ *such that*

$$I_k(\psi) \le P_k(\|\psi\|_h), \qquad k = 1, 2, \ldots, m$$

for all $\psi \in Q_h$;

(c) *let*

$$\sup_{c\in(0,+\infty)} \frac{c}{\|e\|_l + q_0 f_1(\|\Phi\|_h + 3lc) + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^m (P_k(\|\Phi\|_h + 3lc)/(\delta + \gamma\tau_\infty(t_k)))} > 1,$$

where $q_0 = \sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t,s)q(s)ds < +\infty$;

(d) *let*

$$\lim_{u\to+\infty} \frac{f_2(u)}{u} = +\infty. \tag{4.1}$$

Then Eq. (2.1) *has at least two nonnegative solutions in* $PC_l([0, +\infty), R)$.

Proof. First we will prove that $JQ \subseteq Q$. For $x \in Q$, it is easy to see that $(Jx)(t) \ge 0$ for all $t \in [0, +\infty)$, and by virtue of (2.8), (2.9), and (2.10), for $t \in [a^*, b^*]$

$$\begin{aligned}
(Jx)(t) &= e(t) + \int_0^{+\infty} G(t,s) f(s, x(s))ds + [\delta + \gamma\tau_\infty(t)] \sum_{k=1}^m \frac{(I_k(x_{t_k})}{\delta + \gamma\tau_\infty(t_k)} \\
&\ge c^* \Bigg[e(u) + \int_0^{+\infty} G(u,s) f(s, x(s))ds + [\delta + \gamma\tau_\infty(u)] \\
&\qquad \times \sum_{k=1}^m \frac{I_k x)(t_k)}{\delta + \gamma\tau_\infty(t_k)} \Bigg] \\
&= c^*(Jx)(u)
\end{aligned}$$

for all $u \in [0, +\infty)$. Therefore $JQ \subseteq Q$.

By condition (c), we choose an $M_0 > 0$ such that

$$\frac{M_0}{\|e\|_l + q_0 f_1(\|\Phi\|_h + 3lM_0) + [\delta + \gamma\tau_\infty(0)] \sum_{0<t_k<t} (P_k(\|\Phi\|_h + 3lM_0)/(\delta + \gamma\tau_\infty(t_k)))} > 1. \tag{4.2}$$

Condition (d) implies that there exists an $M_1 > M_0$ such that

$$f_2(u) \ge N^* u \tag{4.3}$$

for all $n \ge M_1$, where

$$N^* > \left(c^* \inf_{t\in[a^*, b^*]} \int_{a^*}^{b^*} G(t,s)q(s)ds \right)^{-1}.$$

Let $M' = \max\{M_1, (2M_0/c^*)\}$ and define

$$\Omega_1 = \left\{ x \in PC_l([0, +\infty), R) \mid \|x\|_l < M_0 \right\},$$

and

$$\Omega_2 = \left\{x \in PC_l([0, +\infty), R) \middle| \|x\|_l < M'\right\}.$$

If there exists a $\lambda_0 \in (0, 1]$ and $x \in \partial\Omega_1 \cap Q$ with $x = \lambda_0 Ax$, then for $t \in [0, +\infty)$

$$\begin{aligned}
|x(t)| &\le |e(t)| + |(Ax)(t)| + |(Bx)(t)| \\
&\le |e(t)| + \int_0^{+\infty} G(t,s)q(s)f_1(\|x_s\|_l)ds \\
&\quad + [\delta + \gamma\tau_\infty(t)] \sum_{0<t_k<t} \frac{P_k(\|x_{t_k}\|_h)}{\delta + \gamma\tau_\infty(t_k)} \\
&\le \|e\|_l + \int_0^{+\infty} G(t,s)q(s)f_1(\|\Phi\|_h + 3lM_0)ds \\
&\quad + [\delta + \gamma\tau_\infty(t)] \sum_{0<t_k<t} \frac{P_k(\|\Phi\|_h + 3lM_0)}{\delta + \gamma\tau_\infty(t_k)} \\
&\le \|e\|_l + q_0 f_1(\|\Phi\|_h + 3lM_0) + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^{m} \frac{P_k(\|\Phi\|_h + 3lM_0)}{\delta + \gamma\tau_\infty(t_k)}.
\end{aligned}$$

So

$$\begin{aligned}
M_0 = \sup_{t \in [0, +\infty)} |x(t)| &\le \|e\|_l + q_0 f_1(\|\Phi\|_h + 3lM_0) \\
&\quad + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^{m} \frac{P_k(\| \ Phi\|_h + 3lM_0)}{\delta + \gamma\tau_\infty(t_k)},
\end{aligned}$$

which implies that

$$\frac{M_0}{\|e\|_l + q_0 f_1(\|\Phi\|_h + 3lM_0) + [\delta + \gamma\tau_\infty(0)] \sum_{k=1}^{m} (P_k(\|\Phi\|_h + 3lM_0)/(\delta + \gamma\tau_\infty(t_k)))} \le 1.$$

This contradicts (4.2). Then for any $x \in \partial\Omega_1 \cap Q$ and any $\bar{\lambda} \in (0, 1]$, $x \neq \bar{\lambda} Jx$, which implies that

$$i(J, Q \cap \Omega_1, Q) = 1. \tag{4.4}$$

Now for $x \in \partial\Omega_2 \cap Q$, if $(Jx)(t) \le x(t)$, $t \in [0, +\infty)$, then for $t \in [a^*, b^*]$

$$\begin{aligned}
x(t) &\ge (Jx)(t) \\
&\ge \int_0^{+\infty} G(t, s) f(s, x_s) ds \\
&\ge \int_0^{+\infty} G(t, s)\bar{q}(s) f_2(x(s)) ds
\end{aligned}$$

$$\geq \int_{a^*}^{b^*} G(t,s)\bar{q}(s)f_2(x(s))ds$$
$$> \int_{a^*}^{b^*} G(t,s)\bar{q}(s)N^*x(s)ds$$
$$\geq \int_{a^*}^{b^*} G(t,s)\bar{q}(s)dsN^*c^*\|x\|_l$$
$$> \|x\|_l.$$

This is a contradiction. Then for any $x \in \partial\Omega_2 \cap Q$, $Jx \nleq x$, which implies

$$i(J, \Omega_2 \cap Q, Q) = 0. \tag{4.5}$$

So

$$i(J, (\Omega_2 - \overline{\Omega_1}) \cap Q, Q) = -1. \tag{4.6}$$

By (4.4) and (4.6), J has a fixed point $x_1 \in \Omega_1 \cap Q$ and a fixed point $x_2 \in (\Omega_2 - \overline{\Omega_1}) \cap Q$. x_1 and x_2 are two nonnegative solutions of Eq. (2.1). ∎

5. SOME EXAMPLES

In this section, we will give some examples to illustrate the theorems in Section 4.

EXAMPLE 5.1. Consider the problem

$$\begin{cases} x''(t) + \frac{it^{i-1}}{1+t^i}x'(t) + \phi(t)\big(\ln\big(2 + \int_{-\infty}^0 e^s|x_t(s)|ds\big) \\ \qquad + x^{\frac{1}{3}}(t) + x^{\frac{1}{5}}(t-1)\big) = 0, \qquad t \in (0,+\infty) - \{1\}; \\ \Delta x_{t=1} = \ln\big(1 + \int_{-\infty}^0 e^{2s}|x_1(s)|ds\big); \\ x(0) = 0, \lim_{t\to+\infty}(1+t^i)x'(t) = 0; \end{cases} \tag{5.1}$$

where

$$\phi(t) = \begin{cases} \frac{1}{200}t^{-\frac{1}{2}}, & t \in (0,1]; \\ \frac{1}{50(1+t^i)^2}, & t \in (1,+\infty), \end{cases}$$

$1 < i$, and

$$\Phi(t) = t \cdot sgn(\cos t), \qquad t \in (-\infty, 0].$$

CONCLUSION. *Problem* (5.1) *has at least one solution*.

Proof. Let $h(t) = e^t$, $t \in (-\infty, 0]$. Then $l = \int_{-\infty}^0 e^t dt = 1$ and we can define BM_h-space and PC_l-space. It is easy to see that $p(t) = 1 + t^i$ and

$$f(t, \psi) = \phi(t)\left(\ln\left(2 + \int_{-\infty}^0 e^s|\psi(s)|ds\right) + \psi^{\frac{1}{3}}(0) + \psi^{\frac{1}{5}}(-1)\right)$$

$$I_1(\psi) = \ln\left(1 + \int_{-\infty}^0 e^{2s}|\psi(s)|ds\right)$$

for $t \in (0, +\infty) \times BM_h$. If $y \in PC_l$ is a solution of

$$\begin{cases} (Lx)(t) + \bar{\lambda} f(t, x_t) = 0, & t \neq 1; \\ \Delta x\big|_{t=1} = \bar{\lambda} I_1(x_1); \\ x(0) = 0, \qquad \lim_{t\to+\infty}(1 + t^i)x'(t) = 0; \\ x(t) \text{ is bounded on } [0, +\infty), \end{cases} \tag{5.1$_{\bar\lambda}$}$$

then

$$\begin{aligned} |y(t)| &\leq \int_0^{+\infty} G(t, s)\phi(s)\left(\ln\left(2 + \int_{-\infty}^0 e^s|y_s(r)|dr\right)ds \right.\\ &\qquad \left. + y^{\frac{1}{3}}(s) + y^{\frac{1}{5}}(s-1)\right) + \sum_{0<1<t} I_1(y_1) \\ &\leq \int_0^{+\infty} G(t, s)\phi(s)ds\left(\ln(2 + \|\Phi\|_h + 3\|y\|_l) + \|y\|_h^{\frac{1}{3}} + e^{\frac{1}{5}}\|y\|_h^{\frac{1}{5}}\right) \\ &\quad + \ln(1 + \|\Phi\|_h + 3\|y\|_l), \quad t \in [0, +\infty), \end{aligned}$$

which implies

$$\begin{aligned} \|y\|_l \leq \sup_{t\in[0,+\infty)} \int_0^{+\infty} G(t,s)\phi(s)ds\left(\ln(2+\|\Phi\|_h+3\|y\|_l)+\|y\|_h^{\frac{1}{3}}+e^{\frac{1}{5}}\|y\|_h^{\frac{1}{5}}\right) \\ +\ln(1+\|\Phi\|_h+3\|y\|_l). \end{aligned}$$

Thus there exists a $c > 0$ such that $\|y\|_l < c$ for all y satisfies $(5.1)_{\bar\lambda}$. Now by virtue of Theorem 3.1, Eq. (5.1) has at least one solution. ∎

EXAMPLE 5.2. Consider the problem

$$\begin{cases} x''(t) + \frac{2t}{1+t^2}x'(t) + \phi(t)\left(\ln\left(2 + \int_{-\infty}^0 e^s|x_t(s)|ds\right)\right. & \\ \qquad \left. +x(t) + x(t-1)\right) = 0, & t \in (0, +\infty) - \{1\}; \\ \Delta x_{t=1} = \frac{1}{2}\ln\left(1 + \int_{-\infty}^0 e^{2s}|x_1(s)|ds\right); & \\ x(0) = 0, \lim_{t\to+\infty}(1 + t^i)x'(t) = 0, + & \end{cases} \tag{5.2}$$

where

$$\phi(t) = \frac{1}{4\pi^2(2 + e)^2(1 + t^i)^2}, \qquad t \in [0, +\infty),$$

$$\Phi(t) = t \cdot sgn(\cos t), \qquad t \in (-\infty, 0].$$

CONCLUSION. *Problem* (5.2) *has a unique solution in* $PC_l([0, +\infty), R)$.

Proof. Let $h(t) = e^t$, $t \in (-\infty, 0]$. Then $l = \int_{-\infty}^{0} e^t dt = 1$ and we can define BM_h-space and PC_l-space. It is easy to see that $p(t) = 1 + t^2$ and

$$f(t, \psi) = \phi(t)\left(\ln\left(2 + \int_{-\infty}^{0} e^s|\psi(s)|ds\right) + \psi(0) + \psi(-1)\right),$$

and

$$I_1(\psi) = \frac{1}{2}\ln\left(1 + \int_{-\infty}^{0} e^{2s}|\psi(s)|ds\right)$$

for $t \in (0, +\infty) \times BM_h$ and

$$|f(t, \psi_1) - f(t, \psi_2)| \leq \phi(t)(2 + e)\|\psi_1 - \psi_2\|_h),$$

and

$$|I_1(\psi_1) - I_1(\psi_2)| \leq \frac{1}{2}(\|\psi_1 - \psi_2\|_h).$$

Then

$$k = \sup_{t\in([0, +\infty)}\left[\int_0^{+\infty} G(t, s)\phi(s)ds\right] + \frac{1}{2} < 1.$$

So the conditions of Theorem 3.3 hold. Then Eq. (5.2) has a unique solution in $PC_l([0, +\infty), R)$. ∎

EXAMPLE 5.3. Consider the problem

$$\begin{cases} x''(t) + \frac{2t}{1+t^2}x'(t) + \phi(t)\Big(\ln\Big(2 + \int_{-\infty}^{0} e^s|x_t(s)|ds\Big) & \\ \quad + x^3(t) + x^{\frac{1}{5}}(t-1)\Big) = 0, & t \in (0, +\infty) - \{1\}, \\ \Delta x|_{t=1} = \frac{1}{8}x(1), & \\ x(0) = 0, \lim_{t\to+\infty}(1 + t^2)x'(t) = 0, & \end{cases} \tag{5.3}$$

where

$$\phi(t) = \frac{1}{4\pi^2(\ln 6 + 64 + 4^{1/5}e^{1/5})(1 + t^2)^2}$$

and

$$\Phi(t) = |\sin t|, \qquad t \in (-\infty, 0].$$

CONCLUSION. *Problem* (5.3) *has at least two nonnegative solutions*.

Proof. Let $h(t) = e^t$, $t \in (-\infty, 0]$. Then $l = \int_{-\infty}^{0} e^t dt = 1$ and we can define BM_h-space and PC_l-space. Q is defined in Section 2. It is easy to see that $p(t) = 1 + t^2$ and

$$f(t, \psi) = \phi(t)\left(\ln\left(2 + \int_{-\infty}^{0} e^s|\psi(s)|ds\right) + \psi^3(0) + \psi^{\frac{1}{5}}(-1)\right),$$

$$I_1(\psi) = \frac{1}{8}\psi(0).$$

It is easy to see that

$$\phi(t)\psi^3(0) \leq f(t, \psi) \leq \phi(t)\big(\ln(2 + \|\psi\|_h) + \|\psi\|_h^3 + e^{\frac{1}{5}}\|\psi\|_h^{\frac{1}{5}}\big),$$

$$|I_1(\psi)| \leq \frac{1}{8}\|\psi\|_h$$

for all $t \in [0, +\infty)$ and $\psi \in Q_h$. Then $f_1(x) = \ln(2 + x) + x^3 + e^{1/5}x^{1/5}$, $q(t) = \phi(t)$, $f_2(x) = x^3$, $\bar{q}(t) = \phi(t)$. Now $\lim_{x\to+\infty}(f_2(x)/x) = +\infty$ and

$$\sup_{c\in(0,\,+\infty)} \frac{c}{q_0 f_1(\|\Phi\|_h + 3c) + P_1(\|\Phi\|_h + 3c)} \geq \frac{1}{q_0 f_1(4) + P_1(4)} > 1,$$

where $q_0 = \sup_{t\in[0,\,+\infty)} \int_0^{+\infty} G(t, s)\ \phi(s)ds \leq \frac{1}{4}$. So the conditions of Theorem 4.1 hold. Thus Eq. (5.3) has at least two nonnegative solutions. ∎

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