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Positive Solutions for a Nonhomogeneous Semilinear Elliptic Problem with Supercritical Exponent

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1. INTRODUCTION

Let Ω be a bounded domain in R^N $(N > 2)$ with smooth boundary $\partial\Omega$ and consider the nonhomogeneous semilinear elliptic problem

$$\begin{aligned} -\Delta u &= \lambda + u^p, && x \in \Omega, \\ u &> 0, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where λ and ρ are both real parameters and $p > 0$.

By a solution of (1.1) we mean, unless specifically stated, a classical solution which satisfies (1.1) pointwise. Denote H be Sobolev space $W_0^{1,2}$ with the norm $\|\cdot\|$; if $u \in H$ is a solution of (1.1), let

$$I_\lambda(u) = \frac{1}{2}\|u\|^2 - \lambda \int_\Omega |u|\, dx - \frac{1}{p+1}\int_\Omega |u|^{p+1}\, dx$$

denote the energy of u.

If $p \geq p_N \equiv (N+2)/(N-2)$, $\lambda \leq 0$, and if Ω is star-shaped, it is well-known that (1.1) has no solution; in this situation the solvability of (1.1) is closely related to the geometry and topology of the domain; see [5–7]. In this paper, we always suppose that $\lambda > 0$.

If $p = p_N$ and $\Omega = B$, the unit ball of R^N, Ali and Castro [4] have shown that there exists a continuous function F: $(0,\infty) \to (0,\infty)$ such that u is a solution of (1.1) if and only if $\lambda = F(u(0))$. Moreover, $\lim_{d\to 0} F(d) = 0$, $\lim_{d\to\infty} F(d) = 0$. This tell us that $\{(\lambda, u) : u$ satisfies (1.1)$\}$ is a continuous curve in $R \times L^\infty$, one end of the curve is $(0,0)$, and another end tends to the $|u|_\infty$-axis. So there exists a $\lambda_0 > 0$ such that (1.1) has at least

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two solutions when $\lambda \in (0, \lambda_0)$ (this multiplicity result had been obtained by Tarantello [3] earlier).

For the general nonhomogeneous case see [10].

For $p > p_N$, we know that there exists an unbounded branch of the solution pair (λ, u) of (1.1) by [9]. This branch emanates from $(0,0)$. The question left open is whether it tends to the $|u|_\infty$-axis when $|u|_\infty \to \infty$. This question is closely related to the uniqueness of the solution of (1.1) for small λ.

On the other hand, Ambrosetti *et al.* [1] have considered the problem

$$\begin{aligned} -\Delta u &= \lambda u^q + u^p, && x \in \Omega, \\ u &> 0, && x \in \Omega, \\ u &= 0, && x \in \partial\Omega, \end{aligned} \tag{1.2}$$

where $0 < q < 1$, $p > 1$; they have shown that there exists a constant $\Lambda > 0$ such that (1.2) has a minimal solution if $\lambda \in (0, \Lambda)$, has no solution if $\lambda > \Lambda$, and has a second solution if $p \in (1, p_N]$, $\lambda \in (0, \Lambda)$. Also, they have advanced an open problem: Does (1.2) have a second solution when $\Omega = B$, the unit ball of R^N, $p > p_N$, and $\lambda > 0$ small enough?

This open problem was answered negatively by Zhao and Zhong [11]; we proved that all of the solutions (u, λ) of (1.2) tend to the unique singular solution (ω, λ^*) as their supremum norm tends to infinity with $\lambda^* > 0$, hence there is only one minimal solution of (1.2) for λ small enough.

The purpose of the present paper is to study (1.1) when $\lambda > 0$ and $p > 1$. In Lemmas 2.1–2.3 we give some results very similar to Theorems 2.1–2.4 of [1]. Thus a problem similar to the open problem is presented. By a simple contraction argument as in [2], we prove that the answer to this problem is still negative. Thus, a uniqueness result of (1.1) has been obtained for $\lambda > 0$ small enough.

The paper is organized as follows. Section 2 contains some lemmas and the statements of Theorems 2.1–2.3; in Section 3 we carry out some preliminary transformation and establish the existence and uniqueness as well as the asymptotic behavior of the singular solution, proving Theorem 2.1, and finally in Section 4 we prove Theorems 2.2 and 2.3.

2. SOME LEMMAS AND STATEMENTS OF THE RESULTS

As in [1], by the sub–super solution method, it is easy to obtain the following lemmas:

LEMMA 2.1. *For all $p > 1$ there exists a constant $\Lambda > 0$ such that*:

1. *For all $\lambda \in (0, \Lambda)$ Problem* (1.1) *has a minimal solution u_λ such that $I_\lambda(u_\lambda) < 0$. Moreover, u_λ is increasing with respect to λ.*

2. *For* $\lambda = \Lambda$ *Problem* (1.1) *has at least one weak solution* $u \in H \cap L^{p+1}$.

3. *For all* $\lambda > \Lambda$ *Problem* (1.1) *has no solution.*

Remark 2.1. For $0 < p < 1$, it is easy to verify that $\Lambda = \infty$.

Remark 2.2. For $p = p_N$, there exist at least two solutions for $\lambda > 0$ small enough; see [3].

LEMMA 2.2. *There exists a constant* $A > 0$ *such that for all* $\lambda \in (0, \Lambda)$ *Problem* (1.1) *has at most one solution u such that* $\|u\|_\infty \le A$.

LEMMA 2.3. *Let* $p \ge p_N$ *and suppose that* Ω *is star-shaped. Then* $\|w_\lambda\|_\infty \to \infty$ *as* $\lambda \to 0$, *where* w_λ *is any solution of* (1.1) *distinct from the minimal solution* u_λ.

Now let us consider Problem (1.1) with $\lambda > 0$ small enough. Lemmas 2.1 and 2.2 tell us that the minimal solution u_λ is a unique solution which tends to zero as λ tends to zero. So, for $p = p_N$, by Lemma 2.3, the two solutions mentioned in Remark 2.2 are the minimal solution u_λ and the solution w_λ satisfying $\|w_\lambda\|_\infty \to \infty$ as $\lambda \to 0$. But, for the case $p > p_N$, $\lambda > 0$ small enough, we do not know whether a second solution exists, even if $\Omega = B$. This problem corresponds to the open problem (a) in [1].

From now on we suppose that $p > p_N$ and that $\Omega = B$. By Lemma 2.3, we know that if there exists a second solution w_λ, it must hold that $\|w_\lambda\|_\infty \to \infty$ as $\lambda \to 0$, so we shall study the unbounded branch of the solution pair (λ, u) of (1.1). If the branch is far away from the $|u|_\infty$-axis as $|(\lambda, u)|_{R \times L^\infty} \to \infty$, it seems that (1.1) does not have the second solution; that is, (1.1) may have a unique solution u_λ for $\lambda > 0$ small enough.

By [8], any solution of (1.1) will be radially symmetric. This leads us to use ODE techniques.

From this viewpoint we first introduce the notion of a radial singular solution of (1.1). By this we mean a function $\omega(x)$ which satisfies (1.1) for $x \ne 0$, has radial symmetry, and behaves near the origin as

$$\omega(x) \sim A|x|^{-\theta} \qquad \text{as } x \to 0, \tag{2.1}$$

where A is a constant and $\theta > 0$ is a real number.

Remark 2.3. A simple calculation shows that θ must be $2/(p-1)$.

By some necessary transformation and a simple contraction argument as in [2], we obtain our first main result.

THEOREM 2.1. *Suppose that* $p > p_N$. *Then there exists a unique* $\lambda^* \in (0, \Lambda]$ *such that* (1.1) *with* $\lambda = \lambda^*$ *has a radial singular solution* $\omega(x)$. *Moreover*, $\omega(x)$ *is the unique radial singular solution of* (1.1); *its asymptotic* *behavior near the origin is given by*

$$\omega(x) = A(p, N)|x|^{-2/(p-1)}\{1 + o(|x|^2)\} \qquad \text{as } x \to 0,$$

where

$$A(p, N) = \left[\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right]^{1/(p-1)}. \tag{2.2}$$

About the asymptotic behavior of solutions (λ, u) as $|u|_\infty \to \infty$, we prove Theorem 2.2 by some ODE techniques, energy analysis and contraction principle.

THEOREM 2.2. *Suppose that* $\{(\lambda_n, u_n)\}$ *is a sequence of solutions of* (1.1) *such that* $|u_n|_\infty \to \infty$ *as* $n \to \infty$. *Then* $\lambda_n \to \lambda^*$ *as* $n \to \infty$, *and* $u_n \to \omega$ *as* $n \to \infty$ *uniformly on any compact sets which do not contain the origin. In addition,* $u_n \to \omega$ *as* $n \to \infty$ *in* $L^{1+p}(B)$ *and in* $H(B)$.

Back to our questions, we have a uniqueness result, that is,

THEOREM 2.3. *Suppose that* $p > p_N$. *Then there exists some* $\lambda_0 > 0$ *such that for all* $\lambda \in (0, \lambda_0)$, (1.1) *has only one solution.*

Remark 2.4. For Problem (1.2) with $q = 1$, results similar to those for Theorems 2.1 and 2.2 have been obtained in [2].

Remark 2.5. Theorem 2.3 offers an interesting contrast to the situation where p is critical: $p = p_N$, in which case, by [3] or [4], Eq. (1.1) has at least two solutions for $\lambda > 0$ small enough, but by Theorem 2.3, for $p > p_N$, there is only one (minimal) solution u_λ for $\lambda > 0$ small enough.

3. SOME TRANSFORMATIONS AND THE SINGULAR SOLUTION

Let $\Omega = B$, the unit ball of R^N. It is well-known that all solutions of (1.1) are radially symmetric [8], so we can write $u = u(r)$, where $r = |x|$, and consider instead of Eq. (1.1) the problem

$$(\mathrm{I}) \quad \begin{cases} u_{rr} + \dfrac{N-1}{r}u_r + \lambda + u^p = 0, & 0 < r < 1, \\ u > 0, & 0 \le r < 1, \\ u(0) = a, & u_r(0) = 0, \end{cases}$$

where $a > 0$ is chosen so that $u(1) = 0$.

In view of the asymptotic behavior of the singular solution given in Theorem 2.1, we introduce the variables

$$t = \frac{1}{m} \ln r, \qquad y(t) = A^{-1} r^{2/(p-1)} u(r). \tag{3.1}$$

Then for $y(t)$ we have

$$\text{(II)} \quad \begin{cases} y'' + \alpha y' - y + y^p + \beta e^{m(\theta+2)t} = 0, & -\infty < t < 0, \\ y > 0, & -\infty < t < 0, \\ y(0) = 0,\ e^{-m\theta t} y(t) \to \dfrac{a}{A} & \text{as } t \to -\infty, \end{cases}$$

where

$$\theta = \frac{2}{p-1}, \qquad m = A^{(1-p)/2},$$

$$\alpha = m\left(N - 2 - \frac{4}{p-1}\right), \qquad \beta = \lambda m^2 A^{-1},$$

and A has been defined by (2.2).

For convenience we sometimes consider (II) as a first-order system,

$$\text{(III)} \quad \begin{cases} y' = z, \\ z' = -\alpha z + y - y^p - \beta e^{m(\theta+2)t}, \end{cases}$$

and relate its orbits to those of the associated autonomous system obtained when $t \to -\infty$,

$$(\text{III}_0) \quad \begin{cases} y' = z, \\ z' = -\alpha z + y - y^p. \end{cases}$$

Now we consider the non-autonomous system

$$(\text{II}_1) \quad \begin{cases} y'' + \alpha y' - y + y^p + \beta e^{m(\theta+2)t} = 0, & -\infty < t < 0, \\ y > 0, & -\infty < t < 0, \\ y(0) = 0,\ y(t) \to 1 & \text{as } t \to -\infty. \end{cases} \tag{3.2}$$

By a shift in t,

$$t = \tau - \frac{\ln \beta}{m(\theta + 2)},$$

we may eliminate the eigenvalue β from (3.2) to obtain the problem

$$(\mathrm{II}_2)\quad \begin{cases} y'' + \alpha y' - y + y^p + e^{m(\theta+2)\tau} = 0, & -\infty < \tau < T, \quad (3.3) \\ y > 0, & -\infty < \tau < T, \quad (3.4) \\ y(T) = 0,\ y(\tau) \to 1 & \text{as } \tau \to -\infty, \quad (3.5) \end{cases}$$

where

$$T = \frac{\ln \beta}{m(\theta + 2)}. \tag{3.6}$$

We now proceed as follows: we first show that any solution $y(\tau)$ of (3.3) which converges to 1 as $\tau \to -\infty$ has a certain asymptotic behavior. We then show that there exists precisely one such solution, that it can be continued up to any finite value of τ, and finally, that this solution cannot remain positive for all $\tau \in R$, so it must have a first zero T. The eigenvalue β^* (and also λ^*) then follows from (3.6).

LEMMA 3.1. *Let $y(\tau)$ be a solution of Eq.* (3.3) *with $y(\tau) \to 1$ as $\tau \to -\infty$. Then*

$$y(\tau) = 1 - B(p, N) e^{m(\theta+2)\tau}\left[1 + O(e^{m(\theta+2)\tau})\right] \qquad \text{as } \tau \to -\infty,$$

where

$$B(p, N) = \left[m^2(\theta + 2)^2 + m(\theta + 2)\alpha + p - 1\right]^{-1}.$$

Proof. Set $s = -\tau$ and $\eta(s) = y(\tau) - 1$. Then η satisfies

$$\eta'' - \alpha\eta' + (p - 1)\eta = f(s), \qquad -T < s < \infty,$$

where

$$f(s) = -e^{-m(\theta+2)s} - (1 + \eta)^p + 1 + p\eta.$$

We distinguish three cases:

$$\text{(a)}\quad p - 1 > \frac{\alpha^2}{4}, \qquad \text{(b)}\quad p - 1 = \frac{\alpha^2}{4}, \qquad \text{(c)}\quad p - 1 < \frac{\alpha^2}{4}.$$

For Cases (a) and (c), let $\mu = \sqrt{|p - 1 - \alpha^2/4|}$. Using the method of variation of the constant, we obtain for η the equation

$$\eta(s) = \frac{1}{\mu} e^{(\alpha/2)s} \int_s^\infty e^{-(\alpha/2)\sigma} \sin \mu(\sigma - s) f(\sigma)\, d\sigma. \tag{3.7}$$

For Case (b), we obtain for η the equation

$$\eta(s) = e^{(\alpha/2)s}\int_s^\infty (\sigma - s)e^{-(\alpha/2)}f(\sigma)\,d\sigma. \tag{3.8}$$

Because of the similarity of the arguments, we only discuss Case (a).

Because $\eta(s) \to 0$ as $s \to \infty$, for s large we have

$$|f(s)| \le e^{-m(\theta+2)s} + C\eta^2(s),$$

where, and hereafter, C is some general constant. Set

$$H(s) = Ce^{(\alpha/2)s}\int_s^\infty e^{-(\alpha/2)\sigma}\eta^2(\sigma)\,d\sigma.$$

By (3.7) we have

$$|\eta(s)| \le Ce^{-m(\theta+2)s} + H(s).$$

Let $\delta(s) = -\frac{\alpha}{2}s - C\int_s^\infty|\eta(\sigma)|\,d\sigma$. A direct calculation shows that

$$-\frac{d}{ds}\left(H(s)e^{\delta(s)}\right) \le C|\eta(s)|e^{-m(\theta+2)s}e^{\delta(s)}.$$

Integrating this from s to ∞, and noting that $\lim_{s\to\infty} H(s) = 0$, and $\delta' < 0$, we obtain

$$H(s) \le Ce^{-m(\theta+2)s} \qquad \text{for } s \text{ large enough.}$$

Thus, we have

$$|\eta(s)| \le Ce^{-m(\theta+2)s} \qquad \text{for } s \text{ large enough.} \tag{3.9}$$

By (3.9), it is easy to obtain

$$f(s) = e^{-m(\theta+2)s}\left[1 + O(e^{-m(\theta+2)s})\right] \qquad \text{as } s \to \infty. \tag{3.10}$$

Substituting (3.10) into (3.7), we complete the proof of the lemma.

Below we shall use a simple contraction argument to establish the existence and uniqueness of the singular solution.

LEMMA 3.2. *Suppose $p > p_N$; then there exists a unique solution (y^*, β^*) of Problem* (II_1).

Proof. We first transform (3.2) to an integral equation

$$\eta(s) = F(\eta)(s), \tag{3.11}$$

in which

$$F(\eta)(s) = -h(s) - \frac{1}{\mu}e^{(\alpha/2)s}\int_s^\infty e^{-(\alpha/2)\sigma}\sin\mu(\sigma - s)g(\sigma, \eta(\sigma))\,d\sigma$$

where

$$g(s,\eta) = (1+\eta)^p - 1 - p\eta,$$
$$h(s) = B(p,N)e^{-m(\theta+2)s}.$$

Fix $S > 0$ large enough and let X be the space of continuous functions on (S,∞) supplied with the norm

$$\|\xi\| = \sup\{e^{m(\theta+2)s}|\xi(s)| : s > S\}.$$

Set

$$\Sigma = \{\xi \in X : \|\xi + h\| \le B(p,N)\}.$$

We assert that, for S large enough, F is a contraction on Σ. In fact,

$$|g(s,\eta)| \le p(p-1)(1+\overline{\eta})^{p-2}\eta^2, \tag{3.12}$$

where $|\overline{\eta}| \le |\eta|$. For any $\eta \in \Sigma$, taking note of $|\eta| \le 2B(p,N)e^{-m(\theta+2)s}$, we have

$$|F(\eta)(s) + h(s)| \le Ce^{-2m(\theta+2)s},$$

which implies $F(\eta) \in \Sigma$.

For any $\xi_1, \xi_2 \in \Sigma$, by (3.12),

$$|F(\xi_2)(s) - F(\xi_1)(s)| \le Ce^{-2m(\theta+2)s}|\xi_2(s) - \xi_1(s)|,$$

so

$$\|F(\xi_2) - F(\xi_1)\| \le Ce^{-m(\theta+2)s}\|\xi_2 - \xi_1\|.$$

Thus, F is a contraction on Σ. Hence (3.11) has a unique solution in Σ. That is, (II_2) has a unique continuous solution $y(\tau)$ when $\tau < -S$.

Below we show that $y(\tau)$ can be continued forward and must vanish at some $T < \infty$.

From (3.3) we deduce that, as long as $y \ge 0$,

$$y'' + \alpha y' - y < 0. \tag{3.13}$$

Set

$$z = y' + qy, \tag{3.14}$$

where

$$q = \tfrac{1}{2}\left(-\alpha + \sqrt{\alpha^2 + 4}\right).$$

Then for z Eq. (3.13) implies that

$$z' < (q - \alpha)z,$$

from which the boundedness of z on bounded intervals follows. By (3.14) this means that y is also bounded on bounded intervals, so that $y(\tau)$ can indeed be continued forward as long as it remains nonnegative.

Suppose that, for all τ, $y(\tau) > 0$. Back to the original variables, we obtain a function $u \in C^2(0, \infty)$ satisfying

$$\begin{cases} u_{rr} + \dfrac{N-1}{r} u_r + \lambda + u^p = 0, & 0 < r < \infty, \\ u > 0, & 0 \le r < \infty, \\ u_r(0) = 0. \end{cases}$$

It is well known that $u_r < 0$, so there exists a constant $C \ge 0$ such that $\lim_{r \to \infty} u(r) = C$. Thus $|u_r(r)| < C$ for all $r \in (0, \infty)$. For r large enough,

$$u_{rr}(r) = -\lambda - u^p(r) - \frac{N-1}{r} u_r(r) \le -\lambda + \frac{C}{r}$$

holds, so there exists some $r_0 > 0$ such that for all $r > r_0$, $u_{rr}(r) < 0$ holds. Thus $u_r(r) < u_r(r_0) < 0$ for all $r > r_0$. This contradicts the fact that $\lim_{r \to \infty} u(r) = C$. So, there must exist some T such that

$$T = \sup\{\tau \in R : y > 0 \text{ on } (-\infty, \tau)\} < \infty$$

and $y(T) = 0$. Thus we obtain a unique solution of (II_2) and thereby of (II_1).

Proof of Theorem 2.1. From Lemma 3.2 we know that (II_1) has a unique solution (y^*, β^*). By the transformation $y(t) = A^{-1} r^{2/(p-1)} u(r)$ there exists a unique $u(r)$ satisfying

$$u'' + \frac{N-1}{r} u' + \lambda^* + u^p = 0, \qquad 0 < r < 1,$$

where $\lambda^* = \beta^*/(m^2 A^{-1})$ and $u(r) \sim A r^{(-2)/(p-1)}$ as $r \to 0$. Note that $T < \infty$; by (3.5), we obtain $\beta^* \ne 0$, so $\lambda^* > 0$. Substitute $\omega(x) = u(|x|)$ into the equation in (1.1), and let $x \to 0$. We then obtain

$$\omega(x) = A|x|^{-2/(p-1)}\{1 + o(|x|^2)\} \qquad \text{as } x \to 0,$$

where

$$A = A(p, N) = \left[\frac{2}{p-1}\left(N - 2 - \frac{2}{p-1}\right)\right]^{1/(p-1)}.$$

We complete the proof.

4. ASYMPTOTIC ANALYSIS FOR $|u|_\infty \to \infty$

Writing $|u|_\infty = u(0) = a$, $\theta = 2/(p-1)$, we shall study the behavior of (u, λ) as $a \to \infty$. Set

$$\tau(a) = \frac{\ln(a/A)}{m\theta}. \tag{4.1}$$

LEMMA 4.1. *Let* $y(t,a)$ *be a solution of* (II), *then we have*

$$y(t,a) \le e^{m\theta(t+\tau(a))} \qquad \text{for all } -\infty < t < 0.$$

Proof. Multiplying the equation for y by $e^{m(N-2-\theta)t}$, we obtain

$$\frac{d}{dt}\left(e^{m(N-2-\theta)t}(y' - m\theta y)\right) < 0.$$

Because $e^{m(N-2-\theta)t}(y' - m\theta y) \to 0$ as $t \to -\infty$, the above formula implies that $y' - m\theta y < 0$ for $-\infty < t < 0$, so $d(e^{-m\theta t}y) < 0$. Note that $ye^{-m\theta t} \to a/A$ as $t \to -\infty$, so we have $y(t,a) \le e^{m\theta(t+\tau(a))}$ for $-\infty < t < 0$. ∎

Set $s = t + \tau(a)$ and $\omega(s,a) = y(t,a)$, and then for $\omega(s,a)$ we have

$$\text{(IV)} \quad \begin{cases} \omega'' + \alpha\omega' - \omega + \omega^p + \beta e^{m(\theta+2)(s-\tau(a))} = 0, & -\infty < s < \tau, \\ \omega > 0, & -\infty < s < \tau, \\ \omega(\tau) = 0,\ \omega(s)e^{-m\theta s} \to 1 & \text{as } s \to -\infty. \end{cases}$$

According to Lemma 4.1, we have

$$\omega(s,a) \le e^{m\theta s}, \qquad -\infty < s < 0. \tag{4.2}$$

It's easy to obtain the following (see [2]).

LEMMA 4.2. *Let* $\overline{\omega}$ *be the solution of the autonomous problem associated with* (IV), *and then*

$$\omega(s,a) \to \overline{\omega}(s),\ \omega_s(a,s) \to \overline{\omega}_s(s), \qquad \text{as } a \to \infty,$$

uniformly on half-bounded intervals $(-\infty, S]$, $S \in R$.

By Lemma 4.2, for (III) we have $(y(t), z(t)) \to (1,0)$ as $a \to \infty$ along the lines $t = s - \tau(a)$, provided S is chosen large enough. Below we shall discuss (III) by studying its energy function $E(y,z)$, defined by

$$E(y,z) = \frac{1}{2}z^2 - \frac{1}{2}y^2 + \frac{1}{1+p}y^{1+p}.$$

We can prove that the orbit (y, z) remains near $(1,0)$ until t reaches some T, which is large and negative but independent of a; that is,

LEMMA 4.3. *For every $\varepsilon > 0$, there exists a time $T_\varepsilon < 0$ such that, for any $t_0 < T_\varepsilon$, holds that*

$$(y(t_0), z(t_0)) \in \Gamma_\varepsilon \equiv \{(y, z) : E(y, z) < E(1,0) + \varepsilon\}.$$

Then,

$$(y(t), z(t)) \in \Gamma_{2\varepsilon} \qquad \text{for all } t \in [t_0, T_\varepsilon].$$

Now we can give a global bound for (y, z) on $(-\infty, 0]$.

LEMMA 4.4. *There exists a constant $M > 0$, which does not depend on a, such that*

$$|(y(t), z(t))| \le M \qquad \text{for all } t \le 0. \tag{4.3}$$

Proof. From the definition of the energy function E we conclude that $E(y, z) \ge -K + y^2 + 1/2z^2$ for some constant $K > 0$. Writing $E(t) = E(y(t), z(t))$ we have

$$E' = -\alpha z^2 - \beta e^{m(\theta+2)t} z \le \frac{\beta^2}{4\alpha} e^{2m(\theta+2)t},$$

so

$$E(t) \le E(t_0) + \frac{\beta^2}{4\alpha} \int_{t_0}^{t} e^{2m(\theta+2)t}\, dt$$

holds for any $t > t_0$. Note that $t < 0$, so we have

$$E(t) \le C.$$

Therefore

$$-K + y^2 + \tfrac{1}{2} z^2 \le C.$$

This means there is a constant M, independent of a, such that

$$|(y, z)| \le M \qquad \text{for all } t \le 0.$$

∎

Proof of Theorem 2.2. Let $\{(u_n, \lambda_n)\}$ be a family of solutions of (1.1) such that $u_n(0) = a_n \to \infty$ as $n \to \infty$. Then $\{(y_n, z_n)\}$ is a family of solutions of Problem (III) with the associated eigenvalue $\beta_n = m^2 A^{-1} \lambda_n$. By Lemma 4.4, $\{(y_n, z_n)\}$ is uniformly bounded on $(-\infty, 0]$, so it has a bound in

$C^2(-\infty, 0] \times C^2(\infty, 0]$. It follows from the Arzela–Ascoli Theorem that we may extract a subsequence, again denoted by $\{(y_n, z_n)\}$, which converges in $C^2(I) \times C^2(I)$ to some function $(\bar{y}, \bar{z})$, where I may be any compact subset of $(-\infty, 0]$ and $\lambda_n \to \bar{\lambda} \in [0, \Lambda]$ as $n \to \infty$. By taking the limit in (III) along the subsequence, we find that $(\bar{y}, \bar{z}, \bar{\lambda})$ is a solution of (III) with $\beta = \bar{\beta} = m^2 A^{-1} \bar{\lambda}$, and $\bar{y}(0) = 0$. If we have

$$y(t) \to 1 \qquad \text{as } t \to -\infty, \tag{4.4}$$

then $(\bar{y}, \bar{\beta})$ is a solution of (II_1). By Lemma 3.2, (II_1) has a unique solution (y^*, β^*), so it must be $(\bar{y}, \bar{\beta}) = (y^*, \beta^*)$, hence $\bar{\lambda} = \lambda^* > 0$ and $(y_n, \lambda_n) \to (y^*, \lambda^*)$ as $n \to \infty$. Thus, we have

$$\lambda_n \to \lambda^*, \, y_n \to y^*, \, z_n \to z^* \qquad \text{as } n \to \infty,$$

uniformly on any compact subsets of $(-\infty, 0]$. Back to their original variables, hold

$$u_n \to \omega, \nabla u_n \to \nabla \omega, \qquad \text{as } n \to \infty \tag{4.5}$$

uniformly on any compact subsets which do not contain the origin, where

$$\omega(x) = \omega(|x|) = A(p, N)|x|^{-2/(p-1)} y^*\left(\frac{1}{m} \ln |x|\right). \tag{4.6}$$

To prove (4.4), we suppose to the contrary that (4.4) does not hold. There exists a sequence $\{t_k\}$, such that $t_k \to \infty$ as $k \to \infty$, and a constant $\delta > 0$ so that $(\bar{y}(t_k), \bar{z}(t_k)) \notin \Gamma_\delta$ for all $k \geq 1$. Choose $\varepsilon = \delta/4$ in Lemma 4.2, then there exist numbers S and n_0, both large enough that $(y_n(S - \tau(a_n)), z_n(S - \tau(a_n))) \in \Gamma_\varepsilon$. By Lemma 4.3, it holds that $(y_n(t), z_n(t)) \in \Gamma_{2\varepsilon} \subset \Gamma_\delta$ if $n > n_0$. Of course, we may choose n large enough that for some t_{k_0}, $t_{k_0} \in (S - \tau(a_n), T_\varepsilon)$, where T_ε only depends on ε, holds, thus forcing a contradiction to $(\bar{y}(t_k), \bar{z}(t_k)) \notin \Gamma_\delta$ for all $k \geq 1$.

Now we shall prove that $u_n \to \omega$ as $n \to \infty$ also holds in $L^{p+1}(B)$. In fact, by Lemma 4.4, we have

$$0 < u_n(r) \leq M r^{-2/(p-1)}, \, |u'(r)| \leq M r^{-(2/(p-1))-1} \qquad \text{for all } r \in (0, 1],$$

so

$$\int_{B_\varepsilon} |u_n - \omega|^{p+1} \, dx \leq C \int_0^\varepsilon r^{-2((p+1)/(p-1))+N-1} \, dr = C\varepsilon^\sigma,$$

where

$$\sigma = \frac{1}{p-1}[p(N-2) - (N+2)] > 0, \qquad B_\varepsilon = \{x \in R^N : |x| < \varepsilon\}.$$

Thus

$$\int_{B_\varepsilon} |u_n - \omega|^{p+1}\, dx \to 0 \qquad \text{as } \varepsilon \to 0.$$

By the uniform convergence of u_n on any compact subsets which do not contain the origin, we obtain $u_n \to \omega$ as $n \to \infty$ holds in $L^{p+1}(B)$.

From above, we can prove that the conclusion also holds in $H(B)$. Thus we complete the proof.

Proof of Theorem 2.3. Suppose that for $\lambda > 0$ small enough, (1.1) has a solution v_λ different from u_λ, and then there exists a $\{\lambda_n\}$, $\lambda_n \to 0$ as $n \to \infty$, such that $\|v_{\lambda_n}\|_\infty \to \infty$ as $n \to \infty$. By Theorem 2.2, we have $(\lambda_n, v_{\lambda_n}) \to (\lambda^*, \omega)$ as $n \to \infty$, but $\lambda^* > 0$, a contradiction! This completes the proof.

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