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The contragredient equivalence for several matrices: a set of invariants [1]

J. Gelonch [2]

Dept. of Matemàtica, Univ. de Lleida, 25198 Lleida, Spain

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Abstract

In this paper, the contragredient equivalence is extended to several matrices and a reduced form of the cyclically multiplicable n-tuples of matrices under this equivalence is defined and constructed. An invariant set of elementary divisors (the *spectrum of divisors*) is found. This set describes completely the reduced form of a given cyclically multiplicable n-tuple. © 1999 Elsevier Science Inc. All rights reserved.

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1. Introduction

When A_1 is an $m \times n$ complex matrix and A_2 is an $n \times m$ complex matrix, we say that the pair (A_1, A_2) is *doubly multiplicable*. Then, we can consider the matrices

$$B_1 = A_1 A_2 \quad \text{and} \quad B_2 = A_2 A_1. \tag{1}$$

Flanders [1] found that the matrices (1) are closely related: they have the same elementary divisors with nonzero root and, if $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_p (k'_1 \geqslant k'_2 \geqslant$

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[2] E-mail: jgelonch@matematica.UdL.es

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$\geqslant \cdots \geqslant k'_q$) are the Jordan block sizes associated with the eigenvalue zero in B_1 (B_2), then $|k_i - k'_i| \leqslant 1$ for all i. Furthermore, he found that these conditions are necessary and sufficient in order that two square matrices can be represented as Eq. (1) (Flanders Theorem). Johnson and Schreiner [4] presented a self-contained matrix-theoretic proof of Flanders Theorem. Horn and Merino [3] defined the *contragredient equivalence*: two pairs of doubly multiplicable matrices (A_1, A_2) and (A'_1, A'_2) are contragrediently equivalent if and only if there exist two nonsingular matrices R and S such that $A'_1 = R^{-1}A_1S$ and $A'_2 = S^{-1}A_2R$. Rubió and Gelonch [5] proved that, given any pair of doubly multiplicable matrices, we can find another pair contragrediently equivalent to the first pair such that the new pair has a reduced form; that is, both matrices in the new pair are block diagonal except, maybe, some null rows and/or null columns. There are only four possibilities for the diagonal blocks. Further, the matrices $A'_1A'_2$ and $A'_2A'_1$ are in Jordan form. The same authors constructed a complete set of invariants under contragredient equivalence. This set is formed by the elementary divisors of the polynomial matrices $\lambda I - A_1A_2$, $\lambda I - A_2A_1$, $\lambda A_1 - A_1A_2A_1$ and $\lambda A_2 - A_2A_1A_2$.

Gelonch et al. [2] generalized Flanders Theorem to cyclically multiplicable n-tuples of matrices. Our goal in this paper is to extend the contragredient equivalence to cyclically mutiplicable n-tuples of matrices and to find a canonical form and a complete set of invariants under this relation, using the results of Ref. [2] (Section 3 summarizes the main definitions and results of that paper).

2. Notations and first definitions

We denote by $M_{m,n}$ the set of $m \times n$ complex matrices and write $M_n \equiv M_{n,n}$. The $m \times n$ null matrix is represented by $O_{m,n}$; when there is no possible confusion, the subindices will be omitted. Following the notation of Ref. [3], $J_k(\alpha)$ denotes the upper triangular Jordan block of order k corresponding to a given scalar $\alpha \in \mathbb{C}$. We also define the $(k-1) \times k$ matrices $H_k = [I_{k-1} \quad O_{k-1,1}]$ and $K_k = [O_{k-1,1} \quad I_{k-1}]$. Observe that $H_kK_k^{\mathrm{T}} = J_{k-1}^{\mathrm{T}}(0)$ and $K_k^{\mathrm{T}}H_k = J_k^{\mathrm{T}}(0)$.

Given $A \in M_{m,n}$, the linear transformation of $\mathbb{C}^n$ into $\mathbb{C}^m$ defined by A, as well as its restriction to any subspace, will be represented by A itself.

The empty set is denoted by $\emptyset$. The quotient group of integers modulo n ($n \geqslant 2$) is denoted by $\mathbb{Z}_n$; the null element of this group is represented by n. For language abuse, the same symbol will be used to represent an element in $\mathbb{Z}_n$ or its main representative (between 1 and n, both inclusively) in $\mathbb{Z}^+$, if it is necessary.

Definition 1. We say that a k-tuple $(a_1, a_2, \ldots, a_k) \in (\mathbb{Z}_n)^k$, with $k \geqslant 3$, is weakly cyclic-ordered (WCO) if it satisfies one of the following conditions, where the elements a_i are considered in $\mathbb{Z}^+$:

1. $a_1 < a_2 \leqslant \cdots \leqslant a_k \leqslant a_1 + n$,
2. if $a_1 \geqslant a_2$, then $a_1 < a_2 + n \leqslant \cdots \leqslant a_k + n \leqslant a_1 + n$, or
3. if there exists $1 < j < k$ such that $a_j > a_{j+1}$, then $a_1 < a_2 \leqslant \cdots \leqslant a_j \leqslant a_{j+1} + n \leqslant \cdots \leqslant a_k + n \leqslant a_1 + n$.

Definition 2. We say that a k-tuple $(a_1, a_2, \ldots, a_k) \in (\mathbb{Z}_n)^k$, with $k \geqslant 3$, is strongly cyclic-ordered (SCO) if it satisfies one of the following conditions, where the elements a_i are considered in $\mathbb{Z}^+$:
1. $a_1 < a_2 < \cdots < a_k \leqslant a_1 + n$, or
2. if there exists $1 \leqslant j < k$ such that $a_j \geqslant a_{j+1}$, then $a_1 < a_2 < \cdots < a_j < a_{j+1} + n < \cdots < a_k + n \leqslant a_1 + n$.

Note that, within the above definition, $a_j \neq a_{j+1}$ for all j. In fact, if $a_j = a_{j+1}$, then we have $a_1 < a_j < a_j + n \leqslant a_1 + n$, which is not possible.

Lemma 3. *With the previous definitions, we have:*
1. *If $(a_1, a_2, \ldots, a_k)$ is SCO, then it is also WCO.*
2. *(i, j, i) is WCO for all $j \in \mathbb{Z}_n$; (i, j, i) is SCO for all $j \in \mathbb{Z}_n - \{i\}$.*
3. *If $a_1 \neq a_3$, (a_1, a_2, a_3) is WCO if and only if $(a_1, a_2, a_3 + 1)$ is SCO.*
4. *$(a_1, a_2, a_3, \ldots, a_k)$ is WCO if and only if $(a_2 - 1, a_3, \ldots, a_k, a_1)$ is WCO.*
5. *$(a_1, a_2, \ldots, a_k)$ is SCO if and only if $(a_2, \ldots, a_k, a_1 + 1)$ is SCO.*
6. *For $r_1 \neq r_2$, (i, j, i_2) is SCO for all $j \in \mathbb{Z}_n$ such that $(r_1 - 1, j, r_2)$ is SCO, if and only if (i_1, r_1, r_2, i_2) is WCO.*

For $a, b \in \mathbb{Z}$, $1 \leqslant a \leqslant n$ and $1 \leqslant b \leqslant n$, we denote by $\ell(a, b)$ the value

$$\ell(a,b) = \begin{cases} n + b - a & \text{if } b < a, \\ b - a & \text{if } b \geqslant a. \end{cases}$$

Definition 4. Given a square matrix, the sizes of Jordan blocks associated with the eigenvalue zero are called nilpotence indices of the matrix. The non-increasing sequence of them, made infinite by addition of zeros, is called nilpotence sequence.

Definition 5. An n-tuple of matrices $(A_1, A_2, \ldots, A_n)$ is said to be cyclically multiplicable if the products $A_i A_{i+1}$ make sense for all $i \in \mathbb{Z}_n$.

Definition 6. An n-tuple of square matrices $(B_1, B_2, \ldots, B_n)$, maybe with distinct sizes, is said to be associable if there exists a cyclically multiplicable n-tuple $(A_1, A_2, \ldots, A_n)$ such that $B_i = A_i A_{i+1} \cdots A_{i-1}$ for all $i \in \mathbb{Z}_n$.

Two square matrices with respective nilpotence sequences $\{k_i\}$ and $\{k'_i\}$ satisfy *Flanders condition* if both matrices have the same elementary divisors with nonzero root and $|k_i - k'_i| \leqslant 1$ for all i. Then, we say that an n-tuple of square matrices $(B_1, B_2, \ldots, B_n)$ satisfies *Flanders condition pairwise* if the matrices B_i and B_j satisfy Flanders condition for all i, j.

3. Flanders theorem for more than two matrices

If $(B_1, B_2, \ldots, B_n)$ is associable, then it satisfies Flanders condition pairwise. That is, Flanders condition pairwise is necessary so that an n-tuple of square matrices is associable. Moreover, we have the following result (Theorem 3.1 of Ref. [2]):

Theorem 7. *Let B_1, B_2 and B_3 be square matrices, maybe with distinct sizes. The following statements are equivalent:*
(a) (B_1, B_2, B_3) *is associable.*
(b) (B_1, B_2, B_3) *satisfies Flanders condition pairwise.*

Unfortunately, Flanders condition pairwise is not sufficient for $n \geqslant 4$. Its generalization is given by the next definition (Definition 4.1 of Ref. [2]).

Definition 8. An n-tuple of square matrices $(B_1, B_2, \ldots, B_n)$ is said to be F-related if it satisfies Flanders condition pairwise and there exists an ordering of the nilpotence indices of the matrices B_i, $k_{i,1}, k_{i,2}, \ldots, k_{i,p_i}$, for $i = 1, 2, \ldots, n$ such that either
(a) $k_{i,j} = k_{i+1,j}$ for all $i \in \mathbb{Z}_n$, or
(b) there exist $i_1, i_2 \in \mathbb{Z}_n$ such that $k_{i_1,j} = k_{i_1+1,j} - 1 = \cdots = k_{i_2,j} - 1 = k_{i_2+1,j} = \cdots = k_{i_1,j}$, for all $j = 1, 2, \ldots, p$ with $p = \min_{i=1,2,\ldots,n}\{p_i\}$.

Theorems 4.1 and 4.2 of Ref. [2] prove that the F-related condition is a necessary and sufficient condition in order that an n-tuple of square matrices is associable. Moreover, the proof of both theorems is constructive.

Note that the F-related condition becomes Flanders condition pairwise when $n = 2$ or $n = 3$. So, the F-related condition is a good generalization of Flanders condition pairwise, in the sense that it extends Flanders theorem to cyclically multiplicable n-tuples of matrices.

4. Contragredient equivalence for n-tuples. The reduced form

The definition of contragredient equivalence, given in Ref. [3], is easily generalized for cyclically multiplicable n-tuples of matrices.

Definition 9. Let $(A_1, A_2, \ldots, A_n)$ and $(A'_1, A'_2, \ldots, A'_n)$ be two cyclically multiplicable n-tuples of matrices. We say that they are contragrediently equivalent if there exist nonsingular matrices $T_1, T_2, \ldots, T_n$ such that $A'_i = T_i^{-1} A_i T_{i+1}$ for all $i \in Z_n$. Then, we write $(A_1, A_2, \ldots, A_n) \sim (A'_1, A'_2, \ldots, A'_n)$.

Clearly, we have the next result.

Lemma 10. *If* $(A_1, A_2, \ldots, A_n) \sim (A'_1, A'_2, \ldots, A'_n)$, *then the matrices* $B_i = A_i A_{i+1} \ldots A_{i-1}$ *and* $B'_i = A'_i A'_{i+1} \cdots A'_{i-1}$ *are similar for all* $i \in \mathbb{Z}_n$.

Given a cyclically multiplicable n-tuple of matrices, $(A_1, A_2, \ldots, A_n)$, with $A_i \in M_{m(i),m(i+1)}$ for $i \in \mathbb{Z}_n$, the matrices $B_i = A_i A_{i+1} \ldots A_{i-1}$ are square matrices of order $m(i)$, for all $i \in \mathbb{Z}_n$. We can consider each one of them as a linear transformation of $\mathbb{C}^{m(i)}$ into itself. Let $k_{i,1}, k_{i,2}, \ldots, k_{i,p_i}$ be the nilpotence indices for each matrix B_i, ordered according to Definition 8. Let $p = \min\{p_i\}$. Through the proof of Theorem 4.2 of Ref. [2], we construct a decomposition of each space $\mathbb{C}^{m(i)}$ in direct sum according to the Jordan structure of B_i,

$$\mathbb{C}^{m(i)} = \left(\bigoplus_{j=1}^{q} V_j^{(i)}\right) \oplus \left(\bigoplus_{j=1}^{p_i} E_j^{(i)}\right),$$

where subspaces $V_j^{(i)}$ correspond to the elementary divisors with nonnull root and subspaces $E_j^{(i)}$ correspond to the elementary divisors with null root, satisfying

1. $A_{i-1}(V_j^{(i)}) = V_j^{(i-1)}$ for $j = 1, 2, \ldots, q$,
2. $A_{i-1}(E_j^{(i)}) \subset E_j^{(i-1)}$ for $j = 1, 2, \ldots, p$,
3. $\dim E_j^{(i)} = k_{i,j}$ for $j = 1, 2, \ldots, p$ and $\dim E_j^{(i)} = 1$, for $j = p+1, \ldots, p_i$.

Definition 11. The chain of subspaces $(V_j^{(1)}, V_j^{(2)}, \ldots, V_j^{(n)})$, for $j = 1, \ldots, q$, is said to be a nonsingular chain. Similarly, for $j = 1, 2, \ldots, p$, the chain of subspaces $(E_j^{(1)}, E_j^{(2)}, \ldots, E_j^{(n)})$ is said to be a nilpotent chain.

The construction made in Ref. [2] allows us to affirm that, if we take the ordered union of a determined basis of the subspaces $V_j^{(i)}$ and $E_j^{(i)}$ as basis of each $\mathbb{C}^{m(i)}$, the matrix of each linear transformation $A_i : \mathbb{C}^{m(i+1)} \to \mathbb{C}^{m(i)}$ is a block diagonal matrix except, maybe, some null rows and/or null columns. That is, there exist nonsingular matrices T_i such that the matrices $A'_i = T_i^{-1} A_i T_{i+1}$, for $i = 1, 2, \ldots, n$, are block diagonal matrices except, maybe, some null rows and/or null columns. Moreover, the possible diagonal blocks are only $J_k^{\mathrm{T}}(\alpha)$, I_k, H_k or K_k^{T}. It is clear that the cyclically multiplicable n-tuples of matrices $(A_1, A_2, \ldots, A_n)$ and $(A'_1, A'_2, \ldots, A'_n)$ are contragrediently equivalent.

Definition 12. An n-tuple constructed according to the above process is a reduced form of a given cyclically multiplicable n-tuple of matrices.

Note that if two cyclically multiplicable n-tuples of matrices admit a same reduced form, then they are contragrediently equivalent. Given a cyclically multiplicable n-tuple of matrices and a nilpotent chain of it, $(E_j^{(1)}, E_j^{(2)}, \dots, E_j^{(n)})$, we consider the linear transformations $A_i : E_j^{(i+1)} \to E_j^{(i)}$. From the construction made in Ref. [2], we know that one and only one of these linear transformations is not left invertible. Moreover, the dimensions of the subspaces of each nilpotent chain satisfy the condition of Definition 8, that is, either they are equal or they have an increase by one and a decrease by one. We assign a *type* to each nilpotent chain, according to these facts.

Definition 13. Let $(E_j^{(1)}, E_j^{(2)}, \dots, E_j^{(n)})$ be a nilpotent chain and let $k_{i,j} = \dim E_j^{(i)}$ for $i = 1, 2, \dots, n$. Then, its type is

- $T(i)$ (or, equivalently, $T(i,i)$) if $k_{1,j} = k_{2,j} = \cdots = k_{n,j}$ and the linear transformation $A_i : E_j^{(i+1)} \to E_j^{(i)}$ is not left invertible;
- $T(i_1, i_2)$ *if* $k_{i_1,j} = k_{i_1+1,j} - 1 = \cdots = k_{i_2,j} - 1 = k_{i_2+1,j} = \cdots = k_{i_1,j}$.

Definition 14. The order of a nonsingular or nilpotent chain is the greatest dimension of the subspaces which form the chain.

If $(V_j^{(1)}, V_j^{(2)}, \dots, V_j^{(n)})$ is a nonsingular chain of order k, we can choose a basis of each subspace $V_j^{(i)}$ so that the matrix of the linear transformation $A_1 : V_j^{(2)} \to V_j^{(1)}$ is $J_k^{\mathrm{T}}(\alpha)$ and the remaining linear transformations have I_k as matrix. Similarly, given a nilpotent chain of type $T(i)$ and order k, we can find a basis of each one of its subspaces so that the matrix of the linear transformation A_i is $J_k^{\mathrm{T}}(0)$ and the matrix of A_j is I_k, for all $j \in \mathbb{Z}_n, j \neq i$.

That is,

$$A_j \equiv \begin{cases} J_k^{\mathrm{T}}(0) & \text{if } j = i, \\ I_k & \text{if } j \neq i, \end{cases} \quad \text{and} \quad B_j \equiv J_k^{\mathrm{T}}(0) \quad \text{for all } j \in \mathbb{Z}_n. \tag{2}$$

If the type of the nilpotent chain is $T(i_1, i_2)$ with $i_1 \neq i_2$, we can obtain a basis of its subspaces such that

$$A_j \equiv \begin{cases} H_k & \text{if } j = i_1, \\ K_k^{\mathrm{T}} & \text{if } j = i_2, \\ I_k & \text{if } (i_1, j, i_2) \text{ is SCO}, \\ I_{k-1} & \text{if } (i_2, j, i_1) \text{ is SCO}, \end{cases} \tag{3}$$

$$B_j \equiv \begin{cases} J_k^{\mathrm{T}}(0) & \text{if } (i_1, j, i_2 + 1) \text{ is SCO}, \\ J_{k-1}^{\mathrm{T}}(0) & \text{if } (i_2, j, i_1 + 1) \text{ is SCO}. \end{cases} \tag{4}$$

Definition 15. Given a nilpotent chain of order k, the number of the linear transformations A_i, with matrix I_k over the chain is said to be the length of the nilpotent chain.

It is easy to see that the length of a nilpotent chain is the value $\ell(i_1+1, i_2)$ when the type of the nilpotent chain is $T(i_1, i_2)$.

5. Spectrum of divisors of a cyclically multiplicable n-tuple of matrices

Given a cyclically multiplicable n-tuple of matrices, $(A_1, A_2, \ldots, A_n)$, we can construct the matrices

$$A(r_1, r_2) = \prod_{(r_1-1, j, r_2) \text{ is SCO}} A_j \tag{5}$$

for all $r_1, r_2 \in \mathbb{Z}_n$. If $r_2 = r_1$, there is not j such that $(r_1 - 1, j, r_2)$ is SCO. Then, $A(r_1, r_1)$ is equal to the identity matrix.

Definition 16. Let $(A_1, A_2, \ldots, A_n)$ be a cyclically multiplicable n-tuple of matrices. The characteristic matrices of the given n-tuple are the polynomial matrices defined by $P_{r_1,r_2}(\lambda) = (\lambda I - B_{r_1})A(r_1, r_2)$, for all $r_1, r_2 \in \mathbb{Z}_n$.

Definition 17. The number of matrices which form the product $A(r_1, r_2)$ is said to be the length of the characteristic matrix $P_{r_1,r_2}(\lambda)$.

The length of the characteristic matrix $P_{r_1,r_2}(\lambda)$ coincides with the value $\ell(r_1, r_2)$.

Let $S(P_{r_1}, r_2)$ be the set of the elementary divisors of the characteristic matrix $P_{r_1,r_2}(\lambda)$, ordered according to the following criterium, over its root: (a) greatest modulus, (b) greatest real part, (c) greatest imaginary part and (d) greatest degree of the elementary divisor.

Definition 18. Let $(A_1, A_2, \ldots, A_n)$ be a cyclically multiplicable n-tuple of matrices. The set

$$\Lambda(A_1, A_2, \ldots, A_n) = \{S(P_{r_1,r_2}) \text{ such that } r_1, r_2 \in \mathbb{Z}_n\},$$

where the sets $S(P_{r_1,r_2})$ are non-decreasingly ordered according to the length of the corresponding characteristic matrix and, if the length is equal, according to r_1, is said to be the spectrum of divisors of the given cyclically multiplicable n-tuple of matrices.

Proposition 19. *If $(A_1, A_2, \ldots, A_n) \sim (A'_1, A'_2, \ldots, A'_n)$, then both n-tuples have the same spectrum of divisors.*

Proof. We know that there exist nonsingular matrices $T_1, T_2, \ldots, T_n$ such that $A_i' = T_i^{-1}A_iT_{i+1}$ for all $i \in \mathbb{Z}_n$. Let $P_{r_1,r_2}(\lambda)$ be the characteristic matrices of the first n-tuple and let $P'_{r_1,r_2}(\lambda)$ be those of the second n-tuple. The equality $P'_{r_1,r_2}(\lambda) = T_{r_1}^{-1}P_{r_1,r_2}(\lambda)T_{r_2}$ ensures that the matrices $P'_{r_1,r_2}(\lambda)$ and $P_{r_1,r_2}(\lambda)$ are equivalent for all $r_1, r_2 \in \mathbb{Z}_n$ and, then, they have the same elementary divisors. □

Proposition 19 allows us to consider any cyclically multiplicable n-tuple of matrices in reduced form, without loss of generality, when we work with its spectrum of divisors.

Proposition 20. *Given a cyclically multiplicable n-tuple of matrices, all of its characteristic matrices have the same elementary divisors with nonnull root.*

Proof. Let $(A_1, A_2, \ldots, A_n)$ be a cyclically multiplicable n-tuple of matrices. Consider the n-tuple in reduced form. Since the matrices $B_i = A_iA_{i+1}\cdots A_{i-1}$ satisfy Flanders condition pairwise, for all $i \in \mathbb{Z}_n$, we know that they have the same elementary divisors with nonnull root. Let $(\lambda - \alpha)^k$ be one of these elementary divisor with nonnull root. Then, the matrix A_1 has the diagonal block $J_k^{\mathrm{T}}(\alpha)$ and the matrices A_j, for $j = 2, 3, \ldots, n$, have the diagonal block I_k. Moreover, the matrices B_j have the diagonal block $J_k^{\mathrm{T}}(\alpha)$. Therefore, the characteristic matrices, $P_{r_1,r_2}(\lambda)$, have the diagonal block $[\lambda I_k - J_k^{\mathrm{T}}(\alpha)]J_k^{\mathrm{T}}(\alpha)$, if $(r_1 - 1, 1, r_2)$ is SCO, or $\lambda I_k - J_k^{\mathrm{T}}(\alpha)$, otherwise. Since $J_k^{\mathrm{T}}(\alpha)$ is nonsingular, the elementary divisor which corresponds to both blocks is $(\lambda - \alpha)^k$. □

The rest of this paper is devoted to prove that two cyclically multiplicable n-tuples of matrices are contragrediently equivalent if and only if their spectra of divisors are equal. In what follows, the cyclically multiplicable n-tuples of matrices are considered in reduced form. Then, each nilpotent chain provides a diagonal block to each one of the characteristic matrices, $P_{r_1,r_2}(\lambda)$. Over a nilpotent chain of order k, and using (2)–(4), we know that the matrix $\lambda I - B_{r_1}$ is equal to $\lambda I - J_k^{\mathrm{T}}(0)$ or $\lambda I - J_{k-1}^{\mathrm{T}}(0)$ and the matrix $A(r_1, r_2)$, given by Eq. (5), is equal to $I_k, K_k^{\mathrm{T}}, H_k, J_k^{\mathrm{T}}(0), J_{k-1}^{\mathrm{T}}(0)$ or I_{k-1}. Hence, we have the following result.

Proposition 21. *A nilpotent chain of order k provides to each characteristic matrix one of the following diagonal blocks:*

$$\text{(a) } \lambda I_k - J_k^{\mathrm{T}}(0), \quad \text{(b) } \lambda H_k - J_{k-1}^{\mathrm{T}}(0)H_k, \quad \text{(c) } \lambda K_k^{\mathrm{T}} - J_k^{\mathrm{T}}(0)K_k^{\mathrm{T}},$$

$$\text{(d) } \lambda I_{k-1} - J_{k-1}^{\mathrm{T}}(0), \quad \text{(e) } \lambda J_{k-1}^{\mathrm{T}}(0) - [J_{k-1}^{\mathrm{T}}(0)]^2, \quad \text{(f) } \lambda J_k^{\mathrm{T}}(0) - [J_k^{\mathrm{T}}(0)]^2.$$

Notice that each one of those diagonal blocks has, as elementary divisor, λ^k (case (a)), λ^{k-1} (cases (b), (c), (d) and (f)) or λ^{k-2} (case (e)).

Given two elements of $\mathbb{Z}_n$, r_1, r_2, we define the sets

$$\Delta_1(r_1, r_2) = \{(i, j) \in (\mathbb{Z}_n)^2 \text{ such that } (r_2 - 1, j, i, r_1 - 1) \text{is WCO}\},$$

$$\Delta_2(r_1, r_2) = \{(i, j) \in (\mathbb{Z}_n)^2 \text{ such that } (r_1 - 1, i, j, r_2) \text{is SCO}\}.$$

It is easy to see the next lemma.

Lemma 22. *For all* $r_1, r_2 \in \mathbb{Z}_n$, $\Delta_1(r_1, r_2) \cap \Delta_2(r_1, r_2) = \emptyset$.

We also define the set

$$\Delta_3(r_1, r_2) = (\mathbb{Z}_n)^2 - (\Delta_1(r_1, r_2) \cup \Delta_2(r_1, r_2)).$$

Proposition 23. *For all* $i_1, i_2 \in \mathbb{Z}_n$, *a nilpotent chain of type* $T(i_1, i_2)$ *and order* k *provides to the characteristic matrix* $P_{r_1, r_2}(\lambda)$ *the elementary divisor*

- λ^k *if and only if* $(i_1, i_2) \in \Delta_1(r_1, r_2)$,
- λ^{k-2} *if and only if* $(i_1, i_2) \in \Delta_2(r_1, r_2)$,
- λ^{k-1} *if and only if* $(i_1, i_2) \in \Delta_3(r_1, r_2)$.

Proof. According to Eqs. (2) and (3), the elementary divisor of the matrix $P_{r_1, r_2}(\lambda)$ is λ^k if and only if $B_{r_1} = J_k^{\mathrm{T}}(0)$ and $A_j = I_k$ for all j such that $(r_1 - 1, j, r_2)$ is SCO. We know that $B_{r_1} = J_k^{\mathrm{T}}(0)$ for all r_1, if $i_1 = i_2$, and for all r_1 such that $(i_1, r_1, i_2 + 1)$ is SCO, if $i_1 \neq i_2$; these two conditions can be summarized as (i_1, r_1, i_2) being WCO (recall parts 2 and 3 of Lemma 3). Moreover, $A_j = I_k$ if and only if (i_1, j, i_2) is SCO. Thus, the elementary divisor of P_{r_1, r_2} is λ^k if and only if (i_1, r_1, i_2) is WCO and, when $r_1 \neq r_2, (i_1, j, i_2)$ is SCO for all j such that $(r_1 - 1, j, r_2)$ is SCO. Part 6 of Lemma 3 shows that this last condition is equivalent to (i_1, r_1, r_2, i_2) being WCO. Using twice part 4 of Lemma 3, $(r_2 - 1, j, i, r_1 - 1)$ is WCO, that is, $(i_1, i_2) \in \Delta_1(r_1, r_2)$.

The elementary divisor λ^{k-2} appears only when the corresponding diagonal block is $\lambda J_{k-1}^{\mathrm{T}}(0) - [J_{k-1}^{\mathrm{T}}(0)]^2$ (case (c) of Proposition 21). This block is obtained if and only if $B_{r_1} = J_{k-1}^{\mathrm{T}}(0)$ and $A(r_1, r_2) = J_{k-1}^{\mathrm{T}}(0)$. So that these conditions hold, we need that $i_1 \neq i_2, r_2 \neq r_1$ and $r_2 \neq r_1 + 1$. We know that $B_{r_1} = J_{k-1}^{\mathrm{T}}(0)$ if and only if $(i_2, r_1, i_1 + 1)$ is SCO; using part 5 of Lemma 3, this condition is equivalent to require that $(r_1 - 1, i_1, i_2)$ is SCO. Without loss of generality, we can suppose $r_1 = 1$. Then, $A(r_1, r_2) = A_1, A_2 \ldots A_{r_2 - 1}$ and, according to Eq. (3), this matrix is equal to $J_{k-1}^{\mathrm{T}}(0)$ if and only if $1 \leqslant i_1 \leqslant i_2 \leqslant r_2 - 1$, that is, if and only if $0 < i_1 < i_2 < r_2$. These inequalities are equivalent to $(r_1 - 1, i_1, i_2, r_2)$ being SCO, that is, $(i_1, i_2) \in \Delta_2(r_1, r_2)$. □

For all $i_1, i_2 \in \mathbb{Z}_n$, we denote by $M_k(i_1, i_2)$ the number of elementary divisors λ^k of the matrix $P_{i_1, i_2}(\lambda)$ and by $N_k(i_1, i_2)$ the number of nilpotent chains of order k and type $T(i_1, i_2)$. It is clear that if we know the values $N_k(i_1, i_2)$ for all i_1, i_2 and k, we can construct a reduced form of the given n-tuple $(A_1, A_2, \ldots, A_n)$.

According to Proposition 23, we have the next relation:

$$M_k(r_1, r_2) = \sum_{(i_1,i_2)\in\Delta_3(r_1,r_2)} N_{k+2}(i_1, i_2) + \sum_{(i_1,i_2)\in\Delta_2(r_1,r_2)} N_{k+1}(i_1, i_2) + \sum_{(i_1,i_2)\in\Delta_1(r_1,r_2)} N_k(i_1, i_2). \tag{6}$$

Our aim is to prove that the knowledge of the values $M_k(r_1, r_2)$ for all k, r_1 and r_2 allows us to compute the values $N_k(i_1, i_2)$ for all k, i_1, and i_2. We need the next result.

Lemma 24. *Given a cyclically multiplicable n-tuple of matrices, let C be one of its nilpotent chains, of order k and type $T(i_1, i_2)$, and let $P_{r_1,r_2}(\lambda)$ be one of its characteristic matrices. Also, let L_1 be the length of C and let L_2 be the length of $P_{r_1,r_2}(\lambda)$. Then*

(a) *If $L_1 < L_2$, the elementary divisor of $P_{r_1,r_2}(\lambda)$ provided by C is not λ^k.*

(b) *If $L_1 = L_2$, C provides λ^k as elementary divisor of $P_{r_1,r_2}(\lambda)$ if and only if the type of C is $T(r_1 - 1, r_2)$.*

Proof. (a) According to Proposition 23, the elementary divisor is λ^k if and only if (i_1, r_1, r_2, i_2) is WCO. We can suppose, without loss of generality, $i_1 = n$; that is, we consider that (n, r_1, r_2, i_2) is WCO. Then, $n < r_1 + n \leqslant r_2 + n \leqslant i_2 + n \leqslant 2n$, which implies that

$$1 \leqslant r_1 \leqslant r_2 \leqslant i_2 \tag{7}$$

and $L_1 = \ell(n+1, i_2) = n + i_2 - (n+1) = i_2 - 1 \geqslant r_2 - r_1 = \ell(r_2, r_1) = L_2$.

Thus, if the elementary divisor provided by the nilpoint chain to the characteristic matrix is λ^k, then $L_1 \geqslant L_2$. Hence, $L_1 < L_2$ implies that the elementary divisor is not λ^k.

(b) In order that λ^k be the elementary divisor when $L_1 = L_2$, the equalities $1 = r_1$ and $r_2 = i_2$ in Eq. (7) are necessary. That is, $i_1 = r_1 - 1$ and $i_2 = r_2$. □

Now, note that if k_m is the greatest exponent of the elementary divisors with null root in the spectrum of divisors, then relation (6) becomes

$$M_{k_m}(r_1, r_2) = \sum_{(i_1,i_2)\in\Delta_1(r_1,r_2)} N_{k_m}(i_1, i_2). \tag{8}$$

Moreover, since $\Delta_1(r+1, r) = \{(r, r)\}$, we have

$$N_{k_m}(r,r) = M_{k_m}(r+1,r)$$

for all r. That is, we easily know the number of nilpotent chains of greatest order (k_m) and greatest length ($n-1$). Using Eq. (8) and Lemma 24, we can obtain

$$M_{k_m}(i_1+1,i_2) = N_{k_m}(i_1,i_2) + R, \tag{9}$$

where R represents the rest of the values $N_{k_m}(i,j)$ in Eq. (8), all of them corresponding to nilpotent chains of length greater than $\ell(i_1+1,i_2)$ (the length of the nilpotent chains of type $T(i_1,i_2)$). Since these values are known for the greatest length, we can compute, from Eq. (9) with $i_1 = i_2 + 1$, the values $N_{k_m}(i_2+1,i_2)$ (they correspond to the nilpotent chains of greatest order and length $n-2$), and so on.

Then, relation (6), with $k = k_m - 1$, can be written as

$$\sum_{(i_1,i_2)\in\Delta_1(r_1,r_2)} N_{k_m-1}(i_1,i_2) = M_{k_m-1}(r_1,r_2) - \sum_{(i_1,i_2)\in\Delta_3(r_1,r_2)} N_{k_m}(i_1,i_2),$$

and allows to find the values $N_{k_m-1}(i_1,i_2)$, starting again with the values corresponding to the greatest length, $N_{k_m-1}(r,r)$, because the values $N_{k_m}(i_1,i_2)$ are already known for all i_1 and i_2.

This process can be continued to compute all the values $N_k(i_1,i_2)$. With them, we can construct the reduced forms of the given cyclically multiplicable n-tuple of matrices.

Moreover, we have proved the next results (the main results of this paper).

Theorem 25. *The reduced form of a cyclically multiplicable n-tuple of matrices is uniquely determined up to the order of the chains.*

Theorem 26. *Two cyclically multiplicable n-tuples of matrices are contragrediently equivalent if and only if their spectra of divisors are equal.*

Finally, we present an example of the construction of the reduced form for a cyclically multiplicable n-tuple of matrices.

Example 27. Consider the matrices

$$A_1 = \begin{bmatrix} -6 & -2 & 2 & 4 & -2 & -5 & -3 \\ -3 & 0 & 2 & 3 & 0 & -2 & -2 \\ 0 & 1 & 1 & 1 & 2 & 0 & -1 \\ -4 & 0 & 2 & 3 & 0 & -3 & -3 \\ 3 & 2 & 0 & -2 & 1 & 2 & 1 \\ 2 & 1 & 0 & -1 & 0 & 1 & 1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} -3 & -2 & 2 & 2 & -2 & 1 \\ 3 & 3 & -3 & 0 & 1 & 2 \\ -1 & -3 & 4 & -2 & 1 & -3 \\ 0 & 1 & -1 & 2 & -1 & 3 \\ 1 & 1 & -1 & -1 & 1 & -1 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 3 & 3 & -3 & -2 & 2 & -1 \end{bmatrix},$$

$$A_3 = \begin{bmatrix} -1 & -1 & 2 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ -2 & 0 & 1 & 2 \\ -1 & 0 & -1 & 0 \\ 1 & 0 & -1 & -1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 0 & -1 & 0 & 1 & 1 & 0 \\ -1 & 1 & -2 & 1 & 0 & 0 \\ -1 & 0 & -2 & 2 & 0 & 1 \\ 1 & 0 & 1 & -1 & 1 & 0 \end{bmatrix}.$$

The n-tuple (A_1, A_2, A_3, A_4) is cyclically multiplicable. We compute the matrices

$$B_1 = A_1A_2A_3A_4, \quad B_2 = A_2A_3A_4A_1, \quad B_3 = A_3A_4A_1A_2, \quad B_4 = A_4A_1A_2A_3.$$

In Table 1, we have the list of the characteristic matrices, with its elementary divisors and the values of $M_k(r_1, r_2)$ for $k = 1, 2$ (the greatest exponent of the elementary divisors with null root is 2).

Using the equalities (6), beginning with $k = 2$ and $i_1 = i_2 + 1$, we can compute the values $N_k(i_1, i_2)$; the only nonzero values are $N_2(2, 2) = 1, N_2(4, 3) = 2$ and $N_1(1, 2) = 1$. Then, there are four nilpotent chains: one of order 2 and type $T(2, 2)$, two of order 2 and type $T(4, 3)$ and one of order 1 and type $T(1, 2)$. The

Table 1

Characteristic matrix	Elem. div.	$M_2(r_1, r_2)$	$M_1(r_1, r_2)$
$P_{1,1}(\lambda) = \lambda I - B_1$	$\{\lambda^2, \lambda^2, \lambda^2\}$	3	0
$P_{2,2}(\lambda) = \lambda I - B_2$	$\{\lambda^2, \lambda^2, \lambda^2, \lambda\}$	3	1
$P_{3,3}(\lambda) = \lambda I - B_3$	$\{\lambda^2, \lambda^2, \lambda^2\}$	3	0
$P_{4,4}(\lambda) = \lambda I - B_4$	$\{\lambda^2, \lambda, \lambda\}$	1	2
$P_{1,2}(\lambda) = \lambda A_1 - B_1A_1$	$\{\lambda^2, \lambda^2, \lambda^2\}$	3	0
$P_{2,3}(\lambda) = \lambda A_2 - B_2A_2$	$\{\lambda^2, \lambda^2, \lambda\}$	2	1
$P_{3,4}(\lambda) = \lambda A_3 - B_3A_3$	$\{\lambda^2, \lambda, \lambda\}$	1	2
$P_{4,1}(\lambda) = \lambda A_4 - B_4A_4$	$\{\lambda^2, \lambda, \lambda\}$	1	2
$P_{1,3}(\lambda) = \lambda A_1A_2 - B_1A_1A_2$	$\{\lambda^2, \lambda^2, \lambda\}$	2	1
$P_{2,4}(\lambda) = \lambda A_2A_3 - B_2A_2A_3$	$\{\lambda, \lambda, \lambda\}$	0	3
$P_{3,1}(\lambda) = \lambda A_3A_4 - B_3A_3A_4$	$\{\lambda^2, \lambda, \lambda\}$	1	2
$P_{4,2}(\lambda) = \lambda A_4A_1 - B_4A_4A_1$	$\{\lambda^2, \lambda, \lambda\}$	1	2
$P_{1,4}(\lambda) = \lambda A_1A_2A_3 - B_1A_1A_2A_3$	$\{\lambda, \lambda, \lambda\}$	0	3
$P_{2,1}(\lambda) = \lambda A_2A_3A_4 - B_2A_2A_3A_4$	$\{\lambda, \lambda, \lambda\}$	0	3
$P_{3,2}(\lambda) = \lambda A_3A_4A_1 - B_3A_3A_4A_1$	$\{\lambda^2, \lambda, \lambda\}$	1	2
$P_{4,3}(\lambda) = \lambda A_4A_1A_2 - B_4A_4A_1A_2$	$\{\lambda, \lambda, \lambda\}$	0	3

blocks corresponding to these nilpotent chains, for each one of the matrices A_1, A_2, A_3 and A_4, are $\{I_2, J_2^{\mathrm{T}}(0), I_2, I_2\}$, $\{I_2, I_2, K_2^{\mathrm{T}}, H_2\}$, $\{I_2, I_2, K_2^{\mathrm{T}}, H_2\}$ and $\{H_1, K_1^{\mathrm{T}}, \emptyset, \emptyset\}$, respectively. The "matrix" H_1 adds a null column, without adding rows, and K_1^{T} adds a null row, without adding columns. Hence, there exist nonsingular matrices, T_1, T_2, T_3 and T_4 such that (in Ref. [2], we can find a process to construct these nonsingular matrices)

$$A_1' = T_1^{-1} A_1 T_2 = \begin{bmatrix} 1 & 0 & & & & & \\ 0 & 1 & & & & & \\ & & 1 & 0 & & & \\ & & 0 & 1 & & & \\ & & & & 1 & 0 & \\ & & & & 0 & 1 & 0 \end{bmatrix},$$

$$A_2' = T_2^{-1} A_2 T_3 = \begin{bmatrix} 0 & 0 & & & & \\ 1 & 0 & & & & \\ & & 1 & 0 & & \\ & & 0 & 1 & & \\ & & & & 1 & 0 \\ & & & & 0 & 1 \\ & & & & & 0 \end{bmatrix},$$

$$A_3' = T_3^{-1} A_3 T_4 = \begin{bmatrix} 1 & 0 & & \\ 0 & 1 & & \\ & & 0 & \\ & & 1 & \\ & & & 0 \\ & & & 1 \end{bmatrix},$$

$$A_4' = T_4^{-1} A_4 T_1 = \begin{bmatrix} 1 & 0 & & & & \\ 0 & 1 & & & & \\ & & 1 & 0 & & \\ & & & & 1 & 0 \end{bmatrix}.$$

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