Extending Parikh matrices

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Received 21 March 2003; received in revised form 2 July 2003; accepted 11 July 2003
Communicated by A. Salomaa

Abstract

We introduce the notion of Parikh matrix induced by a word, a natural extension to the notion of Parikh matrix and prove a set of properties for this kind of matrices.

We also study the relation between these two notions. We show that combining properties from both we obtain a more powerful tool for proving algebraic properties of words.

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Keywords: Parikh matrices; Word; Subword; Scattered subword; Number of subwords

1. Preliminaries

1.1. Subwords

Let \( \Sigma \) be an alphabet. The set of all words over \( \Sigma \) is denoted \( \Sigma^* \) and the empty word is \( \lambda \). If \( w \in \Sigma^* \) then \( |w| \) denotes the length of \( w \).

Definition 1. Let \( \Sigma \) be an alphabet and \( u, w \in \Sigma^* \). We say that \( u \) is a scattered subword (or simply subword) of \( w \) if \( w \), as a sequence of letters, contains \( u \) as a subsequence.

Formally, this means that there exist words \( x_1, \ldots, x_k \) and \( y_0, \ldots, y_k \) in \( \Sigma^* \), some of them possibly empty such that

\[
   u = x_1 \ldots x_k \quad \text{and} \quad w = y_0x_1y_1 \ldots x_ky_k.
\]

More formally, \( a_1a_2\ldots a_k \) is a subword of \( b_1b_2\ldots b_n \) (where \( a_i \in \Sigma \) for all \( 1 \leq i \leq k \) and \( b_j \in \Sigma \) for all \( 1 \leq j \leq n \)) if there exists a mapping \( f : \{1, \ldots, k\} \to \{1, \ldots, n\} \) so that \( f(i) < f(i+1) \) for all \( 1 \leq i < k \) and \( b_{f(i)} = a_i \) for all \( 1 \leq i \leq k \).

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We will denote by $|w|_u$ the number of occurrences of word $u$ as a subword in $w$, that is the number of mappings that can be defined with respect to the above definition. For instance,

$$|abba|_{ba} = 2 \quad \text{and} \quad |aabbc|_{abc} = 4.$$ 

In some works [7], the number $|w|_u$ is denoted as the binomial coefficient. Indeed, if the alphabet $\Sigma$ contains only one letter, the number $|w|_u$ reduces to the number of mappings $f : \{1, \ldots, |u|\} \to \{1, \ldots, |w|\}$ so that $f(i) < f(i + 1)$ for all $1 \leq i < |u|$, and that is exactly the binomial coefficient.

It is easy to see that if $|w| < |u|$ then $|w|_u = 0$. Also, if $u = \lambda$ then $|w|_u = 1$ because $\{1, \ldots, |u|\} = \emptyset$ and the inclusion $\emptyset \to \{1, \ldots, |w|\}$ is the only possible mapping (it clearly satisfies the definition).

Let $a, b$ be two letters in an alphabet $\Sigma$. We denote by $\delta_{a,b}$ be the Kronecker symbol regarding letters, that is

$$\delta_{a,b} = \begin{cases} 
1 & \text{if } a = b, \\
0 & \text{if } a \neq b. 
\end{cases}$$

**Fact 2.** It is shown in [7] that the equation

$$|vb|_{ua} = |v|_{ua} + \delta_{a,b}|v|_u, \quad a, b \in \Sigma, \; u, v \in \Sigma^*$$

together with the equations $|w|_{\lambda} = 1$ and $|w|_{\lambda} = 0$ for $|w| < |u|$ suffice to compute all values $|w|_u$.

### 1.2. Parikh matrices

The notion of Parikh matrix was introduced in [3]. All definitions and results presented in this subsection can be found in [3–5].

The definition of the Parikh matrix mapping uses a special type of matrices, called triangle matrices. A triangle matrix is a square matrix $M = (m_{i,j})_{1 \leq i, j \leq k}$, such that $m_{i,j}$ is a nonnegative integer for all $1 \leq i, j \leq k$, $m_{i,j} = 0$ for all $1 \leq j < i \leq k$ and $m_{i,i} = 1$ for all $1 \leq i \leq k$.

The set of all triangle matrices is denoted by $\mathcal{M}$. The set of all triangle matrices of dimension $k \geq 1$ is denoted by $\mathcal{M}_k$. Clearly $(\mathcal{M}_k, \cdot, I_k)$, where $\cdot$ represents the matrix multiplication and $I_k$ is the unit matrix, is a monoid.

An ordered alphabet is an alphabet $\Sigma = \{a_1, \ldots, a_k\}$ with a relation of order $<$ on it. If we have $a_1 < a_2 < \cdots < a_k$, then we use the notation $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$.

**Definition 3.** Let $\Sigma = \{a_1 < \cdots < a_k\}$ be an ordered alphabet. The Parikh matrix mapping, denoted as $\Psi_{\Sigma,k}$, is the monoid morphism

$$\Psi_{\Sigma,k} : (\Sigma^*, \cdot, \lambda) \to (\mathcal{M}_{k+1}, \cdot, I_{k+1})$$

defined by the condition: if $\Psi_{\Sigma,k}(a_q) = (m_{i,j})_{1 \leq i, j \leq (k+1)}$, then for each $1 \leq i \leq (k + 1)$, $m_{i,i} = 1$, $m_{q,q+1} = 1$, and all other elements of the matrix $\Psi_{\Sigma,k}(a_q)$ are 0.
For the ordered alphabet \( \Sigma = \{a_1 < \cdots < a_k\} \), we denote by \( a_{i,j} \) the word \( a_ia_{i+1}\cdots a_j \), where \( 1 \leq i \leq j \leq k \).

The following theorem characterizes the entries of the Parikh matrix:

**Theorem 4.** Let \( \Sigma = \{a_1 < \cdots < a_k\} \) be an ordered alphabet and \( w \in \Sigma^* \). The matrix \( \Psi_{\Sigma,k}(w) = (m_{i,j})_{1 \leq i,j \leq (k+1)} \), has the following properties:

- \( m_{i,j} = 0 \), for all \( 1 \leq j < i \leq (k+1) \),
- \( m_{i,i} = 1 \), for all \( 1 \leq i \leq (k+1) \),
- \( m_{i,j+1} = |w|_{a_{i,j}} \), for all \( 1 \leq i \leq j \leq k \).

We have seen in the above theorem that
\[
m_{i,j+1} = |w|_{a_{i,j}} \text{ for all } 1 \leq i \leq j \leq k.
\]

This means that for any word \( u \) with no repeating letters we can find how many occurrences it has as a subword in another word \( w \) by simply ordering the symbols in \( \Sigma \) in a convenient way
\[
\Sigma = \{a_1 < \cdots < a_k\}, \quad a_i = u_i \forall 1 \leq i \leq |u| \Rightarrow |w|_u = (\Psi_{\Sigma,k}(w))_{1,|u|+1}.
\]

For example, if we want to compute \( \text{cbaaca} \) all we have to do is to consider the following ordering: \( \Sigma = \{c < a < b\} \) and let \( \Psi_{\Sigma,4} \) be the Parikh matrix mapping over \( \Sigma \). Then, \( \text{cbaaca} = (\Psi_{\Sigma,4}(\text{cbaa}))_{1,3} = 2 \). Indeed, computing \( \Psi_{\Sigma,4}(\text{cbaa}) \) we obtain
\[
\Psi_{\Sigma,4}(\text{cbaa}) = \begin{pmatrix}
1 & 1 & 2 & 0 \\
0 & 1 & 2 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

The following two theorems characterize the inverse of an Parikh matrix. But first some notions need to be defined.

The alternate matrix of \( M = (m_{i,j})_{1 \leq i,j \leq k} \), denoted by \( \tilde{M} \), is the matrix \( \tilde{M} = (m'_{i,j})_{1 \leq i,j \leq k} \), where \( m'_{i,j} = (-1)^{i+j}(M)_{i,j} \) for all \( 1 \leq i,j \leq k \).

Given a word \( w = a_1\cdots a_n \) (\( a_i \in \Sigma \) for all \( 1 \leq i \leq n \)), we denote by \( \text{mi}(w) \) the mirror image of word \( w \), that is \( \text{mi}(w) = a_na_{n-1}\cdots a_1 \).

**Theorem 5.** Let \( \Sigma = \{a_1 < a_2 < \cdots < a_k\} \) be an ordered alphabet and let \( w \in \Sigma^* \) be a word. Then
\[
[\Psi_{\Sigma,k}(w)]^{-1} = \text{mi}(\Psi_{\Sigma,k}(w)).
\]

Let \( (A, <) \) be an ordered set. The dual order of the order \( < \), denoted \( <^\circ \), is defined as
\[
a <^\circ b \iff b < a.
\]

Let \( \Sigma = \{a_1 < a_2 < \cdots < a_k\} \) be an ordered alphabet. The dual ordered alphabet, denoted \( \Sigma_\circ \), is \( \Sigma_\circ = \{a_k < a_{k-1} < \cdots < a_1\} \).
The reverse of $M = (m_{i,j})_{1 \leq i,j \leq k}$, denoted by $M^{(\text{rev})}$, is the matrix $M^{(\text{rev})} = (m''_{i,j})_{1 \leq i,j \leq k}$, where $m''_{i,j} = m_{k+1-j,k+1-i}$, for all $1 \leq i,j \leq k$.

**Theorem 6.** Let $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$ be an ordered alphabet and let $w \in \Sigma^*$ be a word. Then

$$[\Psi_{\Sigma,k}(w)]^{-1} = \overline{\Psi_{\Sigma_o,k}(w)}^{(\text{rev})}.$$  

It follows that

**Corollary 7.** $\Psi_{\Sigma_o,k}(w)^{(\text{rev})} = \Psi_{\Sigma,k}(\text{mi}(w))$.

We also mention here a theorem about the minors of a Parikh matrix.

**Theorem 8.** The value of each minor of an arbitrary Parikh matrix is a nonnegative integer.

From the above theorem the next result can be obtained directly.

**Corollary 9.** Consider $\Sigma = \{a_1, \ldots, a_k\}$, and assume that $xyz$ is a factor of the word $u_\Sigma = a_1 \ldots a_k$. Then, for any word $w$,

$$|w|_y|w|_{xyz} \leq |w|_x|w|_{yz}.$$  

Proofs for these theorems as well as further investigations on the Parikh matrices can be found in [3–5].

1.3. Characterization of images of Parikh matrix mappings

We will denote by $A_{i,j}$ the $j$th column of a matrix $A$. Using this notation, it follows that $A = (A_{i,1}, A_{i,2}, \ldots, A_{i,k})$, where $k$ is the number of columns of $A$.

**Definition 10.** On $\mathcal{M}_k$ we introduce the relation $\Rightarrow$ as it follows: if $A = (A_{i,1}, A_{i,2}, \ldots, A_{i,k})$ and $B = (B_{i,1}, B_{i,2}, \ldots, B_{i,k})$ then

$$A \Rightarrow B \text{ iff } \exists 2 \leq j_0 \leq k, \text{ } B_{i,j} = \begin{cases} A_{i,j} & \text{if } j \neq j_0, \\ A_{i,j_0} + A_{i,j_0-1} & \text{if } j = j_0. \end{cases}$$

It is shown in [5] that if $A = \Psi_{\Sigma,k}(w)$ and $B = \Psi_{\Sigma,k}(wa_{j_0})$ then $B_{i,j_0+1} = A_{i,j_0+1} + A_{i,j_0}$ and for all $1 \leq j \leq k+1, j \neq j_0$ we have that $B_{i,j} = A_{i,j}$, which leads to $A \Rightarrow B$.

**Proposition 11.** Let $\Sigma = \{a_1 < a_2 < \cdots < a_k\}$ be an ordered alphabet. Let $N_{k+1}$ be the minimal set of $\mathcal{M}_{k+1}$ which contains $I_{k+1}$ and is closed under relation $\Rightarrow$. Then $N_{k+1} = \Psi_{\Sigma,k}(\Sigma^*)$.

**Proof.** First we will show that for any word $w \in \Sigma^*$, $\Psi_{\Sigma,k}(w)$ is in $N_{k+1}$. We will do that by induction over the length of $w$.
If \( w = \lambda \) then \( \Psi_{\Sigma, k}(w) = I_{k+1} \in N_{k+1} \). If \( |w| > 0 \) then \( w = w' a_j \). By the observation above we have that \( \Psi_{\Sigma, k}(w') \Rightarrow \Psi_{\Sigma, k}(w) \). Because of the induction hypothesis \( \Psi_{\Sigma, k}(w') \in N_{k+1} \) and using that \( N_{k+1} \) is closed under \( \Rightarrow \) it follows that \( \Psi_{\Sigma, k}(w) \in N_{k+1} \).

Now, let \( w \in \Sigma^* \) be an arbitrary word and \( A \) be a matrix such that \( \Psi_{\Sigma, k}(w) \Rightarrow A \). Then there exists \( j_0 \geq 2 \) such that for all \( 1 \leq j \leq k+1 \),

\[
A_j = \begin{cases} 
(\Psi_{\Sigma, k}(w))_{-j} & \text{if } j \neq j_0, \\
(\Psi_{\Sigma, k}(w))_{j_0} + (\Psi_{\Sigma, k}(w))_{j_0-1} & \text{if } j = j_0.
\end{cases}
\]

But this means exactly that \( A = \Psi_{\Sigma, k}(wa_{j_0-1}) \).

We have shown that \( \Psi_{\Sigma, k}(\Sigma^*) \) is closed under \( \Rightarrow \) and that \( \Psi_{\Sigma, k}(\Sigma^*) \subseteq N_{k+1} \). By the minimality of \( N_{k+1} \) we obtain \( N_{k+1} = \Psi_{\Sigma, k}(\Sigma^*) \).

As a consequence of this proposition we have that \( N_{k+1} \) is closed under matrix multiplication and furthermore \( (N_{k+1}, \cdot, I_{k+1}) \) is a monoid.

Also the above proposition gives us an algebraic intuition of \( \Psi_{\Sigma, k}(\Sigma^*) \) and allows us to consider redefining the morphism this way

\[
\Psi_{\Sigma, k} : (\Sigma^*, \cdot, \lambda) \to (N_{k+1}, \cdot, I_{k+1}).
\]

Obviously, with this new definition, \( \Psi_{\Sigma, k} \) is a surjective morphism.

2. Extending Parikh matrices

As we have seen in a previous example, for any word \( u \) with no repeating letters we can compute the value \( |w|_u \) using Parikh matrices. However, \( |w|_u \) cannot be found as an entry of any Parikh matrix if \( u \) has repeating letters (e.g., \( abba_{|abba|} \)). This is one of the reasons for extending the definition of the Parikh matrix as it follows.

2.1. Definition

**Definition 12.** Let \( \Sigma \) be an alphabet and \( u = b_1 \ldots b_{|u|} \) be a word in \( \Sigma^* \) (\( b_i \in \Sigma \) for all \( 1 \leq i \leq |u| \)). The Parikh matrix mapping induced by the word \( u \) over the alphabet \( \Sigma \), denoted \( \Psi_{\Sigma, u} \), is the monoid morphism\(^1\)

\[
\Psi_{\Sigma, u} : (\Sigma^*, \cdot, \lambda) \to (M_{|u|+1}, \cdot, I_{|u|+1}),
\]

defined by the condition: if \( a \in \Sigma \) and \( \Psi_{\Sigma, u}(a) = (m_{i,j})_{1 \leq i, j \leq (|u|+1)} \), then:

\[
m_{i,j} = \begin{cases} 
1 & \text{if } j = i, \\
\delta_{b_i, a} & \text{if } j = i + 1, \\
0 & \text{otherwise}.
\end{cases}
\]

\(^1\)In the notation \( \Psi_{\Sigma, u} \), \( \Sigma \) has to be mentioned because \( u \) can be considered over any alphabet which contains its letters, and we need to know the context we are working in.
If $\Sigma$ is known, then we will use the notation $\Psi_u$ for $\Psi_{\Sigma,u}$, especially in proofs, for reasons of simplicity.

It is clear that if the symbol $a \in \Sigma$ does not occur in $u$, then $\Psi_{\Sigma,u}(a) = I_{|u|+1}$.

Let $u \in \Sigma^*$. We say that $M \in \mathcal{M}_{|u|+1}$ is a Parikh matrix induced by $u$ if there exists a word $w \in \Sigma^*$ such that $M = \Psi_{\Sigma,u}(w)$. Generally, we say that $M \in \mathcal{M}_{k+1}$ is a Parikh matrix induced by a word if there exists an alphabet $\Sigma$ and a word $u \in \Sigma^*$ such that $|u| = k$ and $M$ is a Parikh matrix induced by $u$.

It is easy to see that Definition 3 can be obtained as a particular case of this definition, when $u$ contains all the symbols in $\Sigma$ only once. The ordering of the alphabet is then given by the order in which the symbols appear in $u$.

For example, if $\Sigma = \{b_1 < b_2 < \cdots < b_k\}$ is an ordered alphabet and $u = b_1 b_2 \ldots b_k$, then from Definitions 3 and 12 it can be easily seen that for all $a \in \Sigma$, $\Psi_{\Sigma,k}(a) = \Psi_{\Sigma,u}(a)$. It follows that for all words $w \in \Sigma^*$,

$$\Psi_{\Sigma,k}(w) = \Psi_{\Sigma,u}(w).$$

Similarly, it follows that for all words $w \in \Sigma^*$,

$$\Psi_{\Sigma,\omega}(w) = \Psi_{\Sigma,\mu}(w).$$

Let us now give an example of an $u$-Parikh matrix computation. Let $\Sigma = \{a, b\}$ and $u = aba$. We will compute $\Psi_{\Sigma,u}(abba)$.

We have that $\Psi_{\Sigma,u}(abba) = \Psi_{\Sigma,u}(a)\Psi_{\Sigma,u}(b)\Psi_{\Sigma,u}(b)\Psi_{\Sigma,u}(a)$, which leads to

$$\Psi_{\Sigma,u}(abba) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

2.2. Properties

In this section we show that some theorems that can be proved for the Parikh matrices can also be proved in the more general case of Parikh matrices induced by words.

Similarly to the notation $a_{i,j}$ in the case of an ordered alphabet we introduce the following notation: given the word $u = b_1 \ldots b_n$, we denote by $u_{i,j}$ the word $b_i b_{i+1} \ldots b_j$, where $1 \leq i \leq j \leq n$. 
Using this notation we can give a theorem to characterize the entries of a Parikh matrix induced by a word.

**Theorem 13.** Consider \( u = b_1 \ldots b_{|u|} \in \Sigma^* \) (\( b_i \in \Sigma \) for all \( 1 \leq i \leq |u| \)) and \( w \in \Sigma^* \). The matrix \( \Psi_{\Sigma,u}(w) = (m_{i,j})_{1 \leq i,j \leq (|u|+1)} \), has the following properties:

(i) \( m_{i,j} = 0 \), for all \( 1 \leq j < i \leq (|u|+1) \),

(ii) \( m_{i,i} = 1 \), for all \( 1 \leq i \leq (|u|+1) \),

(iii) \( m_{i,j+1} = |w|_{u_i,j} \), for all \( 1 \leq i \leq j \leq |u| \).

**Proof.** Obviously the first two properties, (i) and (ii) are true (because \( \Psi_u \) is a morphism).

Now we prove the third property (iii). Assume that \( |w| = n \). The proof is done by induction on \( n \). If \( n = 0 \), that is \( w = \lambda \), because \( \Psi_u \) is a monoid morphism, \( \Psi_u(w) = I_{|u|+1} \). For \( n = 1 \) it follows straight from the definition.

Assume now that assertion (iii) is true for all words of length at most \( n \) and let \( w \) be of length \( n + 1 \). Hence, \( w = w'a \), where \( |w| = n \) and \( a \in \Sigma \). It follows that

\[
\Psi_u(w) = \Psi_u(w'a) = \Psi_u(w') \Psi_u(a).
\]

Let \( \Psi_u(w') = (m'_{i,j})_{1 \leq i,j \leq (|w'|+1)} \) and \( \Psi_u(a) = (n_{i,j})_{1 \leq i,j \leq (|u|+1)} \). Using the definition of \( \Psi_u(a) \), it follows that

\[
m_{i,j+1} = \sum_{l=1}^{(|u|+1)} m'_{i,l} m'_{l,j+1} = m'_{i,j+1} + m'_{i,j} \delta_{a,b}
\]

for all \( 1 \leq i \leq j \leq |u| \). From the induction hypothesis we have that

\[
m'_{i,j+1} = |w'|_{u_i,j} \text{ for all } 1 \leq i \leq j \leq |u|.
\]

Replacing in the above relation, we obtain

\[
m_{i,j+1} = |w'|_{u_i,j} + \delta_{a,b} |w'|_{u_i,j-1}
\]

which, from Fact 2, leads to \( m_{i,j+1} = |w|_{u_i,j} \) for all \( 1 \leq i \leq j \leq |u| \). This completes the proof. \( \square \)

**Corollary 14.** For all words \( w \) and \( u \), we have the following relation:

\[
|w|_u = (\Psi_{\Sigma,u}(w))_{1,|u|+1}.
\]

We can now easily compute \( |abba|_{aba} \). From example presented above we have that

\[
\Psi_{aba}(abba) = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.
\]
Using the above corollary we get that

$$|abba|_{aba} = (\Psi_{aba}(abba))_{1,4} = 2.$$  

In Theorem 5 is given an easy way to compute the inverse of a Parikh matrix. We will try further on to do the same thing for Parikh matrices induced by words.

The following proposition is common in matrix theory so we omit the proof:

**Proposition 15.** For all $A, B \in M_k$, $AB = BA$ and $(AB)^{rev} = B^{rev}A^{rev}$.

First, we show that a result similar to Theorem 5 cannot be given for Parikh matrices induced by words: the equation $[\Psi_{u,w}(w)]^{-1} = \Psi_{u,w}(mi(w))$ does not hold in general.

Indeed, for $\Sigma = \{a\}$, $u = aa$ and $w = a$ we have that

$$\Psi_{aa}(a) = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \quad \Psi_{aa}(a) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

$$\Psi_{aa}(a)\Psi_{aa}(a) = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \Psi_{aa}(a)\Psi_{aa}(a) \neq I_3.$$  

However, if $u$ has no consecutive equal letters, then the result holds.

**Theorem 16.** Let $\Sigma$ be an alphabet, $u = b_1 \ldots b_{|u|}$ ($b_i \in \Sigma$ for all $1 \leq i \leq |u|$) with $b_i \neq b_{i+1}$, for all $1 \leq i < |u|$ and $w \in \Sigma^*$. Then

$$[\Psi_{\Sigma,\Sigma}(w)]^{-1} = \Psi_{\Sigma,\Sigma}(mi(w)).$$

**Proof.** This proof is done by induction on $|w|$.

If $|w| = 1$, then $w = a$ for some $a \in \Sigma$. We then obviously have $mi(w) = w$. We will verify that

$$\Psi_u(a)\Psi_u(a) = \Psi_u(a)\Psi_u(a) = I_{|u|+1}.$$  

Let $\Psi_u(a) = (n_{i,j})_{1 \leq i,j \leq |w|+1}$. Then $\Psi_u(a) = ((-1)^{i+j}m_{i,j})_{1 \leq i,j \leq (|u|+1)}$. If $(n_{i,j})_{1 \leq i,j \leq (|u|+1)} = \Psi_u(a)\Psi_u(a)$ and $(n'_{i,j})_{1 \leq i,j \leq (|u|+1)} = \Psi_u(a)\Psi_u(a)$ then

$$n_{i,j} = \sum_{l=1}^{n+1} m_{i,l}(-1)^{i+j}m_{l,j} = (-1)^j \sum_{l=1}^{n+1} (-1)^l m_{i,l}m_{l,j},$$

$$n'_{i,j} = \sum_{l=1}^{n+1} m_{i,l}(-1)^{i+j}m_{l,j} = (-1)^i \sum_{l=1}^{n+1} (-1)^l m_{i,l}m_{l,j}.$$  

We then deduce that $n_{i,j}$ and $n'_{i,j}$ have the same absolute value and also $n_{i,i} = n'_{i,i}$ meaning it is enough to prove only that $(n_{i,j})_{1 \leq i,j \leq (|w|+1)} = I_{|u|+1}$. We will do that by discussion over $i$ and $j$, always having in mind the relation defining $\Psi_u(a)$. 

First, if \( j = 1 \) we have that

\[
n_{i,1} = (-1)^{i+1} \sum_{l=1}^{i} (-1)^l m_{i,l} = m_{i,1} = \begin{cases} 1 & \text{if } \ i = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

If \( j > 1 \), then

\[
n_{i,j} = (-1)^j ((-1)^{j-1} m_{i,j-1} \delta_{b_{j-1},a} + (-1)^j m_{i,j}) = m_{i,j} - m_{i,j-1} \delta_{b_{j-1},a}.
\]

We write the defining relation for \( m_{i,j} \) and \( m_{i,j-1} \) successively

\[
m_{i,j} = \begin{cases} 1 & \text{if } \ j = i, \\ \delta_{b_{j},a} & \text{if } \ j = i + 1, \\ 0 & \text{otherwise}, \end{cases}
\]

\[
m_{i,j-1} = \begin{cases} 1 & \text{if } \ j = i + 1, \\ \delta_{b_{j},a} & \text{if } \ j = i + 2, \\ 0 & \text{otherwise}. \end{cases}
\]

Combining them, we obtain the relation

\[
n_{i,j} = m_{i,j} - m_{i,j-1} \delta_{b_{j-1},a} = \begin{cases} 1 - 0 \cdot \delta_{b_{j-1},a} & \text{if } \ j = i, \\ \delta_{b_{j},a} - 1 \cdot \delta_{b_{j-1},a} & \text{if } \ j = i + 1, \\ 0 - \delta_{b_{j},a} \cdot \delta_{b_{j-1},a} & \text{if } \ j = i + 2, \\ 0 & \text{otherwise}. \end{cases}
\]

Because \( b_{i} \neq b_{i+1} \) for all \( 1 \leq i < |u| \) we have \( \delta_{b_{i},a} \delta_{b_{i+1},a} = 0 \) for all \( 1 \leq i < |u| \) which leads to

\[
n_{i,j} = \begin{cases} 1 & \text{if } \ j = i, \\ 0 & \text{otherwise}. \end{cases}
\]

By combining this result with the one obtained for \( j = 1 \), the theorem is proven for \( n = 1 \).

If \( |w| > 1 \) then \( w = w'a \) with \( |w'| = |w| - 1 \) and

\[
[\Psi_u(w)]^{-1} = [\Psi_u(w'a)]^{-1} = [\Psi_u(w')\Psi_u(a)]^{-1} = [\Psi_u(a)]^{-1}[\Psi_u(w')]^{-1}.
\]

Applying the induction hypothesis and Proposition 15 it follows that

\[
[\Psi_u(a)]^{-1}[\Psi_u(w')]^{-1} = \Psi_u(a)\Psi_u(mi(w')) = \Psi_u(a)\Psi_u(mi(w'))
\]

\[
= \Psi_u(a mi(w')) = \Psi_u(mi(w)).
\]

This proves Theorem 16. \( \square \)
For example, let us compute the inverse of $\Psi_{aba}(abba)$. We have that

$$\Psi_{aba}(abba) = \begin{pmatrix} 1 & 2 & 2 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Because $\text{mi}(abba) = abba$ it follows that

$$[\Psi_{aba}(abba)]^{-1} = \Psi_{aba}(\text{mi}(abba)) = \begin{pmatrix} 1 & -2 & 2 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$ 

Let us now prove another theorem related to the inverse of a Parikh matrix induced by a word.

**Theorem 17.** Let $\Sigma$ be an alphabet, $u = b_1 \ldots b_{|u|}$ ($b_i \in \Sigma$ for all $1 \leq i \leq |u|$) and $w \in \Sigma^*$. Then

$$\Psi_{\Sigma,u}(\text{mi}(w)) = \Psi_{\Sigma,\text{mi}(u)}(w^{(\text{rev})}).$$

**Proof.** This proof is done by induction on $|w|$. If $|w| = 1$ then $w = a, a \in \Sigma$. Let $(m_{i,j})_{1 \leq i,j \leq |u|+1} = \Psi_u(a), (n_{i,j})_{1 \leq i,j \leq |u|+1} = \Psi_{\text{mi}(u)}(a)$ and $(n'_{i,j})_{1 \leq i,j \leq |u|+1} = \Psi_{\text{mi}(u)}(a)^{(\text{rev})}$. Let $\text{mi}(u) = c_1c_2 \ldots c_{|u|}$, where $c_i = b_{|u|+1-i}$ for all $1 \leq i \leq |u|$. We have that

$$n'_{i,j} = n_{|u|+2-j,|u|+2-i} = \begin{cases} \delta_{c_{|u|+2-j},a} & \text{if } |u| + 2 - i = |u| + 2 - j, \\
1 & \text{if } |u| + 2 - i = |u| + 2 - j + 1, \\
0 & \text{otherwise,} \end{cases}$$

$$= \begin{cases} \delta_{b_{i-1,a}} & \text{if } j = i, \\
1 & \text{if } j = i + 1 = m_{i,j}, \\
0 & \text{otherwise.} \end{cases}$$

If $|w| > 1$ then $w = w' a$ with $|w'| = |w| - 1$. We have that

$$\Psi_u(\text{mi}(w)) = \Psi_u(\text{mi}(w'a)) = \Psi_u(a \text{mi}(w')) = \Psi_u(a) \Psi_u(\text{mi}(w')).$$

Applying the induction hypothesis for $a$ and $w$, then by Proposition 15 it follows that

$$\Psi_u(\text{mi}(w)) = \Psi_{\text{mi}(u)}(a)^{(\text{rev})} \Psi_{\text{mi}(u)}(w')^{(\text{rev})} = [\Psi_{\text{mi}(u)}(w') \Psi_{\text{mi}(u)}(a)]^{(\text{rev})} = \Psi_{\text{mi}(u)}(w)^{(\text{rev})}.$$ 

Thus, Theorem 17 is proven. □
The next corollary is an immediate consequence of Theorems 16 and 17.

**Corollary 18.** Let $\Sigma$ be an alphabet, $u = b_1 \ldots b_{|u|}$ ($b_i \in \Sigma$ for all $1 \leq i \leq |u|$) with $b_i \neq b_{i+1}$, for all $1 \leq i < |u|$ and $w \in \Sigma^*$. Then

$$[\Psi_{\Sigma, u}(w)]^{-1} = \Psi_{\Sigma,m(u)}(w)^{(rev)}.$$  

### 2.3. Reduction to Parikh matrices

We will show in this paragraph that the Parikh matrix mapping induced by a word can be obtained as a composition between a classical Parikh matrix mapping and a word substitution morphism.

**Definition 19.** Let $\Sigma$ be an alphabet and $u$ be a word in $\Sigma^*$, $u = b_1 \ldots b_k$ ($b_i \in \Sigma$ for all $1 \leq i \leq |u|$). Let $\Sigma' = \{c_1 < c_2 < \ldots < c_k\}$ be an ordered alphabet.

For each $a \in \Sigma$ the corresponding word for a induced by $u$ in $\Sigma'$ is the word $w = c_i c_i \ldots c_i$, where $n = |u|$, $i_1 < i_{j+1}$ for all $1 \leq j < n$ and $b_i = a$ for all $1 \leq j \leq n$.

The $\Sigma, u, \Sigma'$ substitution morphism is the monoid morphism

$$\sigma_{\Sigma, u, \Sigma'} : (\Sigma^*, \cdot, \lambda) \rightarrow (\Sigma'^*, \cdot, \lambda)$$

which satisfies that for all $a \in \Sigma, \sigma_{\Sigma, u, \Sigma'}(a) = mi(s(a))$, where $s(a)$ is the corresponding word for $a$ induced by $u$ in $\Sigma'$.

When $\Sigma, u, \Sigma'$ are known, we will use the simplified notation $\sigma$ for $\sigma_{\Sigma, u, \Sigma'}$.

For example, if $\Sigma = \{a, b, c, d, e\}$, $u = abac$ and $\Sigma' = \{f < g < h < i\}$, then the corresponding words induced by $abac$ in $\Sigma'$ for the letters of $\Sigma$ are $s(a) = fh$, $s(b) = g$, $s(c) = i$ and $s(d) = s(e) = \lambda$, and

$$\sigma(bad) = \sigma(b)\sigma(a)\sigma(d) = g \cdot h \cdot f \cdot \lambda = ghf.$$  

The main result of this subsection is the following theorem:

**Theorem 20.** Let $\Sigma$ be an alphabet, and $u$ be a word in $\Sigma^*$, $u = b_1 \ldots b_k$ ($b_i \in \Sigma$ for all $1 \leq i \leq |u|$). Let $\Sigma' = \{c_1 < c_2 < \ldots < c_k\}$ be an ordered alphabet. Then

$$\Psi_{\Sigma, u} = \Psi_{\Sigma', k} \circ \sigma_{\Sigma', \Sigma'}.$$  

**Proof.** We will prove the assertion by showing that $\Psi_{\Sigma, u}$ and $\Psi_{\Sigma', k} \circ \sigma$ restricted to $\Sigma$ are equal.

Let $a \in \Sigma$ be a letter. Let $\left(m_{i,j}\right)_{1 \leq i, j \leq k+1} = \Psi_{\Sigma, u}(a)$ and $\left(n_{i,j}\right)_{1 \leq i, j \leq k+1} = \Psi_{k+1, \Sigma'}(\sigma(a))$. We have that

$$m_{i,j} = \begin{cases} 1 & \text{if } i = j \\ \delta_{b_i,a} & \text{if } j = i + 1 \text{ and } n_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i > j, \\ |\sigma(a)|_{c_{i,j-1}} & \text{otherwise}. \end{cases} \end{cases}$$
If $j = i + 1$ then

$$n_{i,j+1} = |\sigma(a)|_{c} = \begin{cases} 1 & \text{if } b_{i} = a \\ 0 & \text{otherwise} \end{cases} = \delta_{b_{i},a}.$$ 

If $j > i + 1$, then $n_{i,j} = 0$ because the letters of $\sigma(a)$ are in reverse order.

This proves Theorem 20. □

This gives us an alternative way to compute Parikh matrices induced by words and shows that the algebraic properties of the Parikh matrices can be transferred to this kind of matrices.

Let us give an example of computing a Parikh matrix induced by a word as a Parikh matrix. Let $\Sigma = \{a, b\}$ and $u = aba$. We want to compute $\Psi_{\Sigma;aba}(abba)$. Let $\Sigma' = \{c < d < e\}$. Then we have $\sigma(a) = ec$ and $\sigma(b) = d$ which leads to $\sigma(abba) = ecdec$. From Theorem 20 we know that $\Psi_{\Sigma;aba}(abba) = \Psi_{\Sigma',3}(ecdec)$.

2.4. Further properties

Theorem 20 shows that the algebraic properties of Parikh matrices can be transferred over Parikh matrices induced by words. Let us give some examples of the possible applications.

From Theorem 8, using Theorem 20 the following result is obvious.

**Corollary 21.** The value of each minor of an arbitrary Parikh matrix induced by a word is a nonnegative integer.

In [3], a Cauchy-like inequality is proven for occurrences of subwords in words, which generalizes Corollary 9, but with a very long and technical proof. In the light of the new definitions and theorems, we give a direct and elegant proof for it.

**Theorem 22.** The inequality $|w|_{xy} |w|_{yz} \leq |w|_{xyz}$ holds for arbitrary words $w, x, y, z$ over an alphabet $\Sigma$.

**Proof.** If $z = \lambda$, the inequality is an identity. We will prove the assertion for $z \neq \lambda$.

Let $u = xyz$. Let $(m_{i,j})_{1 \leq i, j \leq |u| + 1} = \Psi_{u}(w)$. From Theorem 13 we have that $m_{|u|+1,|u|+1} = |w|_{xyz}$, $m_{|x|+1,|xy|+1} = |w|_{y}$, $m_{1,|x|+1} = |w|_{xy}$ and $m_{|x|+1,|u|+1} = |w|_{yz}$. Because

$$\begin{pmatrix} m_{1,|xy|+1} & m_{1,|u|+1} \\ m_{|x|+1,|xy|+1} & m_{|x|+1,|u|+1} \end{pmatrix}$$

is a minor of $\Psi_{u}(w)$, from Corollary 21 we have that $m_{1,|xy|+1}m_{|x|+1,|u|+1} - m_{1,|u|+1}m_{|x|+1,|xy|+1} \geq 0$. Thus the theorem is proven. □

This once again proves that the Parikh matrices induced by words can be a more useful tool for investigating relations between words.
Using Theorem 20, the inverse of an Parikh matrix induced by a word can be computed for any word after reducing it to a Parikh matrix.

**Corollary 23.** Let $\Sigma$ be an alphabet and $u$ and $w$ words in $\Sigma^*$. Let $\Sigma' = \{c_1 < c_2 < \cdots < c_{|u|}\}$ be an ordered alphabet. Then

$$[\Psi_{\Sigma, u}(w)]^{-1} = \Psi_{\Sigma', |u|}(\text{mi}(\sigma_{\Sigma, u}(w))).$$

Let us now reduce the alphabet of a Parikh matrix mapping induced by a word $u$ to contain only the letters from $u$.

If $\Sigma$ is an alphabet and $u \in \Sigma^*$, $u = b_1 b_2 \ldots b_n$ ($b_i \in \Sigma$ for all $1 \leq i \leq |u|$) then denote by $\Sigma_u$ the following set:

$$\Sigma_u = \{b_i \mid 1 \leq i \leq n\}.$$ 

For an alphabet $\Sigma$ and $\Sigma'$ a subset of $\Sigma$, we define the morphism $h_{\Sigma'} : \Sigma^* \rightarrow \Sigma'^*$ given by

$$h_{\Sigma'}(a) = \begin{cases} a & \text{if } a \in \Sigma', \\ \lambda & \text{otherwise} \end{cases}$$

the morphism deleting from any word the letters which are not in $\Sigma'$.

Using Theorem 20 the following corollaries easily follow.

**Corollary 24.** Let $\Sigma$ be an alphabet and $u \in \Sigma^*$ a word. Then for all $w \in \Sigma^*$, $\Psi_{\Sigma, u}(w) = \Psi_{\Sigma', u}(h_{\Sigma'}(w))$.

**Corollary 25.** Let $\Sigma$ be an alphabet, and $u, w$ words over $\Sigma$. Then,

$$|w|_u = |h_{\Sigma'}(w)|_u.$$ 

The above corollary says, for example that $|ccacbdabdd|_{aba} = |ababa|_{aba}$. Furthermore, it says that if $u, v, w$ are words and $v$ has no common letters with $u$ then for any word $x \in w \otimes v$, $|x|_u = |w|_u$, where $\otimes$ denotes the shuffle operation.

### 3. Conclusion

The notion of a Parikh matrix induced by a word, introduced in this paper, is a simple generalization of the original notion of a Parikh matrix. Yet it seems to open new possibilities for research in formal languages and combinatorics on words. We have proved that Parikh matrices over words can be transformed to original Parikh matrices, without reducing their expressive capacity. This shows that properties of both types of matrices can be used according to the requirements of each particular problem.
References