A New Convergence Result for Fixed-Point Iteration in Bounded Interval of $\mathbb{R}^n$

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Abstract—A fixed-point theorem in bounded interval of $\mathbb{R}^n$ and a new fixed-point iteration on Mann's iteration scheme are established to compute a fixed-point of real Lipschitz functions. Two numerical examples are presented to show that the method is feasible and that it produces reasonable results.

Keywords—Fixed-point iterations, Mann's iteration scheme, Lipschitz functions.

1. INTRODUCTION

We call $g : I \rightarrow \mathbb{R}^n$ is a Lipschitz function on the $n$-dimensional rectangle $I$, if

$$||g(x) - g(y)|| \leq L||x - y||, \quad \forall x, y \in I,$$

(1)

where $||x|| = (\sum_{i=1}^{n} |x_i|^p)^{1/p}$, $0 < p \leq \infty$, and $I = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$, $a_i, b_i \in \mathbb{R}$, $i = 1, \ldots, n$. There are various schemes for computing the fixed-point of the mapping $g$. Assume $L < 1$, then $g$ is a contraction. From the basic theorems of this field [1] (and references therein), if $g$ is also self-mapping, then there exists at least a fixed-point of $g(x)$ in $I$, and then each of the two well known fixed-point iteration method (the Mann and Ishikawa iteration methods) converges strongly to the fixed-point of $g$.

However, when $L \geq 1$, what will happen?

As mentioned in [2], one can find a new fixed-point iteration in $\mathbb{R}^1$ in this case, from which we do not need to prerequire that:

(i) $L < 1$; or
(ii) $g(x)$ self-maps $I_1$ into $I_1$ in which $I_1 = [a_1, b_1] \subset \mathbb{R}^1$ in the case of $L \geq 1$.

We need to point out here that $I_1$ is a special case of $n$-dimensional rectangle $I$ while $n = 1$. We will prove that more generally for computing the fixed-point of the mapping $g(x) : \mathbb{R}^n \rightarrow \mathbb{R}^n$, in $I \subset \mathbb{R}^n$ there is a new convergent iteration and a convergence theorem which are not based on the basic theory of [2], but based on [3] or Leray-Schauder Theorem of [1].

2. CONVERGENCE THEOREM

The most important theorem of [2] is the following.

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**Theorem 1.** Let $h : I_1 \rightarrow \mathbb{R}^1$ be a function which satisfies condition $|h(x) - h(y)| \leq L |x - y|, \forall x, y \in I_1$. Let $\{a_n\}$ be a real sequence with the properties

1. $0 < \alpha_n \leq (L + 1)^{-1}$,
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $h(a_1), h(b_1) \in [a_1, b_1]$.

Then, the sequence $\{x_n\}$ defined by $x_1 = a$ or $b$, $x_{n+1} = (1 - \alpha_n)x_n + \alpha_nh(x_n)$ is convergent and its limit point, say $z$, satisfies $z = h(z)$.

In the sequel, we will propose a more general iteration and related convergence theorem of $\mathbb{R}^n$.

**Theorem 2.** Let $g$ be a function which satisfies condition (1) with partial order definition. Let $\{a_n\}$ be a real sequence with the properties

1. $0 < \alpha_n \leq (L + 1)^{-1}$,
2. $\sum_{n=1}^{\infty} \alpha_n = \infty$,
3. $g((z_1, z_2, \ldots, z_n)^T) \in I$, for any given $j \in \mathbb{N}$, $z_j = a_j$ or $b_j$, $z = (z_1, z_2, \ldots, z_n)^T \in I \subset \mathbb{R}^n$, $a = (a_1, a_2, \ldots, a_n)^T$, $b = (b_1, b_2, \ldots, b_n)^T$.

Then, the sequence $\{x_n\}$ defined by

$$x_1 = a = (a_1, a_2, \ldots, a_n)^T, \quad x_{n+1} = (1 - \alpha_n)x_n + \alpha_n g(x_n)$$

is convergent and its limit point, say $z$, satisfies $z = g(z)$.

**Proof.** Let $b = (b_1, b_2, \ldots, b_n)^T$, and let $f(x) = g(x) - x$, then from Condition III, in $I$, the following two conditions $(\alpha)$ and $(\beta)$ occur:

$$(\alpha) \quad f_j(x_1, \ldots, x_{j-1}, a_j, x_{j+1}, \ldots, x_n) \geq 0, \quad \text{for } 1 \leq j \leq n,$$
$$(\beta) \quad f_j(x_1, \ldots, x_{j-1}, b_j, x_{j+1}, \ldots, x_n) \leq 0, \quad \text{for } 1 \leq j \leq n.$$ 

Therefore, from [3,4], there exists at least one point of $I$ where $f_1, \ldots, f_n$ vanish simultaneously. We define $p \in I$ such that $f(p) = g(p) - p$, that is, $p = g(p)$.

Now we will show that following the Mann scheme [5] as (2), the proceeding iteration is

$$a = x_1 < x_2 < \cdots < x_n < p < b.$$  

(Here $\leq$ denotes $\leq$ but not $\equiv$.)

Without loss of generality, let us assume $x_1 \neq p$, i.e., $x_1 \leq p$ (but $x_1 \neq p$). Assume $x_2 > p$, so $0 \leq p - x_1 < x_2 - x_1 = \alpha_1(g(x_1) - x_1),$ 

$$0 < \frac{1}{\alpha_1} ||x_1 - p|| < ||g(x_1) - x_1|| = ||g(x_1) - x_1 + g(p) - g(p)|| = ||g(x_1) - x_1 + p - g(p)|| \leq ||g(x_1) - g(p)|| + ||p - x_1||.$$ 

Hence from Condition I, $||g(x_1) - g(p)|| > (1/\alpha_1 - 1)||p - x_1|| \geq L||p - x_1||$ is a contradiction to assumption (1), so $x_1 < x_2 \leq p$. Similarly with the discussion above for $x_n$, we have (3). Consequently, $\lim_{n \to \infty} (x_n)_j (\forall j)$ exists, say, $\lim_{n \to \infty} (x_n)_j = z_j (\forall j)$ and so $\lim_{n \to \infty} x_n = z$ in which $z = (z_1, \ldots, z_n)^T$.

Note that $g(x_n) \geq x_n (\forall n = 1, 2, \ldots)$, so $g(z) \geq z$. Assume $g(z) \neq z$, i.e., $g(z) > z$, but $g(z) \neq z$, i.e., there is at least $j \in \{1, \ldots, n\}$, such that $g_j(z) > z_j$. For the given $j$, let $\varepsilon = (g_j(z)$
Fixed-Point Iteration

\[ \frac{-z_j}{2} > 0, \text{ hence there exists a natural number } N, \text{ such that while } n \geq N, \left| g(x_n) - x_n \right| \geq 2\varepsilon > \varepsilon, \text{ consequently,} \]

\[ \left[ x_{N+m} - x_N \right]_j = \left[ \sum_{i=N}^{N+m-1} \alpha_i (g(x_i) - x_i) \right]_j = \sum_{i=N}^{N+m-1} \alpha_i \left[ g(x_i) - x_i \right]_j \in \sum_{i=N}^{N+m-1} \alpha_i, \]

yields \( \lim_{m \to \infty} \left[ x_{N+m} - x_N \right]_j \geq \varepsilon \sum_{i=N}^{N+m-1} \alpha_i \) contradict to Condition II.

**Remark 1.** As \( n = 1 \), Condition III will be seen as \( g(a) \in I, g(b) \in I, a, b \in \mathbb{R} \). This shows that reference [2] is a special case of this paper.

From the proof of Theorem 2, if we have known that there exists a fixed-point of \( g(x) \) in \( I \), then Condition III of Theorem 2 is not necessary, and hence, the following corollaries are obvious.

**Corollary 3.** Let \( g \) be a function which satisfies condition (1) with partial-order definition and assume that there exists a fixed-point \( z = g(z) \) of \( g(x) \) in \( I \). Let \( \{a_n\} \) be a real sequence with the properties

1. \( 0 < \alpha_n \leq (L + 1)^{-1} \),
2. \( \sum_{n=1}^{\infty} \alpha_n = \infty \),
3. \( g(a) \in I, \) or \( g(b) \in I, \) \( a = (a_1, a_2, \ldots, a_n)^T, \) and \( b = (b_1, b_2, \ldots, b_n)^T. \)

Then, the sequence \( \{x_n\} \) defined by \( x_1 = a \) or \( b, x_{n+1} = (1 - \alpha_n) x_n + \alpha_n g(x_n) \) is convergent and its limit point, say \( z, z = g(z). \)

**Corollary 4.** Let \( g \) be a function which satisfies condition (1) with partial-order definition and

\( (x - x^0)^T f(x) \geq 0, \forall x \in I, \) and some \( x^0 \in \mathbb{R}^n \), then there exists a fixed-point \( z = g(z) \) in \( I \).

**3. Numerical Results**

In this section, we will present two examples. We use scheme (2) to compute unless \( \|x_n - g(x_n)\|_2 \) is less than the prerequired tolerance \( \varepsilon > 0 \).

**Example 1.**

\[ x = (x_1, x_2)^T \in \mathbb{R}^2, \quad I = [a_1, b_1] \times [a_2, b_2] = [1.2, 2.2] \times [0.0, 1.0] \subset \mathbb{R}^2, \]

let \( a = (1.2, 0.0)^T \in \mathbb{R}^2, b = (2.2, 1.0)^T \in \mathbb{R}^2. \) Let

\[ g(x) = (g_1(x), g_2(x))^T = (0.5x_1^2 - 3\sin(3x_1), 0.5x_2^3 + 0.2x_1)^T \in \mathbb{R}^2. \]

We notice that

\[ g(a) = (2.047561, \ldots, 0.6)^T \in I, \quad g(b) = (1.485376, \ldots, 0.94)^T \in I, \]

but \( g((1.8, 0.5)^T) = (3.93829, \ldots, 0.4225)^T \notin I. \) It is easy to check that \( g(x) \) satisfies Condition III of Theorem 2.

Under the meaning of \( \| \cdot \|_2 \),

\[ L = \max_{\xi = (x_1, x_2)^T \in I} \rho(A(\xi)^T A(\xi)) \approx 9.27083, \ldots, \]
where
\[ A(\xi) = \begin{bmatrix} x_1 - 9 \cos(3x_1) & 0.2 \\ 0 & 1.5x_2^2 \end{bmatrix}. \]

Mann’s iterative scheme is tested by two ways. The first is with \( \alpha_n = 1/(10+n) \), 1309 iterations are required for \( \varepsilon = 10^{-2} \), \( x^* = (2.106911, 4.600806E - 01)^T \). The second is with \( \alpha_n = 1/11 \), 127 iterations are required for \( \varepsilon = 10^{-4} \), \( x^* = (2.106912, 4.748063E - 01)^T \), and 202 iterations for \( \varepsilon = 10^{-6} \), \( x^* = (2.106912, 4.749508E - 01)^T \).

**EXAMPLE 2.**

\[ x = (x_1, x_2)^T \in \mathbb{R}^2, \quad I = [a_1, b_1] \times [a_2, b_2] = [-0.1, 5.0] \times [0.0, 4.0] \subset \mathbb{R}^2, \]

let \( a = (-0.1, 0.0)^T \in \mathbb{R}^2 \), \( b = (5.0, 4.0)^T \in \mathbb{R}^2 \). Let
\[ g(x) = (g_1(x), g_2(x))^T = (0.5x_1 - 2x_2, 0.5x_2 - 0.25x_1)^T \in \mathbb{R}^2. \]

It is clear that \((0,0)^T\) is a fixed-point of \(g(x)\) in \(I\). We notice that \(g(a) = (0.005, 0.0)^T \in I\), \(g(b) = (4.5, 3.0)^T \in I\), but \(g((3.0, 4.0)^T) = (-3.5, 5.0)^T \not\in I\).

Under the meaning of \(\| \cdot \|_2\),
\[ L = \max_{\xi = (x_1, x_2)^T} \rho (A(\xi)^TA(\xi)) \quad \text{(from [6, p. 62, Hirsch Theorem])} \]
\[ \leq \max_{\xi \in I} [2 \max \{|x_1| + 0.25|x_2|, 2 + |x_2 - 0.25x_1|\}] = 2 \max\{5 + 1, 2 + 4\} = 12, \]

where
\[ A(\xi) = \begin{bmatrix} x_1 -0.25x_2 \\ -2 & x_2 -0.25x_1 \end{bmatrix}. \]

The Mann’s iterative scheme is tested by two ways. The first is with \( \alpha_n = 1/(12+n) \), 1131 iterations are required for \( \varepsilon = 10^{-3} \), \( x^* = (-9.991390E - 04, 0.000000E + 00)^T \). The second is with \( \alpha_n = 1/13 \), 87 iterations are required for
\[ \varepsilon = 10^{-4}, \quad x^* = (-9.716833E - 05, 0.000000E + 00)^T, \]

and 145 iterations for
\[ \varepsilon = 10^{-6}, \quad x^* = (-9.360519E - 07, 0.000000E + 00)^T. \]

**REMARK 2.** Here we present a fixed-point iteration for bounded interval \(I\) of \(\mathbb{R}^n\), which is not necessary to check whether the function \(g\) is mapping \(I\) into \(I\) in practice or not.

**REMARK 3.** Here we define the Lipschitz condition (1) under the meaning of \(\| \cdot \|_2\), which pre-requires the existence of \(\rho(A(\xi)^TA(\xi))\). Since \(A(\xi)^TA(\xi)\) is a real symmetric matrix, this pre-condition is very obvious. If one defines (1) under other \(\| \cdot \|_p\), there should be other requirements for \(p \neq 2, 0 < p \leq \infty\).

**REFERENCES**