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Integrability and Regularity of 3D Euler and Equations for Uniformly Rotating Fluids

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Abstract—We consider 3D Euler and Navier-Stokes equations describing dynamics of uniformly rotating fluids. Periodic boundary conditions are imposed, and the ratio of domain periods is assumed to be generic (nonresonant). We show that solutions of 3D Euler/Navier-Stokes equations can be decomposed as $\mathbf{U}(t,x_1,x_2,x_3) = \tilde{\mathbf{U}}(t,x_1,x_2) + \mathbf{V}(t,x_1,x_2,x_3) + \mathbf{r}$ where $\tilde{\mathbf{U}}$ is a solution of the 2D Euler/Navier-Stokes system with vertically averaged initial data (axis of rotation is taken along the vertical e_3). Here $\mathbf{r}$ is a remainder of order $\mathrm{Ro}_a^{1/2}$ where $\mathrm{Ro}_a = (H_0U_0)/(\Omega_0L_0^2)$ is the *anisotropic Rossby number* (H_0—height, L_0—horizontal length scale, Ω_0—angular velocity of background rotation, U_0—horizontal velocity scale); $\mathrm{Ro}_a = (H_0/L_0)\mathrm{Ro}$ where H_0/L_0 is the aspect ratio and $\mathrm{Ro} = U_0/(\Omega_0L_0)$ is a Rossby number based on the horizontal length scale L_0. The vector field $\mathbf{V}(t,x_1,x_2,x_3)$ which is *exactly solved* in terms of 2D dynamics of vertically averaged fields is phase-locked to the phases $2\Omega_0t$, $\tau_1(t)$, and $\tau_2(t)$. The last two are defined in terms of passively advected scalars by 2D turbulence. The phases $\tau_1(t)$ and $\tau_2(t)$ are associated with vertically averaged vertical vorticity $\operatorname{curl}\overline{\mathbf{U}}(t)\cdot e_3$ and velocity $\overline{U}^3(t)$; the last is weighted (in Fourier space) by a classical nonlocal wave operator. We show that 3D rotating turbulence decouples into **phase turbulence** for $\mathbf{V}(t,x_1,x_2,x_3)$ and 2D turbulence for vertically averaged fields $\overline{\mathbf{U}}(t,x_1,x_2)$ if the anisotropic Rossby number Ro_a is small. The mathematically rigorous control of the error $\mathbf{r}$ is used to prove existence on a long time interval T^* of regular solutions to 3D Euler equations ($T^* \to +\infty$, as $\mathrm{Ro}_a \to 0$); and global existence of regular solutions for 3D Navier-Stokes equations in the small anisotropic Rossby number case.

Keywords—Euler equations, Navier-Stokes equations, Rotating fluids, Phase turbulence.

There are many important engineering and geophysical problems in which rotation significantly modifies properties of fluid flows. In particular, large-scale atmospheric and oceanic flows are dominated by rotational effects and the impact of shallowness on the large scale dynamics. Long-time computation of geophysical flows using unmodified Euler equations is prohibitive due to severe accuracy and time step restrictions. The disparity of time scales leads to problems in the numerical solution of the equations because the Courant number is determined by the fastest

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Typeset by $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-TEX time scale $1/\Omega_0$ (Ω_0—angular velocity of background rotation) and therefore limits the time step which makes explicit solution impractical.

We consider flows uniformly rotating with a constant angular velocity Ω_0 about the x_3-axis. Let L_0 be the characteristic horizontal length scale and U_0 be the characteristic velocity. The problem is made dimensionless by referring lengths to L_0, velocities to U_0 and time to L_0/U_0, which leads to the introduction of a dimensionless rotation rate Ω (Ro $= 1/\Omega$ is the Rossby number), defined by $\Omega = (\Omega_0 L_0)/U_0$.

In a frame of reference rotating with constant angular velocity Ω about the x_3-axis, the inviscid 3D Euler equations have the form

$$\begin{aligned} \partial_t \mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U} + 2\Omega\mathbf{J}\mathbf{U} &= -\nabla p, \\ \nabla\cdot\mathbf{U} &= 0. \end{aligned} \tag{1}$$

Here $\mathbf{U} = (U^1, U^2, U^3)$ is the velocity field, $\mathbf{J}$ is the rotation matrix, $2\Omega\mathbf{J}\mathbf{U} = 2\Omega e_3 \times \mathbf{U}$ is the Coriolis force term, and p is a modified pressure. We consider solutions of equation (1) $2\pi a_j$-periodic in every x_j taken along Cartesian axes e_j, $j = 1,2,3$; $a_1 = 1$.

In our previous work [1], we have introduced Poincaré velocity variables and then formally averaging Euler equations in time, derived new extended Euler/Navier-Stokes equations. The structure of these equations is determined by the structure of resonant cones in Fourier space of wavenumbers; after time averaging over the fast time scale $1/\Omega$, these resonant cones generate nontrivial limit to Euler as $\Omega \to +\infty$. The method is closely related to time-averaging methods in celestial mechanics [2–4]. We have fully resolved the small divisor problem for resonances in rotating Euler equations. The resonant set in the space of wavenumbers is the intersection of cones with a lattice which depends on a_2, a_3; we call this set resonant cones. We show that the resonant cones in Fourier space have the simplest structure if the domain is nonresonant; that is the ratios a_2, a_3 of lengths of edges of the period parallelepiped are nonresonant (*resonant domain ratios are of zero Lebesgue measure; we assume everywhere below that such domain ratios are nonresonant*). An important case of nonresonant domains is small a_3 (thin domains).

We show that solutions of 3D Euler/Navier-Stokes equations can be decomposed as $\mathbf{U}(t, x_1, x_2, x_3) = \tilde{\mathbf{U}}(t, x_1, x_2) + \mathbf{V}(t, x_1, x_2, x_3) + \mathbf{r}$ where $\tilde{\mathbf{U}}$ is a solution of the 2D Euler system with vertically averaged initial data, and $\mathbf{r}$ is a remainder of order $\mathrm{Ro}_a^{1/2}$, for smooth enough initial data. The vector field $\mathbf{V}(t)$ is related to $\mathbf{v}(t)$ by a Van der Pol type transformation based on the Poincaré propagator and $\mathbf{v}(t)$ is a solution of a new system which we call Extended Euler/Navier-Stokes system. The Poincaré propagator [5] is the unitary group solution $\mathbf{E}(\Omega t)$ ($\mathbf{E}(0) = \mathbf{Id}$) to the Poincaré problem:

$$\begin{gathered} \partial_t \mathbf{\Phi} + 2\Omega\mathbf{J}\mathbf{\Phi} = -\nabla p, \qquad \nabla\cdot\mathbf{\Phi} = 0, \\ \mathbf{\Phi}(t) = \mathbf{E}(-\Omega t)\mathbf{\Phi}(0). \end{gathered} \tag{2}$$

Poincaré [5] solutions can be written in Fourier space as

$$\mathbf{E}_n(\Omega t) = \{\mathbf{E}(\Omega t)\}_n = \cos\left(\frac{n_3}{a_3|\hat{n}|}2\Omega t\right)\mathbf{I} + \frac{1}{|\hat{n}|}\sin\left(\frac{n_3}{a_3\,|\hat{n}|}2\Omega t\right)\mathbf{R}_n, \tag{3}$$

where the matrix $i\mathbf{R}_n$ is the Fourier transform of the curl operator; we denote $|\hat{n}|^2 = n_1^2 + n_2^2/a_2^2 + n_3^2/a_3^2$. The mathematical theory of the Poincaré problem (2) (in particular, in bounded domains) has attracted a considerable amount of attention starting from the work of Sobolev [6].

We use Fourier series expansions for velocity fields $\mathbf{U}(x) = (U^1(x), U^2(x), U^3(x))$

$$\mathbf{U}(x) = \sum_n \exp\left(i\left(n_1 x_1 + \frac{n_2 x_2}{a_2} + \frac{n_3 x_3}{a_3}\right)\right)\mathbf{U}_n, \tag{4}$$

where $\mathbf{U}_n$ are the Fourier coefficients, and $(n_1, n_2, n_3) \in \mathbf{Z}^3$ are wavenumbers. We assume that functions have zero average over the periodic parallelepiped.

The Poincaré velocity $\mathbf{v}(t)$ is related to $\mathbf{V}(t)$ by $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$, which is a canonical unitary transformation preserving energy, helicity and divergence free property. For fields which depend only on x_1, x_2 but not on x_3, $\mathbf{E}(\Omega t) = \mathbf{Id}$. Before any averaging, the equations for $\mathbf{v}$ are Euler-like, but with time-dependent coefficients. The extended Euler equations (which are obtained after time-averaging for nonresonant domains) are for the velocity fields which are three-dimensional and three-component (3D-3C: three components, dependence on three variables x_1, x_2, x_3). The extended system possesses infinitely many conservation laws. These conserved quantities are adiabatic invariants for the classical 3D Euler equations in a rotating frame in the small anisotropic Rossby number case. If the vertically averaged initial data are reflectionally symmetric (that is satisfy zero flux boundary conditions), we can fully integrate the extended system and obtain the exact solutions for $\mathbf{v}(t)$ and $\mathbf{V}(t)$. We underline that the splitting is global, the initial data is not assumed to be small and strong vertical shearing is allowed. The estimate of the error $\mathbf{r}$ is rigorously proven. This enables us to prove long time existence of regular solutions for 3D Euler and global existence for Navier-Stokes equations in the small anisotropic Rossby number case. The anisotropic Rossby number Ro_a is defined as

$$\mathrm{Ro}_a = \frac{a_3}{\Omega} = a_3 \mathrm{Ro}. \tag{5}$$

The aspect ratio a_3 in general is not assumed to be small; as a matter of fact, its smallness improves error estimates in convergence theorems.

As usual, we denote x_3-averaging of a function U by $\overline{U}$, and for vector fields $\overline{\mathbf{U}}(x_1, x_2) = \frac{1}{2\pi a_3}\int_0^{2\pi a_3} \mathbf{U}(x_1, x_2, x_3)\, dx_3$. We prove the theorem on behavior of solutions of 3D Euler equations which gives a rigorous sense to the Taylor-Proudman theorem for time-dependent flows. Namely, we have the following theorem.

THEOREM 1. *Let a_2, a_3 be nonresonant. Let $\mathbf{U}(t)$ be an exact solution of 3D Euler equations (1) with initial data $\mathbf{U}(0) = \mathbf{U}_0(x_1, x_2, x_3)$. Let $\sigma - 10 \geq \alpha > 5/2$, and let $M_\sigma > 0$. Then there exists $T_1 > 0$ such that if $\|\mathbf{U}(0)\|_\sigma \leq M_\sigma$, $\tilde{\mathbf{U}}(0) = \overline{\mathbf{U}}(0) = \overline{\mathbf{U}}_0(x_1, x_2)$, then*

$$\left\|\overline{\mathbf{U}}(t) - \tilde{\mathbf{U}}(t)\right\|_\alpha^2 \leq C\,\mathrm{Ro}_a$$

for $|t| \leq T_1$; C, T_1 depend only on M_σ. Here $\|\cdot\|_\sigma$ is the norm in a Sobolev space $\mathbf{H}_\sigma$ (of divergence free periodic functions) and $\tilde{\mathbf{U}}(t)$ is a solution of 2D-3C Euler equations with vertically averaged initial data $\tilde{\mathbf{U}}(0) = \overline{\mathbf{U}}(0) = \overline{\mathbf{U}}_0(x_1, x_2)$; the 2D-3C Euler equations are for velocity fields independent of x_3:

$$\partial_t \tilde{\mathbf{U}} + \tilde{\mathbf{U}} \cdot \nabla \tilde{\mathbf{U}} = -\nabla \tilde{p}, \qquad \nabla \cdot \tilde{\mathbf{U}} = 0, \tag{6}$$

where $\tilde{\mathbf{U}} \cdot \nabla$ is the classical 2D Euler advection operator. In particular,

$$\partial_t \tilde{U}^3 + \left(\tilde{U}^1 \frac{\partial}{\partial x_1} + \tilde{U}^2 \frac{\partial}{\partial x_2}\right) \tilde{U}^3 = 0. \tag{7}$$

This theorem shows that the exact x_3-averaged vector field $\overline{\mathbf{U}}(t)$ is close to a solution $\tilde{\mathbf{U}}(t)$ of 2D-3C Euler equations (6),(7) with averaged initial data $\tilde{\mathbf{U}}(0) = \overline{\mathbf{U}}_0(x_1, x_2)$. We use the terminology of Reynolds and Kassinos [7]. In their terminology, 2D-3C fields have three components and depend on two variables x_1 and x_2. A direct corollary of this theorem is separate approximate conservation of the energy of components $\overline{\mathbf{U}}(t)$ and $\mathbf{U}(t) - \tilde{\mathbf{U}}(t)$ of the flow. This implies, in particular, that $\mathbf{U}(t) - \tilde{\mathbf{U}}(t)$ is not small. The existence of classical conserved quantities for 2D Euler equations (integrals of functions of the vertical component of curl) and the above theorem imply existence of approximate conservation laws (adiabatic invariants [8]) for 3D Euler equations in the case of small Ro_a.

We note that the dependence of initial velocity fields on x_3 can be arbitrary and, in particular, we allow strongly sheared flows in x_3 (provided the fields are smooth).

The difference $\mathbf{U}(t) - \tilde{\mathbf{U}}(t)$ (or $\mathbf{U}(t) - \overline{\mathbf{U}}(t)$) which is not small can be rigorously approximated by $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$ where $\mathbf{v}(t)$ is a solution of the extended Euler equations. These equations are for the velocity fields which are three-dimensional and three-component (3D-3C). They have the following form, where $\tilde{\mathbf{U}}$ is the solution of the 2D-3C Euler equations (6),(7) with x_3-averaged initial data:

$$\partial_t \mathbf{v} = \mathbf{B}_{ex}\left(\tilde{\mathbf{U}}, \mathbf{v}\right); \tag{8}$$

here $\mathbf{B}_{ex}(\tilde{\mathbf{U}}, \mathbf{v})$ is a bilinear operator restricted to some resonant cones in Fourier space and $\overline{\mathbf{v}} = 0$. After $\tilde{\mathbf{U}}$ is found from equations (6),(7), $\mathbf{v}$ is obtained from equation (8) which is a linear equation for $\mathbf{v}$ with time and space dependent variable coefficients $\tilde{\mathbf{U}}$ (the latter with three components).

The bilinear operator $\mathbf{B}_{ex}(\tilde{\mathbf{U}}, \mathbf{v})$ is a nonlocal bilinear operator for the x_3-dependent $\mathbf{v}$ field and $\tilde{\mathbf{U}}$. It comes from a nonlocal restriction of Euler-like operators to a special class of wavenumber interactions determined by the resonance conditions. These Euler-like operators were explicitly written and its properties discussed in our previous paper [1]. Similar extended equations were written for the Navier-Stokes equations.

The remarkable property of the extended system is that (for typical, nonresonant a_2, a_3) *it splits into an infinite sequence of independent (uncoupled) subsystems of linear 12-component ordinary differential equations (ODE's)* for the corresponding Fourier modes. For a given index (n_1, n_2, n_3), $n_3 \neq 0$, they describe interactions of quadruplets $(\pm n_1, \pm n_2, n_3)$ of Fourier modes. Each subsystem couples 12 nonautonomous ODE's. The equations are (in the case of nonresonant domains)

$$\partial_t \mathbf{v}_n = i \sum_{\substack{m_1 = \pm n_1,\, m_2 = \pm n_2 \\ m_3 = n_3,\, k+m=n}} \left[-\mathbf{P}_n \left(\tilde{\mathbf{U}}_k \cdot \hat{m} \right) \mathbf{v}_m + \frac{\hat{n}}{2\,|\hat{n}|^2} \times \left(\left(\hat{k} \times \tilde{\mathbf{U}}_k \right) \times \left(\hat{m} \times \mathbf{v}_m \right) \right) \right], \tag{9}$$

where $\hat{n} = (n_1, n_2/a_2, n_3/a_3)$ and similarly for $\hat{k}$, $\hat{m}$; $\mathbf{P}_n$ is the matrix of Leray projection onto the space of divergence free periodic vector fields. The coefficients of this system are determined by the solution $\tilde{\mathbf{U}}(t, x_1, x_2)$ of 2D-3C Euler equations (6),(7). The equations are very simple and can be used for numerical computations of 3D rotating flows, since the subsystems are decoupled in a very convenient way for parallel computing.

Every subsystem (9) preserves energy and helicity separately, which gives infinitely many conservation laws for the whole system; moreover, the 3D H_s-Sobolev norms, for every s, are conserved for $\mathbf{v}$ (including enstrophy). The existence of an infinite number of conservation laws for the extended Euler equations is in contrast to the classical 3D Euler equations, where only energy and helicity are conserved [9,10]. *These conserved quantities of the extended Euler equations are the approximate adiabatic invariants [8] of 3D Euler equations in the small anisotropic Rossby number situation.*

Being so elegant and even explicitly integrable as we describe below, the extended equations describe exact 3D Euler flows with high accuracy if Ro_a is small. The vector field $\mathbf{v}(t)$ is related to $\mathbf{V}(t)$ by means of the explicitly given linear unitary operator $\mathbf{E}(\Omega t)$ (the Poincaré propagator), $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$. We show that $\mathbf{U}(t) - \tilde{\mathbf{U}}(t)$ can be rigorously approximated by $\mathbf{V}(t)$ where $\mathbf{v}(t)$ is a solution of the extended equations (8),(9). We prove the following theorem delineating the structure of exact solutions of 3D Euler equations in the small anisotropic Rossby number situation; effectively, we show that $\tilde{\mathbf{U}} + \mathbf{V} \to \mathbf{U}$ as $\mathrm{Ro}_a \to 0$, uniformly in time and strongly in some Sobolev norms.

THEOREM 2. *Let $\mathbf{U}(t)$ be an exact solution of 3D Euler equations (1) with $\mathbf{U}(0) = \mathbf{U}_0(x_1, x_2, x_3)$. Let conditions of Theorem 1 hold, $\sigma - \alpha > 39$, $\alpha > 5/2$. Let $\mathbf{E}(\Omega t)$ be the Poincaré propagator, and $\mathbf{v}(t)$ be the solution of extended Euler system with initial data $\mathbf{v}(0) = \mathbf{U}(0) - \tilde{\mathbf{U}}(0)$. Then the difference $\mathbf{U}(t) - \tilde{\mathbf{U}}(t) - \mathbf{E}(-\Omega t)\mathbf{v}(t)$ satisfies the estimate*

$$\left\|\mathbf{U}(t) - \tilde{\mathbf{U}}(t) - \mathbf{E}(-\Omega t)\mathbf{v}(t)\right\|_\alpha^2 \leq C\mathrm{Ro}_a \qquad \textit{for } |t| \leq T_1, \tag{10}$$

where T_1, C depend on M_σ.

A similar estimate holds for solutions of 3D Navier-Stokes and extended Navier-Stokes equations.

Existence of global strong solutions of 2D Euler equations (see [11–15]) imply global existence of regular solutions of extended Euler equations. For 3D Euler equations, there was no long time regularity results until recently Marsden, Ratiu and Raugel [16] proved long time existence of regular solutions for 3D Euler equations in thin spherical layers with initial data close to 2D but arbitrary large. In the small anisotropic Rossby number situation, we prove analogous results in the case of periodic boundary conditions not assuming that initial data are close to two-dimensional ones. Using Theorem 2 on approximation of solutions of 3D Euler equations by solutions of extended Euler equations, we have proven regularity of solutions of 3D Euler equations with arbitrary large initial data on arbitrary long time intervals in the small anisotropic Rossby number situation.

THEOREM 3. *Let the domain be nonresonant; $M > 0$, $T^* > 0$ both given a priori. Then there exists $\mathrm{Ro}_a^* = \mathrm{Ro}_a^*(M, T^*)$ such that if $\|\mathbf{U}(0)\|_{42} \leq M$, then for every $\mathrm{Ro}_a : 0 < \mathrm{Ro}_a \leq \mathrm{Ro}_a^*$, there exists a unique regular solution $\mathbf{U}(t)$, $0 < t < T^*$ of 3D Euler equations which belongs to H_{42} as $0 \leq t \leq T^*$. For M fixed, $T^* \to +\infty$ with $\mathrm{Ro}_a^* \to 0$. Simultaneously we can take arbitrary large (but bounded) sets of initial data: $M \to +\infty$ if $\mathrm{Ro}_a^* \to 0$.*

The problem of global regularity of solutions of 3D Navier-Stokes equations has been extensively studied by many mathematicians ([17–27] among many others) and still is an outstanding unsolved problem of applied analysis. Using the detailed description of dynamics of the extended Navier-Stokes equations and error estimates (analogue of Theorem 2 for Navier-Stokes equations), we have solved the problem in the situation of a small anisotropic Rossby number. Namely, we consider the 3D Navier-Stokes equations with the forcing $\mathbf{f}$ on the infinite time interval

$$\begin{aligned} \partial_t \mathbf{U} + \nu \,\mathrm{curl}\,\mathrm{curl}\,\mathbf{U} + \mathbf{U}\cdot\nabla\mathbf{U} + 2\Omega\mathbf{J}\mathbf{U} &= -\nabla p + \mathbf{f}, \\ \nabla\cdot\mathbf{U} &= 0. \end{aligned} \tag{11}$$

We prove the following theorem.

THEOREM 4. *Let the domain be nonresonant. Let $\mathbf{f} \in H_{42}$, $M > 0$. There exists $\mathrm{Ro}_a^0(M, \nu, \|\mathbf{f}\|_{42})$ such that if $\|\mathbf{U}(0)\|_1 \leq M$, then for every $\mathrm{Ro}_a : 0 < \mathrm{Ro}_a \leq \mathrm{Ro}_a^0$, there exists a unique regular solution $\mathbf{U}(t)$, $0 < t < +\infty$ of 3D Navier-Stokes equations which belongs to H_1 as $t \geq 0$, $\|\mathbf{U}(t)\|_1 \leq C_1(M, \nu, \|\mathbf{f}\|_{42})$, and to H_{42} as $t > 0$; and $\|\mathbf{U}(t)\|_{42} \leq C_{42}(M, \nu, \|\mathbf{f}\|_{42}, t_0)$ for every $t \geq t_0$, t_0 finite. The constants C_1, C_{42} are uniform in time.*

In Theorem 4, we can take arbitrary large M, $\|\mathbf{f}\|_{42}$ and a solution is regular if Ro_a is small enough; but the question whether or not for a small *fixed* Ro_a solutions with arbitrary large initial data blow up in H_1 in finite time is still open. In all cases, the energy estimate $(1/T)\int_0^T \|\mathbf{U}(t)\|_1^2\, dt \leq \nu^{-1}\|\mathbf{U}(0)\|^2/T + \nu^{-2}\|\mathbf{f}\|_{-1}^2$ holds. From [22], a weak solution is H_1-regular on a full measure of t. If we take $T \geq T^* = \|\mathbf{U}(0)\|^2/\nu$, $M_1^2 = 2\nu^{-2}\|\mathbf{f}\|_{-1}^2 + 2$, then $\|\mathbf{U}(t_c)\|_1 \leq M_1$ for some $t_c \in [0, T)$. Then applying Theorem 4 with a shift in time and using the equality of a weak solution of 3D Navier-Stokes equations with the regular one, we conclude that *a weak solution with arbitrary large initial data in H_0 is regular for $t > T^*$ and is attracted to the global attractor if $0 < \mathrm{Ro}_a \leq \mathrm{Ro}_a^0(M_1, \nu, \|\mathbf{f}\|_{42})$.*

Contrary to the paper by Raugel and Sell [27] where first global results on global existence of 3D Navier-Stokes equations for general enough sets of initial data were obtained, we do not need to impose conditions of the thin domain type and do not assume that initial data are close to 2D initial data in some Sobolev space. *Flows corresponding to initial conditions may have strong x_3-dependence (strong x_3-shearing) and large U^3-component.*

Surprisingly, if the vertically averaged initial data $\overline{\mathbf{U}}(0)$ satisfies zero flux boundary conditions (which are the natural boundary conditions for the 2D Euler equations in a rectangular domain), we can fully integrate the extended Euler system.

Now we give explicit formulas which express solutions of the extended Euler equations (8),(9) in terms of its time-dependent coefficients. The corresponding $\tilde{\mathbf{U}}(t,x_1,x_2)+\mathbf{V}(t,x_1,x_2,x_3)$ are in fact exact explicit solutions of 3D Euler equations in the small anisotropic Rossby number limit.

Consider the case when x_3-averaged initial data have the reflection symmetry $\bar{U}^j(-x_i+\pi a_i) = \bar{U}^j(x_i+\pi a_i)$, $i \neq j$, $\bar{U}^i(-x_i+\pi a_i) = -\bar{U}^i(x_i+\pi a_i)$, $i=1,2$ at $t=0$; it corresponds to the usual Euler zero-flux boundary condition $\bar{U}^j = 0$ as $x_j = 0$ and $x_j = \pi a_j$, $j=1,2$, at the boundary of the box. Moreover, we take $\overline{U}^3(0)$ odd with respect to reflection symmetries in x_1 and x_2 centered at $x_1 = \pi$, $x_2 = \pi a_2$; this corresponds to no-slip condition for the third averaged component. For (n_1,n_2,n_3), we denote $L(n_1,n_2,n_3) = (-n_1,-n_2,n_3)$, $n' = (n_1,n_2,0)$.

In the case of zero flux boundary conditions for averaged data, the extended Euler equations decouple into systems of ODE's describing interactions between the Fourier coefficients $\mathbf{v}_n$ and $\mathbf{v}_{Ln}$. Let

$$\xi_n(t) = \frac{1}{2}\left(\operatorname{curl}\tilde{\mathbf{U}}\right)_{2n'} \cdot e_3$$

be the Fourier coefficient of the vertical component of curl corresponding to the wavenumber $2n' = (2n_1, 2n_2, 0)$ and

$$\eta_n(t) = \frac{n_1^2 + n_2^2/a_2^2 - n_3^2/a_3^2}{|\hat{n}|}\,\tilde{U}^3_{2n'}(t),$$

where $\tilde{U}^3_{2n'}(t)$ is the third component of the solution of 2D-3C Euler system, $|\hat{n}|^2 = n_1^2 + n_2^2/a_2^2 + n_3^2/a_3^2$. The functions ξ and η are real-valued.

Then equation (8) splits into equations of the form

$$\partial_t \begin{pmatrix} \mathbf{v}_n \\ \check{\mathbf{v}}_n \end{pmatrix} = \frac{n_3}{a_3|\hat{n}|^2}\left(\xi_n(t)\mathbf{R}_n' - \eta_n(t)\, i\,|\hat{n}|\,\mathbf{I}'\right)\begin{pmatrix} \mathbf{v}_n \\ \check{\mathbf{v}}_n \end{pmatrix}, \tag{12}$$

where $\check{\mathbf{v}}_n = L\mathbf{v}_{Ln} = (-v^1_{Ln}, -v^2_{Ln}, v^3_{Ln})$ and

$$\mathbf{I}' = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \mathbf{R}_n' = \begin{pmatrix} \mathbf{0} & \mathbf{R}_n \\ \mathbf{R}_n & \mathbf{0} \end{pmatrix}, \quad \mathbf{R}_n = \begin{pmatrix} 0 & -\dfrac{n_3}{a_3} & \dfrac{n_2}{a_2} \\ \dfrac{n_3}{a_3} & 0 & -n_1 \\ -\dfrac{n_2}{a_2} & n_1 & 0 \end{pmatrix}. \tag{13}$$

Here $\mathbf{I}$ is the identity matrix and $i\mathbf{R}_n$ is the curl in Fourier space; $i\mathbf{I}'$ and $\mathbf{R}_n'$ are skew-symmetric matrices which *commute*. The commutativity property allows us to write general solutions of the system (12) explicitly. The solution is given by

$$\begin{pmatrix} \mathbf{v}_n(t) \\ \check{\mathbf{v}}_n(t) \end{pmatrix} = \exp\left(\frac{n_3}{a_3|\hat{n}|^2}\left[\tau_1(t)\mathbf{R}_n' - \tau_2(t)\, i\,|\hat{n}|\,\mathbf{I}'\right]\right)\begin{pmatrix} \mathbf{v}_n(0) \\ \check{\mathbf{v}}_n(0) \end{pmatrix}, \tag{14}$$

where

$$\tau_1(t) = \int_0^t \xi_n(s)\,ds, \quad \tau_2(t) = \int_0^t \eta_n(s)\,ds. \tag{15}$$

We note that

$$\begin{aligned} \exp\left(\frac{n_3}{a_3|\hat{n}|^2}\tau_1(t)\mathbf{R}_n'\right) &= \cos\left(\frac{n_3}{a_3|\hat{n}|}\tau_1(t)\right)\mathbf{Id} + \sin\left(\frac{n_3}{a_3|\hat{n}|}\tau_1(t)\right)\frac{1}{|\hat{n}|}\mathbf{R}_n', \\ \exp\left(\frac{n_3}{a_3|\hat{n}|}\tau_2(t) i\mathbf{I}'\right) &= \cos\left(\frac{n_3}{a_3|\hat{n}|}\tau_2(t)\right)\mathbf{Id} + \sin\left(\frac{n_3}{a_3|\hat{n}|}\tau_2(t)\right) i\mathbf{I}', \end{aligned} \tag{16}$$

where

$$\mathbf{Id} = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}, \qquad |\hat{n}|^2 = n_1^2 + \frac{n_2^2}{a_2^2} + \frac{n_3^2}{a_3^2};$$

we always take $\mathbf{v}_n \in \mathbf{H}_n$. For Navier-Stokes equations without force, the right-hand side of (14) has to be multiplied by $\exp(-\nu|\hat{n}|^2 t)$.

We recall that $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$. An important observation is that the Poincaré propagator $\mathbf{E}(\Omega t)$ commutes with $\exp((n_3/a_3|\hat{n}|^2)\tau_1(t)\mathbf{R}'_n)$ and $\exp((n_3/a_3|\hat{n}|^2)\tau_2(t)i|\hat{n}|\mathbf{I}')$ and, therefore, the phases $2\Omega t$, $\tau_1(t)$, and $\tau_2(t)$ are added in the final formulas for $\mathbf{V}(t)$. Thus, the vector field $\mathbf{V}(t)$ which approximates $\mathbf{U}(t) - \overline{\mathbf{U}}(t)$ is phase-locked to the phases $2\Omega t$, $\tau_1(t)$, and $\tau_2(t)$. The phases $\tau_1(t)$ and $\tau_2(t)$ are associated with vertically averaged vertical vorticity $\operatorname{curl}\overline{\mathbf{U}}(t) \cdot e_3$ and velocity $\overline{U}^3(t)$; the latter is multiplied (in Fourier space) by the nonlocal wave operator $(n_1^2 + n_2^2/a_2^2 - n_3^2/a_3^2)/|\hat{n}|$. The phase formulas (16) are strikingly similar to the Poincaré propagator formula (3), except that they describe mode coupling and $2\Omega t$ is replaced by $\tau_1(t)$, respectively, $\tau_2(t)$, which are phases associated to the passive scalars $\operatorname{curl}\overline{\mathbf{U}}(t) \cdot e_3$, respectively, $\overline{U}^3(t)$. The latter are passively advected scalars by 2D turbulence. Three-dimensional rotating turbulence decouples into phase turbulence for $\mathbf{V}(t, x_1, x_2, x_3)$ and 2D turbulence for vertically averaged fields, in the small anisotropic Rossby number situation.

REMARK. The formula $\mathbf{V}(t) = \mathbf{E}(-\Omega t)\mathbf{v}(t)$ is formally similar to the usual quasi-classical ansatz $e^{-i\Omega\phi(t,x)}a(t,x)$ in the theory of hyperbolic equations, where ϕ is the phase function and a is the amplitude. But in our case, the operator $\mathbf{E}(-\Omega t)$ is nonlocal; it does not depend on initial data and does not influence the dynamics of $\mathbf{v}(t)$; the formula gives the approximation uniformly for every bounded set of initial data. The determining role in the dynamics of our "amplitude" $\mathbf{v}(t)$ is played by the resonant equations on the resonant sets of wavenumbers instead of a Hamiltonian flow generated by the Hamilton-Jacobi equation.

In a recent work, Grenier [28] considers the Navier-Stokes problem (11) without a force and obtains some preliminary results; he does not explicit the equations for $\mathbf{v}$, whereas the latter were already obtained and presented by Babin *et al.* in 1994 [1]; he gives no information on $\mathbf{v}(t)$. He proves that $\mathbf{U}(t) \to \tilde{\mathbf{U}}(t)$ (the 2D-3C field) weakly in $L_2([0,T], H_1)$ using linear phase theorems. Of course this is not true in $C([0,T], H_1)$ since $\mathbf{v}(t)$ does not depend on Ω and $\mathbf{r} \to 0$ strongly. This property reflects the influence of fast background oscillations described by the Poincaré propagator $\mathbf{E}(-\Omega t)$ and is typical of a situation where fast oscillations modulated by a slowly varying amplitude are present (like in the quasi-classical case described in the previous remark). The property points to (but does not help to resolve) the difficulty in finding the dynamics of $\mathbf{v}(t)$. The similar difficulty is overcome in the quasi-classical approximation using the classical Hamilton-Jacobi theory. In our case, we solved the problem delineating the structure of resonant sets and small divisors.

The approach developed in our paper is applicable to different equations in a rotating frame, for example, to shallow water equations and Boussinesq equations, and to equations in magnetohydrodynamics. In particular, the Boussinesq case follows easily from a similar analysis, and $\tau_2(t)$ plays an essential role in the Boussinesq dynamics [29].

REFERENCES

1. A. Babin, A. Mahalov and B. Nicolaenko, Long-time averaged Euler and Navier-Stokes equations for rotating fluids, In *Structure and Dynamics of Nonlinear Waves in Fluids*, 1994 IUTAM Conference, (Edited by K. Kirchgässner and A. Mielke), pp. 145–157, World Scientific, (1995).
2. H. Poincaré, *Les Méthodes Nouvelles de la Mécanique Celeste*, Gauthier-Villars, Paris, (1892).
3. J. Moser, *Stable and Random Motions in Dynamical Systems*, Princeton University Press, Princeton, NJ, (1973).
4. C.L. Siegel and J. Moser, *Lectures on Celestial Mechanics*, Springer-Verlag, (1971).
5. H. Poincaré, Sur la précession des corps déformables, *Bull. Astronomique* **27**, 321 (1910).

6. S.L. Sobolev, Ob odnoi novoi zadache matematicheskoi fiziki, *Izvestiia Akademii Nauk SSSR, Ser. Matematicheskaia* **18** (1), 3–50 (1954).
7. W.C. Reynolds and S.C. Kassinos, A one-point model for the evolution of the Reynolds stress and structure tensors in rapidly deformed homogeneous turbulence, Presented at the *Osborne Reynolds Centernary Symposium*, UMIST, Manchester, (1994).
8. V.I. Arnold, *Mathematical Methods of Classical Mechanics*, Springer-Verlag, (1978).
9. D. Serre, First order invariants for 3D Euler equations, *Physica D* **13**, 105–136 (1984).
10. D. Serre, Invariants and symplectic degeneracies of Euler's equations for perfect incompressible fluids, *Proc. Acad. Sc. Paris* **298**, 349–352 (1984).
11. V.I. Yudovitch, Non-stationary flows of an ideal incompressible fluid, *Zh. Vychisl. Mat. i Fiziki* **3**, 1032–1066 (1963).
12. T. Kato, On classical solutions of the two-dimensional non-stationary Euler equation, *Arch. Rational Mech. Anal.* **25**, 188–200 (1968).
13. D.G. Ebin and J.E. Marsden, Groups of diffeomorphisms and the motion of an incompressible fluid, *Ann. of Math.* **92**, 102–163 (1970).
14. R. Temam, On the Euler equations of incompressible perfect fluids, *J. of Funct. Anal.* **20**, 32–43 (1975).
15. C. Bardos and S. Benachour, Domaine d'analycite des solutions de l'equation d'Euler dans ouvert de R^n, *Annali della Scuola Normale Superiore di Pisa* **4**, 647–687 (1977).
16. J.E. Marsden, T. Ratiu and G. Raugel, Équations d'Euler dans une coque sphérique mince, *Proc. Paris Acad. Sc.* (1995) (to appear).
17. J. Leray, Essai sur les mouvements plans d'un liquide visqueux que limitent des parois, *J. Math. Pures Appl.* **13** (série 9), 331–418 (1934).
18. J. Leray, Sur le mouvement d'un liquide visqueux emplissant l'espace, *Acta Math.* **63**, 193–248 (1934).
19. E. Hopf, Über die Anfangswertaufgabe für die hydrodynamischen Grundgleichungen, *Math. Nachrichten* **4**, 213–231 (1950–1951).
20. C. Foias and G. Prodi, Sur le comportement global des solutions non stationnaries des équations de Navier-Stokes en dimension 2, *Rend. Sem. Mat. Univ. Padova* **39**, 1–34 (1967).
21. O. Ladyzhenskaya, *Mathematical Theory of Viscous Incompressible Fluids*, Gordon and Breach, (1971).
22. L. Caffarelli, R. Kohn and L. Nirenberg, Partial regularity of suitable weak solutions of the Navier-Stokes equations, *Comm. Pure Appl. Math.* **35**, 771–831 (1982).
23. P. Constantin and C. Foias, *Navier-Stokes Equations*, University of Chicago Press, Chicago, (1988).
24. R. Temam, Navier-Stokes equations and nonlinear functional analysis, In *CBMS-NSF Regional Conference Series in Applied Mathematics*, SIAM, Philadelphia, (1983).
25. A.V. Babin and M.I. Vishik, *Attractors of Evolution Equations*, North-Holland, Amsterdam, (1992).
26. A. Eden, C. Foias, B. Nicolaenko and R. Temam, Exponential attractors for dissipative evolution equations, In *Research in Applied Mathematics Monographs*, Volume 37, John Wiley, (1994).
27. G. Raugel and G. Sell, Navier-Stokes equations on thin 3D domains. I. Global attractors and global regularity of solutions, *J. Amer. Math. Soc.* **6** (3), 503–568 (1993).
28. E. Grenier, Oscillatory perturbations of the Navier-Stokes equations, Report LMENS-95-6, Ecole Normale Superieure, Paris, (1995).
29. A. Babin, A. Mahalov and B. Nicolaenko, The mathematical theory of phase turbulence in Boussinesq equations, (submitted).