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Existence and Approximation Results Involving Regularized Constrained Stackelberg Problems

ABDELMALEK ABOUSSOROR

Université de Bourgogne, Département de mathématiques, Laboratoire d'Analyse Appliquée et Optimisation, BP 138-21004 Dijon Cedex, France

AND

PIERRE LORIDAN

Université de Paris 1, C. E. R. M. S. E. M., 90, rue de Tolbiac, 75634 Paris Cedex 13, France

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We are concerned with two-level optimization problems, corresponding to non-zero-sum noncooperative games, in which the set of optimal solutions to the lower level problem is not a singleton. We are interested in Stackelberg solutions first introduced by H. Von Stackelberg in the context of economic competition. Our aim is to extend some approximation and existence results given by Loridan and Morgan by considering the case when the set of constraints in the upper level problem depends on the set of optimal solutions to the lower level problem.

1. INTRODUCTION

Let X and Y be two Haussdorff topological spaces and f_i and g_i, $i = 1, 2$, be real valued functions defined on $X \times Y$. We consider the following constrained Stackelberg problem (S) in which the set of solutions to the lower level problem is not a singleton:

$$\operatorname*{Inf}_{x \in K} \sup_{y \in M_2(x)} f_1(x, y),$$

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where $M_2(x)$ is the set of optimal solutions to the lower level problem

$$P(x): \quad \operatorname*{Inf}_{\substack{y \in Y \\ g_2(x,y) \leq 0}} f_2(x, y)$$

and

$$K = \left\{ x \in X: \sup_{y \in M_2(x)} g_1(x, y) \leq 0 \right\}.$$

This formulation generalizes the one considered by Loridan and Morgan in [8]. It corresponds to a two-player Stackelberg game in which the first player called the leader has the leadership in playing the game. The leader knows everything about the second player called the follower. He knows his objective and constraint functions and that he reacts optimally, whereas the follower only knows the strategy announced by the leader. The aim of the two players is to minimize their objective functions. Since the optimal reaction set $M_2(x)$ is not a singleton, the leader has to provide himself against the worst choice of optimal strategies in $M_2(x)$ and the possible violation of his constraint by the follower, by minimizing the function $\sup_{y \in M_2(x)} f_1(x, y)$ under the constraint $u(x) \leq 0$, where $u(x) = \sup_{y \in M_2(x)} g_1(x, y)$. The problem (S) may fail to have a solution even if the decision variables x and y range over a compact set, whereas f_1 and f_2 are continuous (see Basar and Olsder [2]). This fact mainly results from the lack of lower semicontinuity for the multifunction M_2. The aim of this paper is to use two regularizations of (S) as in [7, 9] in order to obtain an approximation of (S), together with existence results for the regularized problems, generalizing the results given by Loridan and Morgan in [7–9].

The paper is organized as follows: In Section 2 we recall some definitions, present the first regularization called ε-regularization for which we need convexity and quasiconvexity assumptions in the lower level problems to give convergence results for approximate solutions and approximate marginal values. In Section 3 we recall other definitions and give the second regularization procedure based on the concept of strict ε-solutions ([7]). For such a regularization, we prove the existence of solutions to the corresponding regularized Stackelberg problem and the convergence of approximate marginal values, without convexity and quasiconvexity assumptions.

2. The ε-Regularized Problem (S_ε)

2.1. *Notations and Definitions*

Let us introduce the following notations:

$$v_2(x) = \inf_{\substack{y \in Y \\ g_2(x,y) \leq 0}} f_2(x, y), \qquad w_1(x) = \sup_{y \in M_2(x)} f_1(x, y), \qquad v_1 = \inf_{x \in K} w_1(x).$$

We assume that v_1 and $v_2(x)$, are finite numbers, for any $x \in X$:

$$M_2(x) = \{y \in Y : g_2(x, y) \leq 0 \text{ and } f_2(x, y) = v_2(x)\}$$
$$M_1 = \{x \in K : w_1(x) = v_1\}.$$

For $\varepsilon > 0$,

$$M_2(x, \varepsilon) = \{y \in Y : g_2(x, y) \leq 0 \text{ and } f_2(x, y) \leq v_2(x) + \varepsilon\}$$
$$K_\varepsilon = \left\{x \in X : \sup_{y \in M_2(x,\varepsilon)} g_1(x, y) \leqslant 0\right\}.$$

We shall assume that K_ε is nonempty for ε sufficiently small:

$$w_1(x, \varepsilon) = \sup_{y \in M_2(x,\varepsilon)} f_1(x, y), \qquad v_1(\varepsilon) = \inf_{x \in K_\varepsilon} w_1(x, \varepsilon)$$
$$M_1(\varepsilon) = \{x \in K_\varepsilon : w_1(x, \varepsilon) = v_1(\varepsilon)\}.$$

DEFINITION 2.1.1. Any $\bar{x} \in K$ verifying $v_1 = w_1(\bar{x})$ is called a Stackelberg solution to (S). This definition corresponds to the one given in [4].

DEFINITION 2.1.2. Any pair $(\bar{x}, \bar{y}) \in K \times Y$ verifying $v_1 = w_1(\bar{x})$ and $\bar{y} \in M_2(\bar{x})$ is called a Stackelberg equilibrium pair. This concept appeared in [14] for static economic problems. For the dynamic case, see for example [2, 12, 13].

Let U be a topological space. We recall the following definitions.

DEFINITION 2.1.3. If $A \subset U$, $\overline{A}^S$ is the subset of points $y \in U$ limit of a sequence of points of A; A is sequentially closed if and only if $A = \overline{A}^S$. A is sequentially compact if and only if any sequence x_n, $n \in \mathbb{N}$, of points of A, has a subsequence converging to a point of A.

DEFINITION 2.1.4. Let A_n, $n \in \mathbb{N}$, be a sequence of subsets of U:

$y \in \liminf\limits_{n \to +\infty} A_n$ if and only if there exists a sequence y_n, $n \in \mathbb{N}$, converging to y in U such that $y_n \in A_n$ for large $n \in \mathbb{N}$.

$y \in \limsup\limits_{n \to +\infty} A_n$ if and only if there exists an infinite subsequence y_n, $n \in N' \subset \mathbb{N}$, converging to y in U such that $y_n \in A_n$ for any $n \in N'$.

For $\varepsilon > 0$ let (S_ε):

$$\operatorname*{Inf}_{x \in K_\varepsilon} \sup_{y \in M_2(x,\varepsilon)} f_1(x, y),$$ where $M_2(x, \varepsilon)$ is the set of ε-approximate solutions to the lower level problem $P(x)$, previously defined;

$(S\varepsilon)$ is called the ε-regularized problem of (S), see [9, Remark 2.3].

2.2. *Assumptions and Properties for the Lower Level Problems*

We consider the following assumptions for the follower:

(F_1) f_2 is a real-valued sequentially continuous function defined on $X \times Y$.

(F_2) Y is sequentially compact.

(F_3) g_2 is a real-valued sequentially lower semicontinuous function defined on $X \times Y$.

(F_4) For any $(x, y) \in X \times Y$ such that $g_2(x, y) \leq 0$, and for any sequence x_n converging to x in X, there exists a sequence $y_n \in Y$ converging to y in Y such that $g_2(x_n, y_n) \leq 0$ for large $n \in \mathbb{N}$.

(F_5) Y is a convex subset of a real vector topological space.

(F_6) For any $x \in X$, the function $y \to f_2(x, y)$ is strictly quasiconvex on Y (Mangasarian [11]).

(F_7) For any $x \in X$, the function $y \to g_2(x, y)$ is convex on Y.

PROPOSITION 2.2.1. *If the assumptions* (F_1) *to* (F_4) *are satisfied, then for any $x \in X$ and for any sequence x_n converging to x in X, we have:*

(i) $\lim_{n \to +\infty} v_2(x_n) = v_2(x)$

(ii) $\limsup_{n \to +\infty} M_2(x_n, \varepsilon) \subset M_2(x, \varepsilon)$, *for* $\varepsilon \geq 0$

(iii) $\limsup_{n \to +\infty} M_2(x_n, \varepsilon_n) \subset M_2(x)$, *for any sequence* $\varepsilon_n \to 0$, *with* $\varepsilon_n > 0$ *for any* $n \in \mathbb{N}$.

We note that the limits of sets are given in a sequential sense, according to Definition 2.1.4.

Proof. (i) The proof is an adaptation of the one in [9, Proposition 4.3] by taking into account the inequality constraint $g_2(x, y) \leq 0$. From (F_1), (F_2), and (F_3) we deduce that $v_2(x)$ is a finite number (and $M_2(x) \neq \emptyset$), for any $x \in X$. Let $x \in X$ and x_n be a sequence converging to x in X. Let $y \in Y$ verifying $g_2(x, y) \leq 0$. From (F_4), there exists a sequence y_n converging to y in Y verifying $g_2(x_n, y_n) \leq 0$ for large $n \in \mathbb{N}$. But f_2 is sequentially continuous, so

$$\lim_{n \to +\infty} f_2(x_n, y_n) = f_2(x, y).$$

Since $v_2(x_n) \leq f_2(x_n, y_n)$ for large $n \in \mathbb{N}$, we deduce that

$$\limsup_{n \to +\infty} v_2(x_n) \leq f_2(x, y)$$

for any $y \in Y$ such that $g_2(x, y) \leq 0$. Hence,

$$\limsup_{n \to +\infty} v_2(x_n) \leq v_2(x)$$

and the sequence $v_2(x_n)$, $n \in \mathbb{N}$, is bounded above. Furthermore, it is bounded from below. If not, there would exist an infinite subsequence $v_2(x_n)$, $n \in N' \subset \mathbb{N}$, converging to $-\infty$. Then for any $A > 0$, there would exist $n_0 \in N'$ such that $v_2(x_n) < -A$ for any $n \in N'$, $n \geq n_0$. So, for any $n \geq n_0$, $n \in N'$, there would exist $y_n \in Y$, verifying $f_2(x_n, yn) \leq -A$ and $g_2(x_n, y_n) \leq 0$. From the sequential compactness of Y, there would exist a subsequence y_n, $n \in N'' \subset N'$, converging to $\bar{y} \in Y$. So

$$g_2(x, \bar{y}) \leq \liminf_{\substack{n \to +\infty \\ n \in N''}} g_2(x_{n,} y_n) \leq 0$$

and

$$-A \geq \lim_{\substack{n \to +\infty \\ n \in N''}} f_2(x_n, y_n) = f_2(x, \bar{y}) \geq v_2(x)$$

and we get a contradition by choosing $-A < v_2(x)$. Then the sequence $v_2(x_n)$, $n \in \mathbb{N}$, is bounded in $\mathbb{R}$.

Now, let $v_2(x_n)$, $n \in N_1 \subset \mathbb{N}$, an infinite subsequence converging to $\bar{v} \in \mathbb{R}$. Let $\alpha > 0$. For any $n \in N_1$, there exists $y_n^\alpha \in Y$, verifying $f_2(x_n, y_n^\alpha) \leq v_2(x_n) + \alpha$ and $g_2(x_n, y_n^\alpha) \leq 0$. From the sequential compactness of Y, there exists an infinite subsequence y_n^α, $n \in N_2 \subset N_1$, converging to $y^\alpha \in Y$. By using, respectively, (F_3) and (F_1) we get

$$g_2(x, y^\alpha) \leq \liminf_{\substack{n \to +\infty \\ n \in N_2}} g_2(x_n, y_n^\alpha) \leq 0$$

and

$$v_2(x) \leq f_2(x, y^\alpha) = \lim_{\substack{n \to +\infty \\ n \in N_2}} f_2(x_n, y_n^\alpha) \leq \lim_{\substack{n \to +\infty \\ n \in N_2}} v_2(x_n) + \alpha = \bar{v} + \alpha.$$

Since we have $v_2(x) \leq \bar{v} + \alpha$ for any $\alpha > 0$, we deduce that $v_2(x) \leq \bar{v}$. From the first part of the proof we have

$$\limsup_{n \to +\infty} v_2(x_n) \leq v_2(x)$$

and

$$\bar{v} = \lim_{\substack{n \to +\infty \\ n \in N_1}} v_2(x_n) \leq \limsup_{n \to +\infty} v_2(x_n) \leq v_2(x)$$

and $v_2(x) = \bar{v}$. Then the entire sequence $v_2(x_n)$, $n \in \mathbb{N}$, converges to $v_2(x)$.

(ii) Let $x \in X$ and let x_n be a sequence converging to x in X. We have $\limsup_{n \to +\infty} M_2(x_n, \varepsilon) \neq \emptyset$, because $M_2(x_n, \varepsilon) \neq \emptyset$, for all $n \in \mathbb{N}$, and Y is sequentially compact. Let $\bar{y} \in \limsup_{n \to +\infty} M_2(x_n, \varepsilon)$. There exists an infinite subsequence y_n, $n \in N' \subset \mathbb{N}$, converging to $\bar{y}$ in Y, verifying $y_n \in M_2(x_n, \varepsilon)$ for any $n \in N'$; that is to say, $f_2(x_n, y_n) \leq v_2(x_n) + \varepsilon$ and $g_2(x_n, y_n) \leq 0$ for any $n \in N'$. By using (i), (F_1), and (F_3) we get

$$f_2(x, \bar{y}) \leq \lim_{\substack{n \to +\infty \\ n \in N'}} v_2(x_n) + \varepsilon = v_2(x) + \varepsilon$$

and

$$g_2(x, \bar{y}) \leq \liminf_{\substack{n \to +\infty \\ n \in N'}} g_2(x_n, y_n) \leq 0$$

which means that $\bar{y} \in M_2(x, \varepsilon)$.

(iii) It is similar to the previous proof.

Remark 2.2.1. The result (ii) means that the multifunction $M_2(\cdot, \varepsilon)$ is sequentially closed graph on X (see [5]).

For $\varepsilon > 0$ let $\tilde{M}_2(x, \varepsilon) = \{y \in Y : g_2(x, y) \leq 0 \text{ and } f_2(x, y) < v_2(x) + \varepsilon\}$, the set of strict ε-solutions to $P(x)$. Then, we have the following result.

COROLLARY 2.2.1. *With the previous assumptions, for any $x \in X$, for any sequence x_n converging to x in X, and for any sequence $\varepsilon_n \to 0$ with $\varepsilon_n > 0$ for any $n \in \mathbb{N}$, we have*

$$\limsup_{n \to +\infty} \tilde{M}_2(x_n, \varepsilon_n) \subset M_2(x).$$

Proof. We have $\tilde{M}_2(x_n, \varepsilon_n) \subset M_2(x_n, \varepsilon_n)$, then, from Proposition 2.2.1 we get

$$\limsup_{n \to +\infty} \tilde{M}_2(x_n, \varepsilon_n) \subset \limsup_{n \to +\infty} M_2(x_n, \varepsilon_n) \subset M_2(x).$$

PROPOSITION 2.2.2. *We suppose that the assumptions* (F_1) *to* (F_7) *are* *satisfied. Then for any* $\varepsilon > 0$, *for any* $x \in X$, *and for any sequence* x_n *converging to* x *in* X, *we have*

$$M_2(x, \varepsilon) \subset \overline{\liminf_{n \to +\infty} M_2(x_n, \varepsilon)}^S.$$

Proof. It is an adaptation of the one given by Loridan and Morgan in [9, Proposition 6.2] for lower level problems without explicit contraints. First, let us prove that

$$\tilde{M}_2(x, \varepsilon) \subset \liminf_{n \to +\infty} M_2(x_n, \varepsilon).$$

Let $y \in \tilde{M}_2(x, \varepsilon)$; that is to say, $f_2(x, y) < v_2(x) + \varepsilon$ and $g_2(x, y) \le 0$. From (F_4) there exists a sequence y_n converging to y in Y, such that $g_2(x_n, y_n) \le 0$ for large $n \in \mathbb{N}$. By using Proposition 2.2.1 and (F_1) we get

$$\lim_{n \to +\infty} v_2(x_n) = v_2(x), \qquad \lim_{n \to +\infty} f_2(x_n, y_n) = f_2(x, y).$$

So $\lim_{n \to +\infty} (f_2(x_n, y_n) - v_2(x)) < \varepsilon$ and we deduce that $f_2(x_n, y_n) < v_2(x_n) + \varepsilon$ and $g_2(x_n, y_n) \le 0$ for large $n \in \mathbb{N}$, which means that $y_n \in \tilde{M}_2(x_n, \varepsilon)$, for large $n \in \mathbb{N}$. Then $y \in \liminf_{n \to +\infty} \tilde{M}_2(x_n, \varepsilon)$.

Now, let us prove that $M_2(x, \varepsilon) = \overline{\tilde{M}_2(x, \varepsilon)}^S$. From (F_1) and (F_3), $M_2(x, \varepsilon)$ is sequentially closed, so $\overline{\tilde{M}_2(x, \varepsilon)}^S \subset M_2(x, \varepsilon)$. Let $\bar{y} \in M_2(x, \varepsilon)$. If $\bar{y} \in \tilde{M}_2(x, \varepsilon)$ the result is obvious. So, it suffices to consider the case when $\bar{y}$ satisfies $f_2(x, \bar{y}) = v_2(x) + \varepsilon$.

Let $z \in \tilde{M}_2(x, \varepsilon)$. For any $k \in \mathbb{N}^*$ we let $\bar{y}_k = (1/k)z + (1 - 1/k)\bar{y}$. For any neighbourhood V of the origin, $\bar{y}_k - \bar{y} = (1/k)(z - \bar{y})$ belongs to V for large k. So $\bar{y}_k \to \bar{y}$ as $k \to +\infty$. From convexity assumptions we have $\bar{y}_k \in Y$ and $g_2(x, \bar{y}_k) \le (1/k)g_2(x, z) + (1 - 1/k)g_2(x, \bar{y}) \le 0 \; \forall k \ne 0$, and from the strict quasiconvexity of the function $y \to f_2(x, y)$ we have

$$f_2(x, \bar{y}_k) < \max\{f_2(x, z), f_2(x, \bar{y})\} = v_2(x) + \varepsilon$$

which means that $\bar{y}_k \in \tilde{M}_2(x, \varepsilon)$. Then $\bar{y} \in \overline{\tilde{M}_2(x, \varepsilon)}^S$ and $M_2(x, \varepsilon) = \overline{\tilde{M}_2(x, \varepsilon)}^S$ and we deduce that

$$M_2(x, \varepsilon) \subset \overline{\liminf_{n \to +\infty} M_2(x_n, \varepsilon)}^S.$$

Remark 2.2.2. We recall that the topological and the sequential definitions of lim sup and lim inf of sets are equivalent in the case when Y is a first countable space [3]. In such a case, the sequential closure is equal to the topological one.

PROPOSITION 2.2.3. *If the assumptions* (F_1) *to* (F_4) *are satisfied, then, for any* $\varepsilon > 0$, *for any* $x \in X$, *and for any sequence* x_n *converging to* x *in* X,

$$\tilde{M}_2(x, \varepsilon) \subset \liminf_{n \to +\infty} \tilde{M}_2(x_n, \varepsilon).$$

Proof. See the first part of the proof of Proposition 2.2.2.

Remark 2.2.3. The result of Proposition 2.2.3 means that the multifunction $\tilde{M}_2(\cdot, \varepsilon)$ is sequentially lower semicontinuous on X (see [5]).

Remark 2.2.4. From (F_1) and (F_6) we deduce that the function $y \to f_2(x, y)$ is quasiconvex and sequentially continuous for any $x \in X$. If the assumptions (F_1) to (F_7) (except (F_4)) are fulfilled, then $M_2(x, \varepsilon)$ and $M_2(x)$ are sequentially compact convex sets. In particular, $v_2(x)$ is a finite number for any $x \in X$.

2.3. *Assumptions and Properties for the Upper Level Problems*

We consider the following assumptions for the leader:

(L_1) g_1 is a real-valued sequentially upper semicontinuous function defined on $X \times Y$.

(L_2) For any $x \in X$ such that $\sup_{y \in M_2(x)} g_1(x, y) = 0$, there exists a sequence $x_n \in X$ converging to x in X such that $\sup_{y \in M_2(x_n)} g_1(x_n, y) < 0$ for large $n \in \mathbb{N}$.

Remark 2.3.1. The second assumption generalizes the following one which was used by Aiyoshi and Shimizu in [1] and Loridan and Morgan in [8] in the case where $M_2(x) = \{\bar{y}(x)\}$, namely:

For any $x \in X$ such that $g_1(x, \bar{y}(x)) = 0$ there exists a sequence x_n converging to x verifying $g_1(x_n, \bar{y}(x_n)) < 0$ for all $n \in \mathbb{N}$; it means that the constraint g_1 of the upper level problem is consistent with the collection of the lower level problems $P(x)$, $x \in X$.

Let us consider the following example of spaces and functions verifying (L_2).

EXAMPLE 2.3.1. Let $X = [-4, 4]$, $Y = [-5, 5]$, $g_1(x, y) = x - y - 2$, $g_2(x, y) = x + 2y - 2$ and let f_2 be the following function:

if $x \leq 0$, $f_2(x, y) = 0$ for any $y \in Y$

if $x > 0$, $f_2(x, y) = 0$ for $y = 0$; xy for $y \neq 0$.

We note that f_2 is continuous on $X \times Y$. Furthermore:

if $x \leq 0$, we have $v_2(x) = 0$ and $M_2(x) = [-5, (2 - x)/2]$

if $x > 0$, we have $v_2(x) = -5x$ and $M_2(x) = \{-5\}$.

Let $K^* = \{x \in X: \sup_{y \in M_2(x)} g_1(x, y) = 0\}$. Then we have $K = \{x \in X: x \leq -3\}$ and $K^* = \{-3\}$. We take the sequence $x_n = -3 - 1/n$, $n \in \mathbb{N}^*$, which converges to -3. Then $v_2(x_n) = 0$, $M_2(x_n) = [-5, (2 - x_n)/2]$, and $\sup_{y \in M_2(x_n)} g_1(x_n, y) = -1/n < 0$. Hence ($L_2$) is satisfied.

LEMMA 2.3.1. *If the assumptions* (L_1) *and* (L_2) *and the assumptions of Proposition* 2.2.1 *are satisfied, if X is such that:*

(D) $x_k = \lim_{n \to +\infty} x_{k,n}$ *and* $x = \lim_{k \to +\infty} x_k$ *imply that there exists a mapping* $k \to k(n)$ *increasing to* $+\infty$ *such that* $x = \lim_{n \to +\infty} x_{k(n),n}$;

then we have

($\tilde{L}_2$) *For any* $x \in X$ *such that* $\sup_{y \in M_2(x)} g_1(x, y) \leq 0$, *for any sequence* $\varepsilon_n \to 0$, *with* $\varepsilon_n > 0$ *for any* $n \in \mathbb{N}$, *there exists a sequence* $x_n \in X$ *converging to* $x \in X$ *such that* $\sup_{y \in M_2(x_n, \varepsilon_n)} g_1(x_n, y) \leq 0$ *for large* $n \in \mathbb{N}$.

Remark 2.3.2. (i) The condition (D) is satisfied if X is a first countable topological space [6].

(ii) The result ($\tilde{L}_2$) means that: for any sequence $\varepsilon_n \to 0$, with $\varepsilon_n > 0$ for any $n \in \mathbb{N}$, we have $K \subset \liminf_{n \to +\infty} K_{\varepsilon_n}$.

Proof of Lemma 2.3.1. *First case*. Let $x \in X$ such that $\sup_{y \in M_2(x)} g_1(x, y) < 0$, that is to say for any $y \in M_2(x)$, we have $g_1(x, y) < 0$. Let $\varepsilon_n \to 0$, with $\varepsilon_n > 0$ for any $n \in \mathbb{N}$. Then, there exists a sequence x_n converging to x in X such that $\sup_{y \in M_2(x_n, \varepsilon_n)} g_1(x_n, y) \leq 0$ for large $n \in \mathbb{N}$. In effect, if it is not true, then there exists a sequence $\varepsilon_n \to 0$, with $\varepsilon_n > 0$ for any $n \in \mathbb{N}$, such that for any sequence $x_n \in X$ converging to $x \in X$, there exists an infinite subsequence x_n, $n \in N' \subset \mathbb{N}$, verifying $\sup_{y \in M_2(x_n, \varepsilon_n)} g_1(x_n, y) > 0$. So, for each $n \in N'$, there exists $y_n \in M_2(x_n, \varepsilon_n)$ such that $g_1(x_n, y_n) > 0$. From the sequential compactness of Y, there exists an infinite subsequence y_n, $n \in N'' \subset N'$, converging to $\bar{y} \in M_2(x)$ (from Proposition 2.2.1). The function g_1 is sequentially upper semicontinuous; then $g_1(x, \bar{y}) \geq 0$, which contradicts $g_1(x, \bar{y}) < 0$.

Second case. Let $x \in X$ such that $\sup_{y \in M_2(x)} g_1(x, y) = 0$. From ($L_2$), there exists a sequence x_k, $k \in \mathbb{N}$, converging to x in X, verifying $\sup_{y \in M_2(x_k)} g_1(x_k, y) < 0$, for large $k \in \mathbb{N}$. From the first case, for any sequence $\varepsilon_n \to 0$, with $\varepsilon_n > 0$, for any $n \in \mathbb{N}$, there exists a sequence $x_{k,n} \in X$, $x_{k,n} \to x_k$ as $n \to +\infty$, verifying $\sup_{y \in M_2(x_{k,n}, \varepsilon_n)} g_1(x_{k,n}, y) \leq 0$ for large $n \in \mathbb{N}$. From (D), there exists a selection of integers $k(n)$ such that $x = \lim_{n \to +\infty} x_{k(n),n}$ and $k(n) \to +\infty$ as $n \to +\infty$. Let $z_n = x_{k(n),n}$, then we have $z_n \to x$ as $n \to +\infty$ and $\sup_{y \in M_2(z_n, \varepsilon_n)} g_1(z_n, y) \leq 0$ for large $n \in \mathbb{N}$.

DEFINITION 2.3.1 [5]. Let $M: X \rightrightarrows Y$ a multifunction. M is nearly sequentially lower semicontinuous at $x \in X$ if for any sequence x_n, $n \in \mathbb{N}$, converging to x in X we have

$$M(x) \subset \overline{\liminf_{n \to +\infty} M(x_n)}^S;$$

that is to say:

—either $M(x) = \emptyset$

—or for any sequence x_n, $n \in \mathbb{N}$, converging to x in X and any $y \in M(x)$, there exists a sequence y_m, $m \in \mathbb{N}$, converging to y in Y and a sequence $y_{m,n}$, $n \in \mathbb{N}$, converging to y_m in Y such that $y_{m,n} \in M(x_n)$ for large n.

M is nearly sequentially lower semicontinuous on X if it is nearly sequentially lower semicontinuous at any $x \in X$.

PROPOSITION 2.3.1. *If the assumptions of Proposition* 2.2.2 *are satisfied, if X is sequentially compact and f_1 and g_1 are sequentially lower semicontinuous, then the ε-regularized problem (S_ε) has at least one solution.*

Proof. From Proposition 2.2.2, for any $\varepsilon > 0$, for any $x \in X$ and any sequence x_n converging to x in X, we have

$$M_2(x, \varepsilon) \subset \overline{\liminf_{n \to +\infty} M_2(x_n, \varepsilon)}^S.$$

Let $u(x, \varepsilon) = \sup_{y \in M_2(x,\varepsilon)} g_1(x, y)$ and $w_1(x, \varepsilon) = \sup_{y \in M_2(x,\varepsilon)} f_1(x, y)$. The functions f_1 and g_1 are sequentially lower semicontinuous and the multifunction $M_2(\cdot, \varepsilon)$ is nearly lower semicontinuous on X, then $w_1(\cdot, \varepsilon)$ and $u(\cdot, \varepsilon)$ are sequentially lower semicontinuous on X (see [5]). So, the set $K_\varepsilon = \{x \in X : \sup_{y \in M_2(x,\varepsilon)} g_1(x, y) \leq 0\}$ is sequentially closed in the sequentially compact space X, and it is also sequentially compact. Then, $w_1(\cdot, \varepsilon)$ is sequentially lower semicontinuous on K_ε sequentially compact and the existence of solutions to (S_ε) is derived.

COROLLARY 2.3.1. *If the assumptions of Proposition* 2.3.1. *are fulfilled and if, moreover, f_1 is a sequentially upper semicontinuous function, then: for any $\varepsilon > 0$, there exists an ε-regularized Stackelberg equilibrium pair $(\bar{x}_\varepsilon, \bar{y}_\varepsilon)$ such that*

$$f_1(\bar{x}_\varepsilon, \bar{y}_\varepsilon) = \inf_{x \in K_\varepsilon} \sup_{y \in M_2(x,\varepsilon)} f_1(x, y) = v_1(\varepsilon).$$

Proof. With the previous assumptions in Proposition 2.3.1. the marginal function $w_1(x, \varepsilon) = \sup_{y \in M_2(x,\varepsilon)} f_1(x, y)$ is sequentially lower semicontinuous on K_ε sequentially compact, then there exists $\bar{x}_\varepsilon \in K_\varepsilon$ such that $w_1(\bar{x}_\varepsilon, \varepsilon) = v_1(\varepsilon)$. Finally, from the sequential compactness of $M_2(\bar{x}_\varepsilon, \varepsilon)$

and, since f_1 is a sequentially continuous function, there exists $\bar{y}_\varepsilon \in M_2(\bar{x}_\varepsilon, \varepsilon)$ such that $f_1(\bar{x}_\varepsilon, \bar{y}_\varepsilon) = w_1(\bar{x}_\varepsilon, \varepsilon) = v_1(\varepsilon)$.

Remark 2.3.3. The previous result generalizes the one given in [9, Proposition 7.4].

PROPOSITION 2.3.2. *If the assumptions of Lemma* 2.3.1 *are satisfied, if f_1 is sequentially upper semicontinuous, then*

$$\lim_{\varepsilon \to 0^+} v_1(\varepsilon) = v_1.$$

Proof. We have $K_\varepsilon \subset K \; \forall \varepsilon > 0$. Let $\varepsilon_n \to 0$, with $\varepsilon_n > 0$ for any $n \in \mathbb{N}$. We shall prove that $\lim_{n \to +\infty} v_1(\varepsilon_n) = v_1$. We have

$$w_1(x) = \sup_{y \in M_2(x)} f_1(x, y) \leq \sup_{y \in M_2(x, \varepsilon)} f_1(x, y) = w_1(x, \varepsilon)$$

so

$$v_1 = \inf_{x \in K} w_1(x) \leq \inf_{x \in K} w_1(x, \varepsilon) \leq \inf_{x \in K_\varepsilon} w_1(x, \varepsilon) = v_1(\varepsilon)$$

and we deduce that

$$v_1 \leq \liminf_{n \to +\infty} v_1(\varepsilon_n).$$

Now, let $x \in K$ and let us prove that $\limsup_{n \to +\infty} v_1(\varepsilon_n) \leq v_1$. From Lemma 2.3.1, there exists a sequence x_n converging to x in X such that $\sup_{y \in M_2(x_n, \varepsilon_n)} g_1(x_n, y) \leq 0$ for large $n \in \mathbb{N}$. Then for this sequence x_n we have

$$\limsup_{n \to +\infty} w_1(x_n, \varepsilon_n) \leq w_1(x).$$

In effect, if it is not true, then there exists $\alpha \in \mathbb{R}$, verifying

$$w_1(x) < \alpha < \limsup_{n \to +\infty} w_1(x_n, \varepsilon_n).$$

Since there exists an infinite subsequence $w_1(x_n, \varepsilon_n)$, $n \in N' \subset \mathbb{N}$, such that

$$\lim_{\substack{n \to +\infty \\ n \in N'}} w_1(x_n, \varepsilon_n) = \limsup_{n \to +\infty} w_1(x_n, \varepsilon_n),$$

there exists $n_0 \in N'$ such that $w_1(x_n, \varepsilon_n) > \alpha$ for $n \geq n_0$, $n \in N'$. So, for each $n \in N'$, $n \geq n_0$, there exists $y_n \in M_2(x_n, \varepsilon_n)$ such that $f_1(x_n, y_n) > \alpha$. From the sequential compactness of Y, there exists a subsequence y_n, $n \in N'' \subset N'$ converging to $\bar{y} \in M_2(x)$. So

$$\alpha \leq \limsup_{\substack{n \to +\infty \\ n \in N''}} f_1(x_n, y_n) \leq f_1(x, \bar{y}) \leq w_1(x)$$

which contradicts $w_1(x) < \alpha$. Then we have

$$\limsup_{n \to +\infty} w_1(x_n, \varepsilon_n) \leq w_1(x), \qquad \limsup_{n \to +\infty} v_1(\varepsilon_n) \leq w_1(x) \quad \forall x \in K$$

because $v_1(\varepsilon_n) \leq w_1(x_n, \varepsilon_n)$, since $x_n \in K_{\varepsilon_n}$ for large $n \in \mathbb{N}$. So, we obtain

$$\limsup_{n \to +\infty} v_1(\varepsilon_n) \leq v_1$$

and from the first part we deduce that $\lim_{n \to +\infty} v_1(\varepsilon_n) = v_1$. Then $\lim_{\varepsilon \to 0^+} v_1(\varepsilon) = v_1$.

COROLLARY 2.3.2. *If the assumptions of Proposition* 2.3.1 *and Proposition* 2.3.2 *are satisfied, then for any* $\varepsilon > 0$, (S_ε) *has at least one solution* x_ε *and* $\lim_{\varepsilon \to 0^+} v_1(\varepsilon) = \lim_{\varepsilon \to 0^+} w_1(x_\varepsilon, \varepsilon) = v_1$.

Proof. From Proposition 2.3.1, (S_ε) has at least one solution, $x_\varepsilon \in K_\varepsilon$. So $v_1(\varepsilon) = w_1(x_\varepsilon, \varepsilon)$ and from Proposition 2.3.2,

$$\lim_{\varepsilon \to 0^+} v_1(\varepsilon) = \lim_{\varepsilon \to 0^+} w_1(x_\varepsilon, \varepsilon) = v_1.$$

Let $M^\circ = \{x \in \overline{K}^S : \text{there exists } y \in M_2(x) \text{ verifying } f_1(x, y) = v_1\}$. Then, we have the following result.

PROPOSITION 2.3.3. *If the assumption of Proposition* 2.3.2 *are fulfilled and if, moreover,* f_1 *is sequentially lower semicontinuous, then for any sequence* $\varepsilon_n \to 0$, *with* $\varepsilon_n > 0$ *for any* $n \in \mathbb{N}$, *we have*

$$\limsup_{n \to +\infty} M_1(\varepsilon_n) \subset M^\circ.$$

Proof. If $\limsup_{n \to +\infty} M_1(\varepsilon_n) = \emptyset$, the result is obvious. Otherwise, let $\bar{x}$ be an element of $\limsup_{n \to +\infty} M_1(\varepsilon_n)$. There exists an infinite subsequence $\bar{x}_n \in X$, $n \in N' \subset \mathbb{N}$, converging to $\bar{x} \in X$, verifying $\bar{x}_n \in M_1(\varepsilon_n)$, for any $n \in N'$; that is to say, $\bar{x}_n \in K_{\varepsilon_n}$ and $w_1(\bar{x}_n, \varepsilon_n) = v_1(\varepsilon_n)$ for any $n \in N'$. So $\bar{x} \in \limsup_{n \to +\infty} K_{\varepsilon_n} \subset \overline{K}^S$, because $K_{\varepsilon_n} \subset K$ for any $n \in \mathbb{N}$. But the function f_1 is sequentially continuous and $M_2(\bar{x}_n)$ is sequentially compact for any $n \in N'$. Then for any $n \in N'$ there exists $\bar{y}_n \in M_2(\bar{x}_n)$ such that $w_1(\bar{x}_n) = f_1(\bar{x}_n, \bar{y}_n)$. From the sequential compactness of Y, there exists a subsequence $\bar{y}_n$, $n \in N'' \subset N'$, converging to $\bar{y} \in M_2(\bar{x})$ (from Proposition 2.2.1). By using the inequalities

$$v_1 \leq w_1(\bar{x}_n) = f_1(\bar{x}_n, \bar{y}_n) \leq w_1(\bar{x}_n, \varepsilon_n) = v_1(\varepsilon_n) \qquad \text{for all } n \in N''$$

and Proposition 2.3.2, we get

$$v_1 \leq \lim_{\substack{n \to +\infty \\ n \in N''}} w_1(\bar{x}_n) = \lim_{\substack{n \to +\infty \\ n \in N''}} f_1(\bar{x}_n, \bar{y}_n) = f_1(\bar{x}, \bar{y}) \leq \lim_{\substack{n \to +\infty \\ n \in N''}} v_1(\varepsilon_n) = v_1.$$

Then $f_1(\bar{x}, \bar{y}) = v_1$ and $\bar{x} \in \overline{K}^S$, that is to say, $\bar{x} \in M^\circ$.

Remark 2.3.4. Generally, the previous result does not hold when M° is replaced by M_1 (the set of solutions to (S)). This mainly results from the lack of lower semicontinuity for M_2 (see [10]).

3. The Strict ε-Regularized Problem ($\tilde{S}_\varepsilon$)

In order to avoid convexity and strict quasiconvexity assumptions we consider the set $\tilde{M}_2(x, \varepsilon)$, instead of $M_2(x, \varepsilon)$. For $\varepsilon > 0$, we define the strictly ε-regularized problem, ($\tilde{S}_\varepsilon$):

$$\operatorname*{Inf}_{x \in \tilde{K}_\varepsilon} \sup_{y \in \tilde{M}_2(x,\varepsilon)} f_1(x, y), \qquad \text{where } \tilde{M}_2(x, \varepsilon) \text{ is the set of strict}$$

ε-solutions to the lower level problem

$P(x)$ previously defined

and

$$\tilde{K}_\varepsilon = \left\{ x \in X : \sup_{y \in \tilde{M}_2(x,\varepsilon)} g_1(x, y) \leq 0 \right\}.$$

Let us denote

$$\tilde{w}_1(x, \varepsilon) = \sup_{y \in \tilde{M}_2(x,\varepsilon)} f_1(x, y), \qquad \tilde{v}_1(\varepsilon) = \inf_{x \in \tilde{K}_\varepsilon} \tilde{w}_1(x, \varepsilon).$$

Proposition 3.1. *With the assumptions of Proposition* 2.3.2 *we have*

$$\lim_{\varepsilon \to 0^+} \tilde{v}_1(\varepsilon) = v_1.$$

Proof. For any $x \in X$ and any $\varepsilon > 0$ we have $M_2(x) \subset \tilde{M}_2(x, \varepsilon) \subset M_2(x, \varepsilon)$. So

$$w_1(x) \le \tilde{w}_1(x, \varepsilon) \le w_1(x, \varepsilon).$$

Since $K_\varepsilon \subset \tilde{K}_\varepsilon \subset K$, we get

$$\inf_{x \in K} w_1(x) = v_1 \le \inf_{x \in K} \tilde{w}_1(x, \varepsilon) \le \inf_{x \in \tilde{K}_\varepsilon} \tilde{w}_1(x, \varepsilon) = \tilde{v}_1(\varepsilon)$$

and

$$\inf_{x \in \tilde{K}_\varepsilon} \tilde{w}_1(x, \varepsilon) = \tilde{v}_1(\varepsilon) \le \inf_{x \in \tilde{K}_\varepsilon} w_1(x, \varepsilon) \le \inf_{x \in K_\varepsilon} w_1(x, \varepsilon) = v_1(\varepsilon);$$

then we get $v_1 \le \tilde{v}_1(\varepsilon) \le v_1(\varepsilon)$, and from Proposition 2.3.2 we deduce that $\lim_{\varepsilon \to 0^+} \tilde{v}_1(\varepsilon) = v_1$.

PROPOSITION 3.2. *Let $\varepsilon > 0$. Supposing that the assumptions of Proposition 2.2.3 are fulfilled, together with X sequentially compact and the functions f_1 and g_1 sequentially lower semicontinuous at every $(x, y) \in X \times Y$, such that $y \in \tilde{M}_2(x, \varepsilon)$, then the problem $(\tilde{S}_\varepsilon)$ has at least one solution.*

Proof. Let us show that the marginal function $\tilde{w}_1(x, \varepsilon) = \sup_{y \in \tilde{M}_2(x,\varepsilon)} f_1(x, y)$ is sequentially lower semicontinuous on X. Let $x \in X$ and let x_n be a sequence converging to x in X and $y \in \tilde{M}_2(x, \varepsilon)$. From Proposition 2.2.3, there exists a sequence y_n converging to y such that $y_n \in \tilde{M}_2(x_n, \varepsilon)$ for large $n \in \mathbb{N}$. So

$$\tilde{w}_1(x_n, \varepsilon) = \sup_{y \in \tilde{M}_2(x_n,\varepsilon)} f_1(x_n, y) \ge f_1(x_n, y_n) \qquad \text{for large } n \in \mathbb{N}$$

and

$$\liminf_{n \to +\infty} \tilde{w}_1(x_n, \varepsilon) \ge \liminf_{n \to +\infty} f_1(x_n, y_n) \ge f_1(x, y).$$

Since $\liminf_{n \to +\infty} \tilde{w}_1(x_n, \varepsilon) \ge f_1(x, y)$ for any $y \in \tilde{M}_2(x, \varepsilon)$, we get

$$\liminf_{n \to +\infty} \tilde{w}_1(x_n, \varepsilon) \ge \tilde{w}_1(x, \varepsilon).$$

Let $\tilde{u}(x, \varepsilon) = \sup_{y \in \tilde{M}_2(x,\varepsilon)} g_1(x, y)$. By using the previous arguments we can also show that the marginal function $\tilde{u}(\cdot, \varepsilon)$ is sequentially lower semicontinuous. So, the set $\tilde{K}_\varepsilon$ is sequentially closed in the sequentially compact space X. Then $\tilde{K}_\varepsilon$ is also sequentially compact and we deduce that $(\tilde{S}_\varepsilon)$ has at least one solution.

DEFINITION 3.1 [6]. Let $M : X \rightrightarrows Y$ a multifunction. M is sequentially open graph at $x \in X$ if for any $y \in M(x)$, any sequence x_n converging to x in X, and any sequence y_n converging to y in Y, we have $y_n \in M(x_n)$ for large $n \in \mathbb{N}$. M is sequentially open graph on X if it is sequentially open graph at any $x \in X$.

PROPOSITION 3.3. *If the assumptions* (F_1) *to* (F_3) *are satisfied and if the function* g_2 *satisfies:*

$(\tilde{F}_4)$ *For any* $(x, y) \in X \times Y$ *such that* $g_2(x, y) \leq 0$, *for any sequence* (x_n, y_n) *converging to* (x, y) *in* $X \times Y$, *we have* $g_2(x_n, y_n) \leq 0$ *for large* $n \in \mathbb{N}$;

then the multifunction $\tilde{M}_2(\cdot, \varepsilon)$ *is sequentially open graph on* X $\forall \varepsilon > 0$.

Proof. The proof is merely an adaptation of the one given in [7, Proposition 3.5] in the case where there are no explicit constraints.

Remark 3.1. If the assumption $(\tilde{F}_4)$ is satisfied, then (F_4) is satisfied (take $y_n = y$ $\forall n \in \mathbb{N}$ in (F_4)).

PROPOSITION 3.4. *Let* $\varepsilon > 0$. *Suppose that the assumptions of Proposition* 3.3 *are satisfied. If, morever,* X *is sequentially compact and if we suppose that:*

(H_1) *For any* $x \in X$, *for any* $y \in \tilde{M}_2(x, \varepsilon)$, *and for any sequence* x_n *converging to* x *in* X, *there exists a sequence* y_n *converging to* y *in* Y *such that*

$$f_1(x, y) \leq \liminf_{n \to +\infty} f_1(x_n, y_n);$$

(H_2) *For any* $x \in X$, *for any* $y \in \tilde{M}_2(x, \varepsilon)$, *and for any sequence* x_n *converging to* x *in* X, *there exists a sequence* y_n *converging to* y *in* Y *such that*

$$g_1(x, y) \leq \liminf_{n \to +\infty} g_1(x_n, y_n);$$

then, there exists at least a solution to the problem $(\tilde{S}_\varepsilon)$.

Proof. It is sufficient to prove that the marginal function $\tilde{w}_1(x, \varepsilon) = \sup_{y \in \tilde{M}_2(x,\varepsilon)} f_1(x, y)$ is sequentially lower semicontinuous on X and that the set $\tilde{K}_\varepsilon$ is sequentially compact in X.

Let $x \in X$ and x_n be a sequence converging to x in X. Let $y \in \tilde{M}_2(x, \varepsilon)$. From (H_1) there exists a sequence y_n converging to y in Y such that

$$f_1(x, y) \leq \liminf_{n \to +\infty} f_1(x_n, y_n).$$

From Proposition 3.3 we have $\tilde{M}_2(\cdot, \varepsilon)$ sequentially open graph on X, so $y_n \in \tilde{M}_2(x_n, \varepsilon)$ for large $n \in N$. Since $\tilde{w}_1(x_n, \varepsilon) \geq f_1(x_n, y_n)$ for large $n \in \mathbb{N}$, we deduce that

$$\liminf_{n \to +\infty} \tilde{w}_1(x_n, \varepsilon) \geq \liminf_{n \to +\infty} f_1(x_n, y_n) \geq f_1(x, y).$$

So we get

$$\liminf_{n \to +\infty} \tilde{w}_1(x_n, \varepsilon) \geq f_1(x, y) \qquad \text{for any } y \in \tilde{M}_2(x, \varepsilon);$$

then

$$\liminf_{n \to +\infty} \tilde{w}_1(x_n, \varepsilon) \geq \tilde{w}_1(x, \varepsilon).$$

By using the previous arguments we can also prove that the marginal function $\tilde{u}(x, \varepsilon) = \sup_{y \in \tilde{M}_2(x,\varepsilon)} g_1(x, y)$ is sequentially lower semicontinuous on X and, finally, we deduce that the set $\tilde{K}_\varepsilon$ is sequentially compact. From the sequential compactness of $\tilde{K}_\varepsilon$ and the sequential lower semicontinuity of the marginal function $\tilde{w}_1(\cdot, \varepsilon)$ we deduce that the problem $(\tilde{S}_\varepsilon)$ has at least one solution.

Remark 3.2. The Stackelberg value $\tilde{v}_1(\varepsilon)$ of $(\tilde{S}_\varepsilon)$ is a real number for any $\varepsilon > 0$ but $(\tilde{S}_\varepsilon)$ does not necessarily possess a Stackelberg equilibrium pair $(\bar{x}_\varepsilon, \bar{y}_\varepsilon)$, verifying a property analogous to the one of Corollary 2.3.1, that is to say,

$$f_1(\bar{x}_\varepsilon, \bar{y}_\varepsilon) = \inf_{x \in \tilde{K}_\varepsilon} \sup_{y \in \tilde{M}_2(x,\varepsilon)} f_1(x, y) = \tilde{v}_1(\varepsilon)$$

which is contrary to (S_ε). This fact mainly results from the lack of closedness of the set $\tilde{M}_2(x, \varepsilon)$.

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