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Global well-posedness and scattering for the energy-critical, defocusing Hartree equation for radial data

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Abstract

We consider the defocusing, $\dot{H}^1$-critical Hartree equation for the radial data in all dimensions ($n \geqslant 5$). We show the global well-posedness and scattering results in the energy space. The new ingredient in this paper is that we first take advantage of the term $-\int_I \int_{|x|\leqslant A|I|^{1/2}} |u|^2 \Delta(\frac{1}{|x|})\,dx\,dt$ in the localized Morawetz identity to rule out the possibility of energy concentration, instead of the classical Morawetz estimate dependent of the nonlinearity.

Keywords: Hartree equation; Global well-posedness; Scattering; Morawetz estimate

1. Introduction

In this paper, we study the Cauchy problem for the Hartree equation

$$\begin{cases} iu_t + \Delta u = f(u), & \text{in } \mathbb{R}^n \times \mathbb{R},\ n \geqslant 5, \\ u(0) = \varphi(x), & \text{in } \mathbb{R}^n. \end{cases} \tag{1.1}$$

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Here $f(u) = (V * |u|^2)u$ is a nonlinear function of Hartree type for $V(x) = |x|^{-\gamma}$, $0 < \gamma < n$, where $*$ denotes the convolution in $\mathbb{R}^n$. In practice, we use the integral formula of (1.1)

$$u(t) = U(t)\varphi - i \int_0^t U(t-s) f\big(u(s)\big)\, ds, \tag{1.2}$$

where $U(t) = e^{it\Delta}$.

If the solution u of (1.1) has sufficient smoothness and decay at infinity, it satisfies two conservation laws:

$$M\big(u(t)\big) = \big\|u(t)\big\|_{L^2} = \|\varphi\|_{L^2},$$
$$E\big(u(t)\big) = \frac{1}{2}\big\|\nabla u(t)\big\|_{L^2}^2 + \frac{1}{4}\iint \frac{1}{|x-y|^\gamma}\big|u(t,x)\big|^2\big|u(t,y)\big|^2\, dx\, dy = E(\varphi). \tag{1.3}$$

As explained in [6], the energy is also conserved for the energy solutions $u \in C_t^0(\mathbb{R}, H^1)$.

From the viewpoint of the fractional integral, we rewrite Eq. (1.1) as

$$iu_t + \Delta u = \big((-\Delta)^{-\frac{n-\gamma}{2}}|u|^2\big)u.$$

For dimension $n \geqslant 5$, the exponent $\gamma = 4$ is the unique exponent which is energy critical in the sense that the natural scale transformation

$$u_\lambda(t,x) = \lambda^{\frac{n-2}{2}} u\big(\lambda^2 t, \lambda x\big),$$

leaves the energy invariant, in other words, the energy $E(u)$ is a dimensionless quantity.

The Cauchy problem of the Hartree equation has been intensively studied [4–10,14,15,17, 18]. With regard to the global well-posedness and scattering results, they all dealt with the $\dot{H}^1$-subcritical case ($2 < \gamma < \min(4,n)$) in the energy space or some weighted spaces. In [15], we obtained the small data scattering result for the $\dot{H}^1$-critical case in the energy space. For the large initial data for the $\dot{H}^1$-critical case ($\gamma = 4$, $n \geqslant 5$) in the energy space, the argument in [15] cannot yield the global well-posedness, even with the conservation of the energy (1.3), because the time of existence given by the local theory depends on the profile of the data as well as on the energy.

Concerning the $\dot{H}^1$-subcritical case ($2 < \gamma < \min(4,n)$), using the method of Morawetz and Strauss [16], J. Ginibre and G. Velo [6] developed the scattering theory in the energy space, where they exploited the properties of Δ and obtained the usual Morawetz estimate

$$-\int_{t_1}^{t_2}\iint \big|u(t,x)\big|^2 \frac{x}{|x|}\frac{1}{|x-y|^\gamma}\nabla\big|u(t,y)\big|^2\, dy\, dx\, dt \lesssim CE(u).$$

Later, K. Nakanishi [17] exploited the properties of $i\partial_t + \Delta$ and used a certain related Sobolev-type inequality to obtain a new Morawetz estimate

$$\iint_{\mathbb{R}^{n+1}} \frac{|t|^{1+\nu}|u(t,x)|^{\frac{2n}{n-2}}}{(|t|+|x|)^{2+\nu}}\,dx\,dt \leqslant C(E,\nu), \quad \text{for any } \nu > 0,$$

which was independent of the nonlinearity.

In this paper, we deal with the Cauchy problem of the Hartree equation with the large data for the $\dot{H}^1$-critical case ($\gamma = 4$, $n \geqslant 5$). Mimicking the approach of Bourgain [1], Killip, Visan and Zhang [13] and Tao [21] in the case of the $\dot{H}^1$-critical Schrödinger equation with the local nonlinear term, we obtain the global well-posedness and scattering results for the Hartree equation for the large radial data in $\dot{H}^1$. The new ingredient is that we take advantage of the following localized estimate for the first time

$$-\int_I \int_{|x|\leqslant A|I|^{1/2}} \left|u(t,x)\right|^2 \Delta\left(\frac{1}{|x|}\right) dx\,dt = (n-3)\int_I \int_{|x|\leqslant A|I|^{1/2}} \frac{|u(t,x)|^2}{|x|^3}\,dx\,dt \leqslant A|I|^{1/2}C(E)$$

to rule out the possibility of energy concentration, instead of the classical Morawetz estimate

$$-\int_I\iiint \left(\frac{x}{|x|}-\frac{y}{|y|}\right)\nabla V(x-y)\left|u(x)\right|^2\left|u(y)\right|^2\,dy\,dx\,dt \lesssim C(E)$$

due to the nonlinear term.

Our main result is the following global well-posedness result in the energy space.

Theorem 1.1. *Let $n \geqslant 5$, and $\varphi \in \dot{H}^1$ be radial. Then there exists a unique global solution $u \in C_t^0(\dot{H}_x^1) \cap L_t^6 L_x^{\frac{6n}{3n-8}}$ to*

$$\begin{cases} iu_t + \Delta u = \left(V * |u|^2\right)u, & \text{in } \mathbb{R}^n \times \mathbb{R}, \\ u(0) = \varphi(x), & \text{in } \mathbb{R}^n, \end{cases} \tag{1.4}$$

where $V(x) = |x|^{-4}$, and on each compact time interval $[t_-, t_+]$, we have

$$\|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}([t_-,t_+]\times\mathbb{R}^n)} \leqslant C\left(\|\varphi\|_{\dot{H}^1}\right). \tag{1.5}$$

As the right-hand side of (1.5) is independent of t_-, t_+, we can obtain the global spacetime estimate. As a direct consequence of the global $L_t^6 L_x^{\frac{6n}{3n-8}}$ estimate, we have scattering, asymptotic completeness, and uniform regularity.

Corollary 1.1. *Let φ be radial and have finite energy, and $u \in C_t^0(\dot{H}_x^1) \cap L_t^6 L_x^{\frac{6n}{3n-8}}$ be the global solution to* (1.4). *Then there exist finite energy solutions $u_\pm(t,x)$ to the free Schrödinger equation $iu_t + \Delta u = 0$ such that*

$$\left\|u_\pm(t) - u(t)\right\|_{\dot{H}^1} \to 0 \quad \text{as } t \to \pm\infty.$$

Furthermore, the maps $\varphi \mapsto u_{\pm}(0)$ *are homeomorphisms from* $\dot{H}^1(\mathbb{R}^n)$ *to* $\dot{H}^1(\mathbb{R}^n)$*. Finally, if* $\varphi \in H^s$ *for some* $s > 1$*, then* $u(t) \in H^s$ *for all time* t*, and one has the uniform bounds*

$$\sup_{t\in\mathbb{R}} \| u(t) \|_{H^s} \leqslant C\big(E(\varphi), s\big) \|\varphi\|_{H^s}.$$

The paper is organized as follows.

In Section 2, we introduce notations and the basic estimates. In Section 3, we derive the local mass conservation and Morawetz inequality. In Section 4, we discuss the local theory for (1.4). In Section 5, we obtain the perturbation theory. Finally, we prove the main theorem in Section 6.

2. Notations and basic estimates

We will often use the notations $a \lesssim b$ and $a = O(b)$ to denote the estimate $a \leqslant Cb$ for some C. The derivative operator ∇ refers to the space variable only. We also occasionally use subscripts to denote the spatial derivatives and use the summation convention over repeated indices.

We define $\langle a, b\rangle = \mathrm{Re}(a\bar{b})$, $\partial = (\partial_t, \nabla)$. For $1 \leqslant p \leqslant \infty$, we denote by p' the dual exponent, that is, $\frac{1}{p} + \frac{1}{p'} = 1$.

For any time interval I, we use $L_t^q L_x^r(I \times \mathbb{R}^n)$ to denote the mixed spacetime Lebesgue norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^n)} := \left(\int_I \|u\|_{L^r(\mathbb{R}^n)}^q \, dt \right)^{1/q}$$

with the usual modifications when $q = \infty$. When $q = r$, we abbreviate $L_t^q L_x^r$ by $L_{t,x}^q$.

We use $U(t) = e^{it\Delta}$ to denote the free group generated by the free Schrödinger equation $iu_t + \Delta u = 0$. It can commute with derivatives, and obeys the inequality

$$\big\| e^{it\Delta} f \big\|_{L^p(\mathbb{R}^n)} \lesssim |t|^{-n(\frac{1}{2}-\frac{1}{p})} \|f\|_{L^{p'}(\mathbb{R}^n)} \tag{2.1}$$

for $t \neq 0$, $2 \leqslant p \leqslant \infty$.

Let $n \geqslant 5$, we say that a pair (q, r) is admissible if

$$\frac{2}{q} = n\left(\frac{1}{2} - \frac{1}{r}\right), \quad \text{for } 2 \leqslant r \leqslant \frac{2n}{n-2}.$$

For a spacetime slab $I \times \mathbb{R}^n$, we define the *Strichartz* norm $\dot{S}^0(I)$ by

$$\|u\|_{\dot{S}^0(I)} := \sup_{(q,r)\ \text{admissible}} \|u\|_{L_t^q L_x^r(I\times\mathbb{R}^n)},$$

and define $\dot{S}^1(I)$ by

$$\|u\|_{\dot{S}^1(I)} := \|\nabla u\|_{\dot{S}^0(I)}.$$

When $n \geqslant 5$, the spaces $(\dot{S}^0(I), \|\cdot\|_{\dot{S}^0(I)})$ and $(\dot{S}^1(I), \|\cdot\|_{\dot{S}^1(I)})$ are Banach spaces, respectively.

Based on the above notations, we have the following *Strichartz* inequalities.

Lemma 2.1. *(See [11,20].) Let u be an $\dot{S}^0$ solution to the Schrödinger equation* (1.1). *Then*

$$\|u\|_{\dot{S}^0} \lesssim \|u(t_0)\|_{L_x^2} + \|f(u)\|_{L_t^{q'} L_x^{r'}(I\times\mathbb{R}^n)}$$

for any $t_0 \in I$ and any admissible pairs (q,r). The implicit constant is independent of the choice of interval I.

From Sobolev embedding, we have

Lemma 2.2. *For any function u on $I \times \mathbb{R}^n$, we have*

$$\|\nabla u\|_{L_t^\infty L_x^2} + \|\nabla u\|_{L_t^3 L_x^{\frac{6n}{3n-4}}} + \|u\|_{L_t^\infty L_x^{\frac{2n}{n-2}}} + \|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}} \lesssim \|u\|_{\dot{S}^1},$$

where all spacetime norms are on $I \times \mathbb{R}^n$.

For convenience, we introduce two abbreviated notations. For a time interval I, we denote

$$\|u\|_{X(I)} := \|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I\times\mathbb{R}^n)}, \qquad \|u\|_{W(I)} := \|\nabla u\|_{L_t^3 L_x^{\frac{6n}{3n-4}}(I\times\mathbb{R}^n)}.$$

Lemma 2.3. *Let $f(u) = (V * |u|^2)u$, where $V(x) = |x|^{-4}$. For any time interval I and $t_0 \in I$, we have*

$$\left\| \int_{t_0}^{t} e^{i(t-s)\Delta} f(u)(s,x)\,ds \right\|_{\dot{S}^1(I)} \lesssim \|u\|_{X(I)}^2 \|u\|_{W(I)}.$$

Proof. By Strichartz estimates, Hardy–Littlewood–Sobolev inequality and Hölder inequality, we have

$$\begin{aligned}
&\left\| \int_{t_0}^{t} e^{i(t-s)\Delta} f(u)(s,x)\,ds \right\|_{\dot{S}^1(I)} \\
&\quad \lesssim \|\nabla f(u)(t,x)\|_{L_t^{\frac{3}{2}} L_x^{\frac{6n}{3n+4}}(I\times\mathbb{R}^n)} \\
&\quad \lesssim \big\|\nabla u\big(V * |u|^2\big)\big\|_{L_t^{\frac{3}{2}} L_x^{\frac{6n}{3n+4}}(I\times\mathbb{R}^n)} + \big\|u\big(V * (u\nabla u)\big)\big\|_{L_t^{\frac{3}{2}} L_x^{\frac{6n}{3n+4}}(I\times\mathbb{R}^n)} \\
&\quad \lesssim \|\nabla u\|_{L_t^3 L_x^{\frac{6n}{3n-4}}(I\times\mathbb{R}^n)} \big\|V * |u|^2\big\|_{L_t^3 L_x^{\frac{3n}{4}}(I\times\mathbb{R}^n)} + \|u\|_{L_t^6 L_x^{\frac{6n}{3n-8}}(I\times\mathbb{R}^n)} \big\|V * (u\nabla u)\big\|_{L_t^2 L_x^{\frac{n}{2}}(I\times\mathbb{R}^n)} \\
&\quad \lesssim \|u\|_{X(I)}^2 \|u\|_{W(I)}. \qquad \square
\end{aligned}$$

3. Local mass conservation and Morawetz inequality

In this section, we will prove two useful estimates. One is a local mass conservation estimate and the other is a Morawetz inequality, which appears in Morawetz identity. The local mass conservation estimate is used to control the flow of mass through a region of space, and the Morawetz inequality is used to prevent concentration.

3.1. Local mass conservation

We recall a local mass conservation law that has appeared in [1,13] and [21]. For completeness, we give the sketch of the proof. Let χ be a bump function supported on the ball $B(0,1)$ that equals 1 on the ball $B(0,1/2)$. Observe that if u is a finite energy solution of (1.4), then

$$\partial_t \left|u(t,x)\right|^2 = -2\nabla \cdot \operatorname{Im}\bigl(\bar{u}\nabla u(t,x)\bigr).$$

We define

$$\operatorname{Mass}\bigl(u(t), B(x_0,R)\bigr) := \int \left|\chi\left(\frac{x-x_0}{R}\right)u(t,x)\right|^2 dx.$$

Differentiating the above quantity with respect to time, we obtain by the integration by parts

$$\begin{aligned}
\partial_t \operatorname{Mass}\bigl(u(t), B(x_0,R)\bigr) &= \int \left|\chi\left(\frac{x-x_0}{R}\right)\right|^2 \partial_t \left|u(t,x)\right|^2 dx \\
&= -2\int \left|\chi\left(\frac{x-x_0}{R}\right)\right|^2 \nabla \cdot \operatorname{Im}(\bar{u}\nabla u)\, dx \\
&= -\frac{4}{R}\int \chi\left(\frac{x-x_0}{R}\right)\nabla\chi\left(\frac{x-x_0}{R}\right)\operatorname{Im}(\bar{u}\nabla u)\, dx \\
&\lesssim \frac{1}{R}\bigl\|\nabla u(t)\bigr\|_{L^2}\bigl(\operatorname{Mass}\bigl(u(t), B(x_0,R)\bigr)\bigr)^{1/2},
\end{aligned}$$

hence, we have

$$\left|\operatorname{Mass}\bigl(u(t_1), B(x_0,R)\bigr)^{1/2} - \operatorname{Mass}\bigl(u(t_2), B(x_0,R)\bigr)^{1/2}\right| \lesssim \frac{1}{R}|t_1-t_2|. \tag{3.1}$$

This implies that if the local mass $\operatorname{Mass}(u(t), B(x_0,R))$ is large for some time t, then it can also be shown to be similarly large for times near t, by increasing the radius R if necessary to reduce the rate of change of the mass.

On the other hand, from Sobolev and Hölder inequalities, we have

$$\operatorname{Mass}\bigl(u(t), B(x_0,R)\bigr) \leqslant \left\|\chi\left(\frac{x-x_0}{R}\right)\right\|_{L_x^n}^2 \|u\|^2_{L_x^{\frac{2n}{n-2}}} \lesssim R^2\|\nabla u\|^2_{L_x^2}. \tag{3.2}$$

This gives the control of mass in small volumes.

3.2. A Morawetz inequality

To prevent the concentration of the energy, we need a Morawetz estimate.

Proposition 3.1 *(Morawetz estimate). Let u be a solution to* (1.4) *on a spacetime slab $I \times \mathbb{R}^n$. Then for any $A \geqslant 1$, we have*

$$\int_I \int_{|x|\leqslant A|I|^{1/2}} \frac{|u|^2}{|x|^3}\,dx\,dt - \int_I \iint_\Omega \left(\frac{x}{|x|} - \frac{y}{|y|}\right)\nabla V(x-y)\big|u(x)\big|^2\big|u(y)\big|^2\,dy\,dx\,dt$$
$$\lesssim A|I|^{1/2}E,$$

where $\Omega = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^n;\ |x| \leqslant A|I|^{1/2};\ |y| \leqslant A|I|^{1/2}\}$.

Remark 3.1. Since

$$-\left(\frac{x}{|x|} - \frac{y}{|y|}\right)\nabla V(x-y) = 4\frac{|x||y| - x\cdot y}{|x-y|^6}\left(\frac{1}{|x|} + \frac{1}{|y|}\right) \geqslant 0,$$

we have

$$-\int_I \iint_\Omega \left(\frac{x}{|x|} - \frac{y}{|y|}\right)\nabla V(x-y)\big|u(x)\big|^2\big|u(y)\big|^2\,dy\,dx\,dt \geqslant 0.$$

Proof of Proposition 3.1. We define $V_0^a(t) = \int a(x)|u(t,x)|^2\,dx$, then

$$M_0^a(t) =: \partial_t V_0^a(t) = 2\,\mathrm{Im}\int a_j u_j \bar{u}\,dx$$

and

$$\begin{aligned}
\partial_t M_0^a(t) &= -2\,\mathrm{Im}\int a_{jj} u_t \bar{u}\,dx - 4\,\mathrm{Im}\int a_j \bar{u}_j u_t\,dx \\
&= -\int \Delta\Delta a|u|^2\,dx + 4\,\mathrm{Re}\int a_{jk}\bar{u}_j u_k\,dx \\
&\quad - 2\,\mathrm{Re}\iint \nabla a(x)\nabla V(x-y)\big|u(y)\big|^2\big|u(x)\big|^2\,dx\,dy \\
&= -\int \Delta\Delta a|u|^2\,dx + 4\,\mathrm{Re}\int a_{jk}\bar{u}_j u_k\,dx \\
&\quad - \mathrm{Re}\iint \big(\nabla a(x) - \nabla a(y)\big)\nabla V(x-y)\big|u(y)\big|^2\big|u(x)\big|^2\,dx\,dy,
\end{aligned}$$

where we use the symmetry of $a(x)$ and $V(x)$. Let $R > 0$ and let ζ be a bump function adapted to the ball $|x| \leqslant R$ which equals 1 on the ball $|x| \leqslant R/2$. We set $a(x) := |x|\zeta(x)$.

For $|x| \leqslant R/2$, we have

$$a_j = \frac{x_j}{|x|}, \quad a_{jk} = \frac{\delta_{jk}}{|x|} - \frac{x_j x_k}{|x|^3}, \quad \Delta a = \frac{n-1}{|x|}, \quad -\Delta\Delta a = \frac{(n-1)(n-3)}{|x|^3},$$

and for $R/2 \leqslant |x| \leqslant R$, we have bounds

$$a_j = O(1), \quad a_{jk} = O\big(R^{-1}\big), \quad \Delta\Delta a = O\big(R^{-3}\big).$$

Thus we have

$$\partial_t M_0^a(t) = (n-1)(n-3)\int_{|x|\leqslant R/2} \frac{|u|^2}{|x|^3}\,dx + 4\int_{|x|\leqslant R/2} \frac{|\nabla u|^2 - |\partial_r u|^2}{|x|}\,dx$$
$$- \iint_{\Omega_1} \left(\frac{x}{|x|} - \frac{y}{|y|}\right)\nabla V(x-y)\big|u(x)\big|^2\big|u(y)\big|^2\,dy\,dx$$
$$+ O\biggl(\int_{|x|\sim R} \left(\frac{|u|^2}{R^3} + \frac{|\nabla u|^2}{R}\right)dx\biggr)$$
$$+ O\biggl(\iint_{\Omega_2} \bigl(a_j(x) - a_j(y)\bigr)\frac{x_j - y_j}{|x-y|^{\gamma+2}}\big|u(x)\big|^2\big|u(y)\big|^2\,dy\,dx\biggr),$$

where $\gamma = 4$,

$$\Omega_1 = \bigl\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n;\ |x| \leqslant R/2,\ |y| \leqslant R/2\bigr\},$$
$$\Omega_2 = \bigl\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n;\ |x| \sim R\bigr\} \cup \bigl\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n;\ |y| \sim R\bigr\}.$$

Meanwhile, using Hardy's inequality, we have

$$\int_{|x|\sim R} \left(\frac{|u|^2}{R^3} + \frac{|\nabla u|^2}{R}\right)dx \lesssim R^{-1}E,$$

$$\left|\iint_{\Omega_2} \bigl(a_j(x) - a_j(y)\bigr)\frac{x_j - y_j}{|x-y|^{\gamma+2}}\big|u(x)\big|^2\big|u(y)\big|^2\,dy\,dx\right|$$
$$\leqslant \left|\int\int_{\Omega_2:\ |x-y|\leqslant R/4} \bigl(a_j(x) - a_j(y)\bigr)\frac{x_j - y_j}{|x-y|^{\gamma+2}}\big|u(x)\big|^2\big|u(y)\big|^2\,dy\,dx\right|$$
$$+ \left|\int\int_{\Omega_2:\ |x-y|\geqslant R/4} \bigl(a_j(x) - a_j(y)\bigr)\frac{x_j - y_j}{|x-y|^{\gamma+2}}\big|u(x)\big|^2\big|u(y)\big|^2\,dy\,dx\right|$$
$$\lesssim R^{-1}\left|\iint_{\Omega_2} \frac{1}{|x-y|^{\gamma}}\big|u(x)\big|^2\big|u(y)\big|^2\,dy\,dx\right|$$
$$\lesssim R^{-1}E.$$

Moreover, from Sobolev and Hölder inequalities, we have

$$M_0^a(t) \lesssim \int_{|x|\lesssim R} |u||\nabla u| \lesssim \|u\|_{L_x^{\frac{2n}{n-2}}}\|\nabla u\|_{L_x^2}\biggl(\int_{|x|\lesssim R} dx\biggr)^{1/n} \lesssim RE.$$

So if we integrate by parts on a time interval I and take $R = 2A|I|^{1/2}$, we obtain

$$\int_I \int_{|x|\leqslant A|I|^{1/2}} \frac{|u|^2}{|x|^3}\,dx\,dt - \iiint_{I\ \Omega} \left(\frac{x}{|x|} - \frac{y}{|y|}\right)\nabla V(x-y)\big|u(x)\big|^2\big|u(y)\big|^2\,dy\,dx\,dt$$
$$\lesssim A|I|^{1/2}E$$

for $n \geqslant 4$. The proof is completed. □

4. Local theory

In this section, we develop a local well-posedness and blow-up criterion for the $\dot H^1$-critical Hartree equation. First, we have

Proposition 4.1 *(Local well-posedness). Let $u(t_0) \in \dot H^1$, and I be a compact time interval that contains t_0 such that*

$$\big\|U(t-t_0)u(t_0)\big\|_{X(I)} \leqslant \eta,$$

for a sufficiently small absolute constant $\eta > 0$. Then there exists a unique solution $u \in C_t^0 \dot H_x^1$ to (1.4) *on $I \times \mathbb{R}^n$ such that*

$$\|u\|_{X(I)} \leqslant C\big(\big\|u(t_0)\big\|_{\dot H^1}\big).$$

Proof. The proof of this proposition is standard and based on the contraction mapping arguments. We define the solution map to be

$$\Phi(u)(t) := U(t-t_0)u(t_0) - i\int_{t_0}^{t} U(t-s)f\big(u(s)\big)\,ds,$$

then Φ is a map from

$$\mathcal{B} = \big\{u\colon\ \|u\|_{X(I)} \leqslant 2\eta,\ \|u\|_{W(I)} \leqslant 2C\big\|u(t_0)\big\|_{\dot H^1}\big\}$$

with the metric

$$\|u\|_{\mathcal{B}} = \|u\|_{X(I)} + \|u\|_{W(I)}$$

onto itself because

$$\big\|\Phi(u)\big\|_{X(I)} \leqslant \big\|U(t-t_0)u(t_0)\big\|_{X(I)} + C\|u\|_{X(I)}^2\|u\|_{W(I)} \leqslant \eta + 8C\eta^2\big\|u(t_0)\big\|_{\dot H^1} \leqslant 2\eta,$$
$$\big\|\Phi(u)\big\|_{W(I)} \leqslant C\big\|u(t_0)\big\|_{\dot H^1} + C\|u\|_{X(I)}^2\|u\|_{W(I)} \leqslant C\big\|u(t_0)\big\|_{\dot H^1} + 8C\eta^2\big\|u(t_0)\big\|_{\dot H^1}$$
$$\leqslant 2C\big\|u(t_0)\big\|_{\dot H^1},$$

as long as η is chosen sufficiently small. It suffices to prove Φ is a contraction map. Let $u, v \in \mathcal{B}$, then

$$\begin{aligned}\left\|\Phi(u)-\Phi(v)\right\|_{W(I)} &\leqslant \left\|\int_0^t U(t-s)\big(V*\big(|u|^2-\bar{v}u\big)\big)u(s,x)\,ds\right\|_{W(I)}\\ &\quad+\left\|\int_0^t U(t-s)\big(V*\big(\bar{v}u-|v|^2\big)\big)u(s,x)\,ds\right\|_{W(I)}\\ &\quad+\left\|\int_0^t U(t-s)\big(V*|v|^2\big)(u-v)(s,x)\,ds\right\|_{W(I)}.\end{aligned}$$

By Lemma 2.3, we have

$$\begin{aligned}\left\|\Phi(u)-\Phi(v)\right\|_{W(I)} &\leqslant \|u-v\|_{X(I)}\big(\|u\|_{W(I)}\|u\|_{X(I)}+\|u\|_{W(I)}\|v\|_{X(I)}+\|v\|_{W(I)}\|v\|_{X(I)}\big)\\ &\quad+\|u-v\|_{W(I)}\big(\|u\|_{X(I)}\|u\|_{X(I)}+\|u\|_{X(I)}\|v\|_{X(I)}+\|v\|_{X(I)}\|v\|_{X(I)}\big)\\ &\leqslant 12C\eta\left\|u(t_0)\right\|_{\dot{H}^1}\|u-v\|_{X(I)}+12\eta^2\|u-v\|_{W(I)}\\ &\leqslant \frac{1}{4}\big(\|u-v\|_{X(I)}+\|u-v\|_{W(I)}\big).\end{aligned}$$

In the same way, we have

$$\begin{aligned}\left\|\Phi(u)-\Phi(v)\right\|_{X(I)} &\leqslant 12C\eta\left\|u(t_0)\right\|_{\dot{H}^1}\|u-v\|_{X(I)}+12\eta^2\|u-v\|_{W(I)}\\ &\leqslant \frac{1}{4}\big(\|u-v\|_{X(I)}+\|u-v\|_{W(I)}\big)\end{aligned}$$

as long as η is chosen sufficiently small. Then the contraction mapping theorem implies the existence of the unique solution to (1.4) on I. In addition, Strichartz estimates also guarantee that the solution $u\in C_t^0\dot{H}_x^1$. □

Next, we give the blow-up criterion of the strong solution for (1.4) (i.e. the solution lies in $C_t^0\dot{H}_x^1$). The usual form is similar to those in [2,12], which is in the form of a maximal interval of existence. For convenience, we obtain

Proposition 4.2 *(Blow-up criterion). Let $\varphi\in\dot{H}^1$, and let u be a strong solution to* (1.4) *on the slab* $[0,T)\times\mathbb{R}^n$ *such that*

$$\|u\|_{X([0,T))}<\infty.$$

Then there exists $\delta>0$ such that the solution u extends to a strong solution to (1.4) *on the slab* $[0,T+\delta]\times\mathbb{R}^n$.

Proof. By the absolute continuity of integrals, there exists $t_0\in[0,T)$, such that

$$\|u\|_{X([t_0,T))}\leqslant\eta/4,$$

then by Lemma 2.3, we have

$$\|u\|_{W([t_0,T))} \lesssim \|u(t_0)\|_{\dot{H}^1} + \|u\|_{X([t_0,T))}^2 \|u\|_{W([t_0,T))},$$

therefore

$$\|u\|_{W([t_0,T))} \lesssim \|u(t_0)\|_{\dot{H}^1}.$$

Now we write

$$U(t-t_0)u(t_0) = u(t) + i\int_{t_0}^{t} U(t-s)\big(V*|u|^2\big)u(s,x)\,ds,$$

then

$$\begin{aligned}\big\|U(t-t_0)u(t_0)\big\|_{X([t_0,T))} &\leqslant \|u\|_{X([t_0,T))} + C\|u\|_{X([t_0,T))}^2\|u\|_{W([t_0,T))}\\ &\leqslant \frac{\eta}{4} + C\eta^2\big\|u(t_0)\big\|_{\dot{H}^1} \leqslant \frac{\eta}{2}.\end{aligned}$$

By the absolute continuity of integrals again, there exists δ, such that

$$\big\|U(t-t_0)u(t_0)\big\|_{X([t_0,T+\delta))} \leqslant \eta.$$

Thus we may apply Proposition 4.1 on the interval $[t_0, T+\delta]$ to complete the proof. □

In other words, this lemma asserts that if $[t_0, T^*)$ is the maximal interval of existence and $T^* < \infty$, then

$$\|u\|_{X([t_0,T^*))} = \infty.$$

5. Perturbation result

In this section, we obtain the perturbation for the Hartree equation, which shows that the solution cannot be large if the linear part of the solution is not large. This is an analogue of Lemma 3.2 in [21], and later, Killip, Visan and Zhang [13] gave the similar perturbation result for the Schrödinger equation with the quadric potentials.

Lemma 5.1 *(Perturbation lemma). Let u be a solution to* (1.4) *on* $I = [t_1, t_2]$ *such that*

$$\frac{1}{2}\eta \leqslant \|u\|_{X(I)} \leqslant \eta,$$

where η *is sufficiently small constant depending on the norm of the initial data, then*

$$\|u\|_{\dot{S}^1(I)} \lesssim 1, \qquad \|u_k\|_{X(I)} \geqslant \frac{1}{4}\eta,$$

where $u_k(t) = U(t-t_k)u(t_k)$ *for* $k = 1, 2$.

Proof. From Strichartz estimate and Lemma 2.3, we obtain

$$\begin{aligned}\|u\|_{\dot S^1(I)} &\lesssim \big\|u(t_1)\big\|_{\dot H^1} + \|u\|_{X(I)}^2 \|u\|_{W(I)} \\ &\lesssim \big\|u(t_1)\big\|_{\dot H^1} + \|u\|_{X(I)}^2 \|u\|_{\dot S^1(I)} \\ &\lesssim \big\|u(t_1)\big\|_{\dot H^1} + \eta^2 \|u\|_{\dot S^1(I)}.\end{aligned}$$

If η is sufficiently small, we have the first claim

$$\|u\|_{\dot S^1(I)} \lesssim 1.$$

As for the second claim, we give the proof for $k=1$, the case $k=2$ is similar. Using Strichartz estimate and Lemma 2.3 again, we have

$$\|u-u_1\|_{X(I)} \lesssim \eta^2 \|u\|_{\dot S^1(I)} \lesssim \eta^2,$$

therefore, the second claim follows by the triangle inequality and choosing η sufficiently small. □

6. Global well-posedness

In this section, we give the proof of Theorem 1.1. The new ingredient is that we first take advantage of the estimate of the term $\int_I \int_{|x|\leqslant A|I|^{1/2}} \frac{|u|^2}{|x|^3}\,dx\,dt$ in the localized Morawetz identity to rule out the possibility of energy concentration, which is independent of the nonlinear term. For the Schrödinger equation, Bourgain [1] and Tao [21] used the classical Morawetz estimate, which depends on the nonlinearity, to prevent the concentration.

For readability, we first take some constants

$$C_1 = 6n, \quad C_2 = 3, \quad C_3 = 18n, \tag{6.1}$$

which come from several constraints in the rest of this section. All implicit constants in this section are permitted to depend on the dimension n and the energy.

Fix E, $[t_-, t_+]$, u. We may assume that the energy is large, $E > c > 0$, otherwise the claim follows from the small energy theory [15]. From the boundedness of energy and Sobolev embedding, we can obtain

$$\big\|u(t)\big\|_{\dot H^1_x} + \big\|u(t)\big\|_{L_x^{\frac{2n}{n-2}}} \lesssim 1 \tag{6.2}$$

for all $t \in [t_-, t_+]$.

Assume that the solution u already exists on $[t_-, t_+]$. By Proposition 4.2, it suffices to obtain a priori estimate

$$\|u\|_{X([t_-,t_+])} \leqslant O(1), \tag{6.3}$$

where $O(1)$ is independent of t_-, t_+.

We may assume that

$$\|u\|_{X([t_-,t_+])} \geqslant 2\eta,$$

otherwise it is trivial. We divide $[t_-, t_+]$ into J subintervals $I_j = [t_j, t_{j+1}]$ for some $J \geqslant 2$ such that

$$\frac{\eta}{2} \leqslant \|u\|_{X(I_j)} \leqslant \eta, \tag{6.4}$$

where η is a small constant depending on the dimension n and the energy. As a consequence, it suffices to estimate the number J.

Now let $u_\pm = U(t - t_\pm)u(t_\pm)$. By Sobolev embedding and Strichartz estimates, we have

$$\|u_\pm\|_{X([t_-,t_+])} \lesssim 1. \tag{6.5}$$

We use the following definition of Tao [21].

Definition 6.1. We call I_j exceptional if

$$\|u_\pm\|_{X(I_j)} > \eta^{C_3}$$

for at least one sign $\pm$. Otherwise, we call I_j unexceptional.

From (6.5), we obtain the upper bound on the number of exceptional intervals, $O(\eta^{-6C_3})$. We may assume that there exist unexceptional intervals, otherwise the claim would follow from this bound and (6.4). Therefore, it suffices to compute the number of unexceptional intervals.

As in [1,13] and [21], we first prove the existence of a bubble of mass concentration in each unexceptional interval.

Proposition 6.1 *(Existence of a bubble). Let I_j be an unexceptional interval. Then there exists $x_j \in \mathbb{R}^n$ such that*

$$\mathrm{Mass}\big(u(t), B\big(x_j, \eta^{-C_1}|I_j|^{1/2}\big)\big) \gtrsim \eta^{C_1}|I_j|$$

for all $t \in I_j$.

Proof. By time translation invariance and scale invariance, we may assume that $I_j = [0, 1]$. We subdivide I_j further into $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. By (6.4) and the pigeonhole principle and time reflection symmetry if necessary, we may assume that

$$\|u\|_{X([\frac{1}{2},1])} \geqslant \frac{\eta}{4}.$$

Thus by Lemma 5.1, we have

$$\left\| U\left(t - \frac{1}{2}\right)u\left(\frac{1}{2}\right)\right\|_{X([\frac{1}{2},1])} \geqslant \frac{\eta}{8}. \tag{6.6}$$

By *Duhamel* formula, we have

$$U\left(t-\frac{1}{2}\right)u\left(\frac{1}{2}\right) = U(t-t_-)u(t_-) - i\int_0^{\frac{1}{2}} U(t-s)f\big(u(s)\big)\,ds - i\int_{t_-}^{0} U(t-s)f\big(u(s)\big)\,ds. \tag{6.7}$$

Since [0, 1] is unexceptional interval, we have

$$\big\|U(t-t_-)u(t_-)\big\|_{X([\frac{1}{2},1])} = \big\|u_-(t)\big\|_{X([\frac{1}{2},1])} \leqslant \eta^{C_3}.$$

On the other hand, by (6.4), Lemmas 2.2, 2.3 and 5.1, we have

$$\left\|\int_0^{\frac{1}{2}} U(t-s)f\big(u(s)\big)\,ds\right\|_{X([\frac{1}{2},1])} \lesssim \|u\|^2_{X([0,\frac{1}{2}])}\|u\|_{W([0,\frac{1}{2}])} \lesssim \eta^2\|u\|_{\dot{S}^1([0,\frac{1}{2}])} \lesssim \eta^2.$$

Thus the triangle inequality implies that

$$\left\|\int_{t_-}^{0} U(t-s)f\big(u(s)\big)\,ds\right\|_{X([\frac{1}{2},1])} \geqslant \frac{1}{100}\eta,$$

provided η is chosen sufficiently small. Hence, if we define

$$v(t) := \int_{t_-}^{0} U(t-s)f\big(u(s)\big)\,ds,$$

then we have

$$\|v\|_{X([\frac{1}{2},1])} \geqslant \frac{1}{100}\eta. \tag{6.8}$$

Next, we estimate the upper bound on v. We have by (6.7) and the triangle inequality

$$\|v\|_{\dot{S}^1([\frac{1}{2},1])} \leqslant \left\|U\left(t-\frac{1}{2}\right)u\left(\frac{1}{2}\right)\right\|_{\dot{S}^1([\frac{1}{2},1])} + \big\|U(t-t_-)u(t_-)\big\|_{\dot{S}^1([\frac{1}{2},1])} + \left\|\int_0^{\frac{1}{2}} U(t-s)f\big(u(s)\big)\,ds\right\|_{\dot{S}^1([\frac{1}{2},1])}$$

$$\begin{aligned}
&\lesssim \|u\|_{\dot H^1} + \|u\|^2_{X([0,\frac12])}\|u\|_{W([0,\frac12])}\\
&\lesssim \|u\|_{\dot H^1} + \|u\|^2_{X([0,\frac12])}\|u\|_{\dot S^1([0,\frac12])}\\
&\lesssim 1, \qquad (6.9)
\end{aligned}$$

where we have used Strichartz estimate, (6.4) and Lemma 5.1. □

We shall need some additional regularity control on v. For any $h \in \mathbb{R}^n$, let $u^{(h)}$ denote the translation of u by h, i.e. $u^{(h)}(t,x) = u(t,x-h)$.

Lemma 6.1. *Let χ be a bump function supported on the ball $B(0,1)$ of total mass one, and define*

$$v_{av}(t,x) = \int \chi(y) v\big(t, x + \eta^{C_2} y\big)\,dy,$$

then we have

$$\|v - v_{av}\|_{X([\frac12,1])} \lesssim \eta^{C_2}.$$

Proof. By the chain rule, Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}
\big\|\nabla f(u)(s)\big\|_{L_x^{\frac{2n}{n+4}}} &\leqslant \big\|\big(V * |u|^2\big)\nabla u\big\|_{L_x^{\frac{2n}{n+4}}} + \big\|u\big(V * \nabla|u|^2\big)\big\|_{L_x^{\frac{2n}{n+4}}}\\
&\leqslant \|\nabla u\|_{L^2}\big\|V * |u|^2\big\|_{L^{\frac n2}} + \|u\|_{L^{\frac{2n}{n-2}}}\big\|V * \nabla|u|^2\big\|_{L^{\frac n3}}\\
&\leqslant \|\nabla u\|_{L^2}\big\||u|^2\big\|_{L^{\frac{n}{n-2}}} + \|u\|_{L^{\frac{2n}{n-2}}}\big\|\nabla|u|^2\big\|_{L^{\frac{n}{n-1}}}\\
&\lesssim 1,
\end{aligned}$$

it follows by (2.1)

$$\|\nabla v\|_{L_t^\infty L_x^{\frac{2n}{n-4}}([\frac12,1]\times\mathbb{R}^n)} \leqslant \sup_{t\in[\frac12,1]} \int_{t_-}^{0} \frac{1}{|t-s|^2}\big\|\nabla f(u)(s)\big\|_{L_x^{\frac{2n}{n+4}}}\,ds \lesssim 1.$$

From (6.9) and interpolation, we have

$$\begin{aligned}
\|\nabla v\|_{L_t^\infty L_x^{\frac{6n}{3n-8}}([\frac12,1]\times\mathbb{R}^n)} &\leqslant \|\nabla v\|^{2/3}_{L_t^\infty L_x^{\frac{2n}{n-4}}([\frac12,1]\times\mathbb{R}^n)}\|\nabla v\|^{1/3}_{L_t^\infty L_x^2([\frac12,1]\times\mathbb{R}^n)}\\
&\lesssim 1.
\end{aligned}$$

From the fundamental theorem of calculus, we have

$$\big\|v - v^{(h)}\big\|_{L_t^\infty L_x^{\frac{6n}{3n-8}}([\frac12,1]\times\mathbb{R}^n)} \lesssim |h|.$$

This implies

$$\begin{aligned}\|v-v_{av}\|_{L_t^\infty L_x^{\frac{6n}{3n-8}}([\frac12,1]\times\mathbb{R}^n)} &\leqslant \int \chi(y)\big\|v\big(t,x+\eta^{C_2}y\big)-v(x)\big\|_{L_t^\infty L_x^{\frac{6n}{3n-8}}([\frac12,1]\times\mathbb{R}^n)}\,dy\\ &\lesssim \int \chi(y)\big|\eta^{C_2}y\big|\,dy\\ &\lesssim \eta^{C_2}.\end{aligned}$$

Hence from Hölder inequality, we obtain

$$\|v-v_{av}\|_{X([\frac12,1])} \lesssim \|v-v_{av}\|_{L_t^\infty L_x^{\frac{6n}{3n-8}}([\frac12,1]\times\mathbb{R}^n)} \lesssim \eta^{C_2}.$$

This completes the proof of lemma. □

Now we return to the proof of Proposition 6.1. By Lemma 6.1 and (6.8), we have

$$\|v_{av}\|_{X([\frac12,1])} \gtrsim \eta. \tag{6.10}$$

On the other hand, by Hölder inequality, Young inequalities and (6.9), we have

$$\begin{aligned}\|v_{av}\|_{L_t^{\frac{2(3n-8)}{n-2}} L_x^{\frac{2n}{n-2}}([\frac12,1]\times\mathbb{R}^n)} &\lesssim \|v_{av}\|_{L_t^\infty L_x^{\frac{2n}{n-2}}([\frac12,1]\times\mathbb{R}^n)}\\ &\lesssim \|v\|_{L_t^\infty L_x^{\frac{2n}{n-2}}([\frac12,1]\times\mathbb{R}^n)}\\ &\lesssim 1.\end{aligned}$$

Interpolating with (6.10) gives

$$\begin{aligned}\|v_{av}\|_{L^\infty_{t,x}([\frac12,1]\times\mathbb{R}^n))} &\gtrsim \|v_{av}\|_{X([\frac12,1])}^{\frac{3n-6}{2}} \|v_{av}\|^{-\frac{3n-8}{2}}_{L_t^{\frac{2(3n-8)}{n-2}} L_x^{\frac{2n}{n-2}}([\frac12,1]\times\mathbb{R}^n)}\\ &\gtrsim \eta^{\frac{3n-6}{2}}.\end{aligned}$$

Thus there exists $(s_j,x_j)\in[\frac12,1]\times\mathbb{R}^n$ such that

$$\big|v_{av}(s_j,x_j)\big| \gtrsim \eta^{\frac{3n-6}{2}}.$$

Hence, by Cauchy–Schwarz inequality, we have

$$\begin{aligned}\big|v_{av}(s_j,x_j)\big| &= \left|\int \chi(y)v\big(s_j,x_j+\eta^{C_2}y\big)\,dy\right|\\ &= \eta^{-nC_2}\left|\int \chi\left(\frac{x-x_j}{\eta^{C_2}}\right)v(s_j,x)\,dx\right|\\ &\lesssim \eta^{-nC_2}\eta^{\frac n2 C_2}\,\mathrm{Mass}\big(v(s_j),B\big(x_j,\eta^{C_2}\big)\big)^{1/2},\end{aligned}$$

that is

$$\mathrm{Mass}\big(v(s_j), B\big(x_j, \eta^{C_2}\big)\big) \gtrsim \eta^{3n-6+nC_2} \gtrsim \eta^{C_1}. \tag{6.11}$$

Observe that (3.1) also holds for v. If we take $R = \eta^{-C_1}$ and choose η sufficiently small, we have

$$\begin{aligned}\mathrm{Mass}\big(v(t), B\big(x_j, \eta^{-C_1}\big)\big) &\gtrsim \left(\mathrm{Mass}\big(v(s_j), B\big(x_j, \eta^{-C_1}\big)\big)^{1/2} - \frac{1}{\eta^{-C_1}}\right)^2 \\ &\gtrsim \big(\mathrm{Mass}\big(v(s_j), B\big(x_j, \eta^{C_2}\big)\big)^{1/2} - \eta^{C_1}\big)^2 \\ &\gtrsim \eta^{C_1}\end{aligned} \tag{6.12}$$

for all $t \in [0, 1]$.

The last step is to show that this mass concentration holds for u. We first show mass concentration for u at time 0.

Since $[0, 1]$ is unexceptional interval, by the pigeonhole principle, there is a $\tau_j \in [0, 1]$ such that

$$\big\|u_-(\tau_j)\big\|_{L_x^{\frac{6n}{3n-8}}} \lesssim \eta^{C_3},$$

and so by Hölder inequality,

$$\begin{aligned}\mathrm{Mass}\big(u_-(\tau_j), B\big(x_j, \eta^{-C_1}\big)\big) &\lesssim \left\|\chi\left(\frac{x - x_j}{\eta^{-C_1}}\right)\right\|^2_{L_x^{\frac{3n}{4}}} \big\|u_-(\tau_j)\big\|^2_{L_x^{\frac{6n}{3n-8}}} \\ &\lesssim \eta^{-\frac{8}{3}C_1+2C_3} \lesssim \eta^{2C_1}.\end{aligned}$$

From (3.1), we have

$$\mathrm{Mass}\big(u_-(0), B\big(x_j, \eta^{-C_1}\big)\big) \lesssim \eta^{2C_1}. \tag{6.13}$$

Recall that $u(0) = u_-(0) - iv(0)$. Combining (6.12) and (6.13) with the triangle inequality, we obtain

$$\mathrm{Mass}\big(u(0), B\big(x_j, \eta^{-C_1}\big)\big) \gtrsim \eta^{C_1}. \tag{6.14}$$

Using (3.1) again, we obtain the result.

Next, we use the radial assumption to show that the bubble of mass concentration must occur at the spatial origin. In the forthcoming paper, we shall use the interaction Morawetz estimate with the frequency localized L^2 almost-conservation law to rule out the possibility of the energy concentration at any place and deal with the non-radial data. The corresponding results for the Schrödinger equation with local nonlinearity, please see [3,19] and [22].

Corollary 6.1 *(Bubble at the origin). Let I_j be an unexceptional interval, and u be a radial solution to* (1.4). *Then*

$$\mathrm{Mass}\big(u(t), B\big(0, \eta^{-3C_1}|I_j|^{1/2}\big)\big) \gtrsim \eta^{C_1}|I_j|$$

for all $t \in I_j$.

Proof. If x_j in Proposition 6.1 is within $\frac{1}{2}\eta^{-3C_1}|I_j|^{1/2}$ of the origin, then the result follows immediately. Otherwise by the radial assumption, there would be at least

$$O\left(\frac{(\eta^{-3C_1}|I_j|^{1/2})^{n-1}}{(\eta^{-C_1}|I_j|^{1/2})^{n-1}}\right) \approx O\big(\eta^{-2(n-1)C_1}\big)$$

disjoint balls each containing at least $\eta^{C_1}|I_j|$ amount of mass. By Hölder inequality, this implies

$$\begin{aligned}
\eta^{-2(n-1)C_1} \times \eta^{C_1}|I_j| &\lesssim \int\limits_{(\eta^{-3C_1}-\eta^{-C_1})|I_j|^{1/2}\leqslant|x|\leqslant(\eta^{-3C_1}+\eta^{-C_1})|I_j|^{1/2}} \big|u(t,x)\big|^2\,dx \\
&\lesssim \|u\|^2_{L_x^{\frac{2n}{n-2}}} \times \left(\int\limits_{(\eta^{-3C_1}-\eta^{-C_1})|I_j|^{1/2}\leqslant|x|\leqslant(\eta^{-3C_1}+\eta^{-C_1})|I_j|^{1/2}} dx\right)^{2/n} \\
&\approx \|u\|^2_{L_x^{\frac{2n}{n-2}}} \times \big(\big(\eta^{-3C_1}|I_j|^{1/2}\big)^{n-1} \times \eta^{-C_1}|I_j|^{1/2}\big)^{\frac{2}{n}},
\end{aligned}$$

that is

$$\|u\|^2_{L_x^{\frac{2n}{n-2}}} \gtrsim \eta^{-\frac{2n^2-9n+4}{2n}C_1}.$$

Because $2n^2-9n+4>0$ for $n \geqslant 5$, this contradicts the boundedness on the energy of (6.2). This completes the proof. □

Next, we use Proposition 3.1 to show that if there are many unexceptional intervals, they must form a cascade and must concentrate at some time t_*.

Corollary 6.2. *Assume that the solution u is spherically symmetric. For any interval $I \subseteq [t_-, t_+]$ and I be a union of consecutive unexceptional intervals I_j. Then*

$$\sum_{I_j \subseteq I} |I_j|^{1/2} \lesssim \eta^{-13C_1}|I|^{1/2},$$

and moreover, there exists a j such that

$$|I_j| \gtrsim \eta^{26C_1}|I|.$$

Proof. For any unexceptional interval I_j, from Hölder inequality and Corollary 6.1, we have

$$\begin{aligned}\eta^{C_1}|I_j| &\lesssim \operatorname{Mass}\big(u(t), B\big(0, \eta^{-3C_1}|I_j|^{1/2}\big)\big)\\ &\lesssim \left\| \left|\chi\left(\frac{x}{\eta^{-3C_1}|I_j|^{1/2}}\right)\right|^2 |x|^3 \right\|_{L_x^\infty} \left\| \left|\chi\left(\frac{x}{2\eta^{-3C_1}|I_j|^{1/2}}\right)\right|^2 \frac{|u(t,x)|^2}{|x|^3} \right\|_{L_x^1}\\ &\lesssim \big(\eta^{-3C_1}|I_j|^{1/2}\big)^3 \int\limits_{|x|\leqslant 2\eta^{-3C_1}|I_j|^{1/2}} \frac{|u(t,x)|^2}{|x|^3}\,dx,\end{aligned}$$

therefore

$$\int\limits_{|x|\leqslant 2\eta^{-3C_1}|I_j|^{1/2}} \frac{|u(t,x)|^2}{|x|^3}\,dx \gtrsim \eta^{10C_1}|I_j|^{-\frac{1}{2}}.$$

We integrate this over each unexceptional interval I_j and sum over j,

$$\begin{aligned}\eta^{10C_1}\sum_{I_j\subseteq I}|I_j|^{1/2} &\lesssim \sum_{I_j\subseteq I}\int_{I_j}\int\limits_{|x|\leqslant 2\eta^{-3C_1}|I_j|^{1/2}} \frac{|u(t,x)|^2}{|x|^3}\,dx\\ &\lesssim \sum_{I_j\subseteq I}\int_{I_j}\int\limits_{|x|\leqslant 2\eta^{-3C_1}|I|^{1/2}} \frac{|u(t,x)|^2}{|x|^3}\,dx\\ &\lesssim \int_I\int\limits_{|x|\leqslant 2\eta^{-3C_1}|I|^{1/2}} \frac{|u(t,x)|^2}{|x|^3}\,dx\\ &\lesssim \eta^{-3C_1}|I|^{1/2}.\end{aligned}$$

The second claim follows from the first and the fact that

$$|I_j|^{1/2} \geqslant |I_j|\Big(\sup_{I_k\subseteq I}|I_k|\Big)^{-1/2}.$$

This completes the proof. □

Proposition 6.2 *(Interval cascade). Let I be an interval tiled by finitely many intervals $I_1,\dots,I_N$. Suppose that for any contiguous family $\{I_j\colon\ j\in\mathcal{J}\}$ of the unexceptional intervals, there exists $j_*\in\mathcal{J}$ such that*

$$|I_{j_*}| \geqslant a\Big|\bigcup_{j\in\mathcal{J}} I_j\Big| \tag{6.15}$$

for some small $a>0$. Then there exist $K\geqslant \log(N)/\log(2a^{-1})$ distinct indices $j_1,\dots,j_K$ such that

$$|I_{j_1}| \geqslant 2|I_{j_2}| \geqslant \cdots \geqslant 2^{K-1}|I_{j_K}|,$$

and for any $t_* \in I_{j_K}$,

$$\operatorname{dist}(I_{j_k}, t_*) \lesssim \frac{1}{a}|I_{j_k}|$$

hold for $1 \leqslant k \leqslant K$.

Proof. Here we use an algorithm in [1,13] and [21] to assign a generation to each I_j. By hypothesis, I contains at least one interval of length $a|I|$. All intervals with length larger than $a|I|/2$ belong to the first generation. By the total measure, we see that there are at most $2a^{-1}-1$ intervals in the first generation. Removing these intervals from I leaves at most $2a^{-1}$ gaps, which are tiled by intervals I_j.

By (6.15) and the contradiction argument, we know that there is not gap with length larger than $|I|/2$.

We now apply this argument recursively to all gaps generated by the previous iteration until every I_j has been labeled with a generation number.

Each iteration of the algorithm removes at most $2a^{-1}-1$ intervals and produces at most $2a^{-1}$ gaps. Suppose that there are N consecutive unexceptional intervals initially, and we perform at most K times iterations. Then the number K obeys

$$\begin{aligned} N &\leqslant \left(2a^{-1}-1\right)+\left(2a^{-1}-1\right)2a^{-1}+\cdots+\left(2a^{-1}-1\right)\left(2a^{-1}\right)^{K-1} \\ &\leqslant \left(2a^{-1}\right)^{K}, \end{aligned}$$

which leads to the claim $K \geqslant \log(N)/\log(2a^{-1})$.

Let $I^{(K)}$ be a collection of intervals obtained after $K-1$ iterations and I_{j_K} be any interval in $I^{(K)}$. For $1 \leqslant i \leqslant K-1$, let $I^{(i)}$ be the $(i-1)$-generation gap which contains the I_{j_K}, and assign the I_{j_i} be any ith generation interval which is contained in $I^{(i)}$ (see Fig. 1). By the construction, for any $t_* \in I_{j_K}$, we have

$$\operatorname{dist}(t_*, I_{j_k}) \leqslant \left|I^{(k)}\right| \leqslant 2a^{-1}|I_{j_k}|$$

for all $1 \leqslant k \leqslant K$. □

Proposition 6.3 *(Energy non-evacuation). Let* $I_{j_1}, \ldots, I_{j_K}$ *be a disjoint family of unexceptional intervals obeying*

$$|I_{j_1}| \geqslant 2|I_{j_2}| \geqslant \cdots \geqslant 2^{K-1}|I_{j_K}| \tag{6.16}$$

and for any $t_* \in I_{j_K}$,

$$\operatorname{dist}(I_{j_k}, t_*) \lesssim \eta^{-26C_1}|I_{j_k}|$$

hold for $1 \leqslant k \leqslant K$. *Then*

$$K \leqslant \eta^{-100C_1}.$$

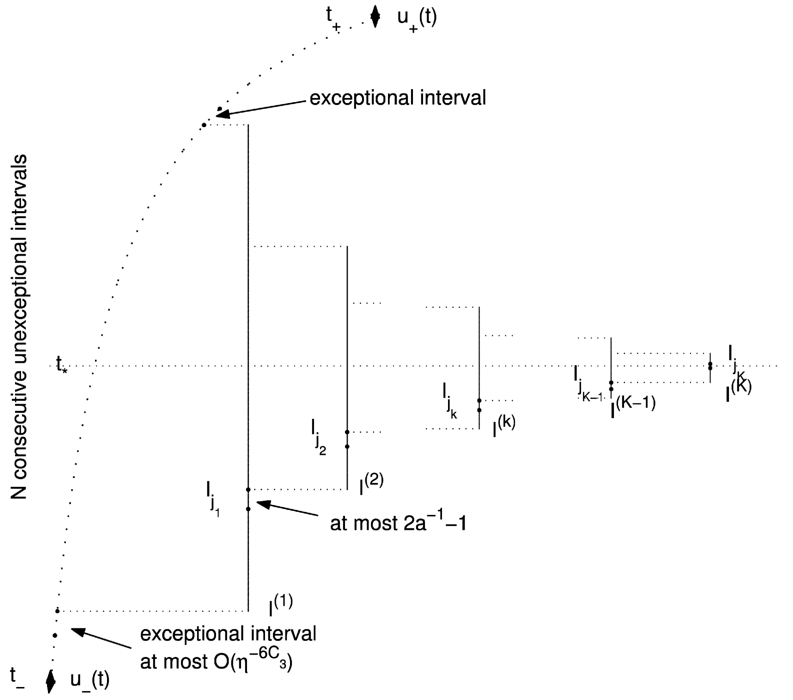

Fig. 1. Iteration process in Proposition 6.2.

Proof. By Corollary 6.1,

$$\mathrm{Mass}\big(u(t), B\big(0, \eta^{-3C_1}|I_{j_k}|^{1/2}\big)\big) \gtrsim \eta^{C_1}|I_{j_k}|$$

for all $t \in I_{j_k}$. By (3.1), we have

$$\begin{aligned}\mathrm{Mass}\big(u(t_*), B\big(0, \eta^{-27C_1}|I_{j_k}|^{1/2}\big)\big) &\gtrsim \left(\big(\eta^{C_1}|I_{j_k}|\big)^{1/2} - \frac{\mathrm{dist}(t_*, I_{j_k})}{\eta^{-27C_1}|I_{j_k}|^{1/2}}\right)^2 \\ &\gtrsim \eta^{C_1}|I_{j_k}|.\end{aligned}$$

On the other hand, from (3.2), we have

$$\mathrm{Mass}\big(u(t_*), B\big(0, 2\eta^{C_1}|I_{j_k}|^{1/2}\big)\big) \lesssim \eta^{2C_1}|I_{j_k}|.$$

Define

$$A(k) = \big\{x\colon\ \eta^{C_1}|I_{j_k}|^{1/2} \leqslant |x| \leqslant \eta^{-27C_1}|I_{j_k}|^{1/2}\big\},$$

then we have

$$\int_{A(k)} \left|u(t_*,x)\right|^2 dx \gtrsim \mathrm{Mass}\big(u(t), B\big(0,\eta^{-27C_1}|I_{j_k}|^{1/2}\big)\big) - \mathrm{Mass}\big(u(t), B\big(0, 2\eta^{C_1}|I_{j_k}|^{1/2}\big)\big)$$
$$\gtrsim \eta^{C_1}|I_{j_k}|.$$

By Hölder inequality, we have

$$\int_{A(k)} \left|u(t_*,x)\right|^{\frac{2n}{n-2}} dx \gtrsim \big(\eta^{C_1}|I_{j_k}|\big)^{\frac{n}{n-2}}\big(\eta^{-27C_1}|I_{j_k}|^{1/2}\big)^{-\frac{2n}{n-2}}$$
$$\gtrsim \eta^{95C_1}.$$

Choosing $M = -56C_1\log\eta$, then we obtain by (6.16)

$$\begin{aligned}\eta^{-27C_1}|I_{j_{M+1}}|^{1/2} &\leqslant \eta^{C_1}|I_{j_1}|^{1/2},\\ \eta^{-27C_1}|I_{j_{2M+1}}|^{1/2} &\leqslant \eta^{C_1}|I_{j_{M+1}}|^{1/2},\\ &\cdots\end{aligned}$$

Hence the annuli $A(k)$ associated to $k = 1, M+1, 2M+1, \ldots,$ are disjoint. The number of such annuli is $O(K/M)$.

Therefore from (6.2), we obtain

$$\frac{K}{M}\eta^{95C_1} \lesssim \int_{\mathbb{R}^n} \left|u(t_*,x)\right|^{\frac{2n}{n-2}} dx \lesssim 1.$$

That is

$$K \lesssim M\eta^{-95C_1} \lesssim \eta^{-100C_1}. \qquad \square$$

We now return to the proof of Theorem 1.1. As explained at the beginning of this section, it suffices to bound the number of the unexceptional intervals.

Note that the number of exceptional interval is at most $O(\eta^{-6C_3})$. We first bound the number N of unexceptional intervals that can occur consecutively.

Let us denote the union of these consecutive unexceptional intervals by I. By Corollary 6.2, the hypotheses of Proposition 6.2 are satisfied with $a = \eta^{26C_1}$ and so we can find a cascade of K intervals and they satisfied the hypotheses of Proposition 6.3. The bound on K implies the bound on N, namely,

$$N \lesssim \big(2a^{-1}\big)^K \approx \big(2\eta^{-26C_1}\big)^{\eta^{-100C_1}}.$$

At last, since there are at most $O(\eta^{-6C_3})$ exceptional intervals, the total number of intervals is

$$J \lesssim \eta^{-6C_3} + \eta^{-6C_3}N \lesssim e^{\eta^{-200C_1}}.$$

This completes the proof of Theorem 1.1.

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