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Green's Function in Some Contributions

Rossana Tazzioli

Dipartimento di Matematica e Informatica, Università di Catania, Viale A. Doria, 6, 95125 Catania, Italy

Many questions in mathematical physics lead to a solution in terms of a harmonic function in a closed region with given continuous boundary values. This problem is known as *Dirichlet's problem*, whose solution is based on an existence principle—the so-called *Dirichlet's principle*. However, in the second half of the 19th century many mathematicians doubted the validity of Dirichlet's principle. They used direct methods in order to overcome the difficulties arising from this principle and also to find an explicit solution of the Dirichlet problem at issue. Many years before, one of these methods had been developed by Green in 1828, which consists in finding a function—called a *Green's function*—satisfying certain conditions and appearing in the analytical expression of the solution of the given Dirichlet problem. Helmholtz, Riemann, Lipschitz, Carl and Franz Neumann, and Betti deduced functions similar to Green's function in order to solve problems in acoustics, electrodynamics, magnetism, theory of heat, and elasticity. © 2001 Academic Press

Molte questioni fisico matematiche conducono a una soluzione in termini di una funzione armonica in una regione chiusa con dati valori continui al contorno. Questo problema è noto come *problema di Dirichlet*, la cui soluzione si basa su un principio di esistenza, il cosiddetto *principio di Dirichlet*. Tuttavia, nella seconda metà del diciannovesimo secolo, molti matematici cominciarono a mettere in dubbio la validità del principio di Dirichlet. Sia per superare le difficoltà sorte da tale principio, sia per trovare una soluzione esplicita del problema di Dirichlet dato, essi presero ad adoperare metodi diretti. Molti anni prima, uno di questi metodi era stato sviluppato da Green nel 1828 e consiste nel trovare una funzione, detta *funzione di Green*, che soddisfa certe condizioni e mediante la quale si rappresenta analiticamente la soluzione del problema di Dirichlet in questione. Helmholtz, Riemann, Lipschitz, Carl e Franz Neumann, e Betti dedussero delle funzioni simili alla funzione di Green allo scopo di risolvere problemi di acustica, elettrodinamica, magnetismo, teoria del calore ed elasticità. © 2001 Academic Press

Nombreuses questions de physique mathématique mènent à une solution en termes d'une fonction harmonique dans une région fermée avec des valeurs continus donnés sur la frontière. Ce problème est connu comme *problème de Dirichlet*, la solution duquel est fondée sur un principe d'existence, le *principe de Dirichlet*. Cependant dans la seconde moitié du dix-neuvième siècle plusieurs mathématiciens mirent en doute la validité du principe de Dirichlet. Alors ils employèrent des méthodes directes soit pour surmonter le difficultés nées de ce principe, soit pour déduire une solution explicite du problème de Dirichlet en question. Avant plusieurs années une de ces méthodes a été développée par Green en 1828 et consiste à trouver une fonction, dite *fonction de Green*, qui satisfait certaines conditions et moyennant laquelle on représente analytiquement la solution du problème de Dirichlet donné. Helmholtz, Riemann, Lipschitz, Carl et Franz Neumann, et Betti déduisirent des fonctions semblables à la fonction de Green pour résoudre de problèmes d'acoustique, électrodynamique, magnétisme, théorie de la chaleur et élasticité. © 2001 Academic Press

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1. INTRODUCTION

In his lectures on potential theory held during the academic year 1856/57 and published posthumously, by F. Grube in 1876, Peter Gustav Lejeune Dirichlet (1805–1859) [Dirichlet 1876] tried to find a harmonic function in a closed region with given continuous boundary values. A function V is harmonic in a domain R if in the interior region of R, V satisfies Laplace's equation:

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

Such a problem, later called Dirichlet's problem, plays an important role in potential theory; its solution is based on an existence principle, actually referred to as Dirichlet's principle, which states that the integral

$$I(V) = \int_R \left[\left(\frac{\partial V}{\partial x}\right)^2 + \left(\frac{\partial V}{\partial y}\right)^2 + \left(\frac{\partial V}{\partial z}\right)^2 \right] dv \tag{1.1}$$

has a minimum value, where R is a connected space and $V(x, y, z)$ is a function in $C^1(R)$ taking the given continuous values on the boundary of R. Moreover, the function V making (1.1) minimum is the solution of the Dirichlet problem.[1] The Dirichlet principle, which was considered one of the most important criteria of existence in mathematics, was put in doubt during the second half of the 19th century. In 1870, Weierstrass exhibited a counterexample to a similar variational problem by constructing a lower bounded functional that does not assume a minimal value. Many years later, Hilbert [1904, 1905] "brought back the Dirichlet principle to life"—as he wrote. By using variational methods, he indeed proved that in the bidimensional case the functional (1.1) has a harmonic function assuming given boundary values as a minimum, under the hypotheses that the boundary is sufficiently regular and the given function is piecewise analytical.

Before the counterexample of Weierstrass—which was published only in 1895—many mathematicians realized that difficulties arose from the Dirichlet principle. Leopold Kronecker (1823–1891), Ludwig Schläfli (1814–1895), Heinrich Weber (1842–1913), Felice Casorati (1835–1890), Hermann Amandus Schwarz (1843–1921) and Carl Gottfried Neumann (1832–1925) tried to put this principle on a rigorous footing (see [Monna 1975, Tazzioli 1994a]). Many of them aimed to re-prove Riemann's mapping theorem in a rigorous way, since in his *Inauguraldissertation* Riemann [1851] had used the Dirichlet principle as an argument in his proof. They realized that the crucial point of Riemann's proof could be reduced to the solution of a particular Dirichlet problem, and constructed its solution without referring to the Dirichlet principle.

Schwarz [1869a, 1869b, 1870a, 1870b, 1870c], one of Weierstrass's students, developed his own method—called the "alternating method"—of proving Riemann's mapping theorem for convex bidimensional domains. Schwarz's proof was independent of the validity of the Dirichlet principle, so that "the well-grounded objections against such a principle could not be raised" [Schwarz 1869a, 83].

[1] Before Dirichlet, Green [1835], and Gauss [1839] had faced such a problem in a similar way.

The Dirichlet problem for three-dimensional convex domains was studied and solved by Carl Neumann. He used a method of approximate solutions, the so-called "method of arithmetic mean" (see [Archibald 1996]), which he developed in two papers published in 1870 [Neumann 1870]. C. Neumann studied problems of conformal mapping and questions of stationary heat and of electrostatic and electrodynamic equilibrium without using the much discussed Dirichlet principle. His method was a *direct* one which allowed him to explicitly exhibit the solution. Moreover, in such a way C. Neumann could "avoid [...] Dirichlet's principle," as he was proud to notice in the *Preface* of his celebrated *Lectures on logarithmic and Newtonian potential* [C. Neumann 1877].

In Italy, in 1868 Felice Casorati (1835–1890) published his *Theory of functions of complex variables* [Casorati 1868], a work which was to be—at least in his intentions—in two parts. However, the second part of the work has never been published. It was to contain the theory of Abelian functions, treated according to a Riemannian approach, that is to say by using the Dirichlet principle. Doubts about the validity of Dirichlet's principle stopped Casorati from finishing his original plan. He discussed these difficulties with Weierstrass, Kronecker, and Schläfli (see [Neuenschwander 1978, Bottazzini 1986, 261–164]) and in 1869 wrote to Giuseppe Battaglini (1826–1894) that "the Dirichlet principle contributes not a little to my delay [in the publication of the second volume of the *Theory of functions*]" [in Neuenschwander 1978, 29].

Enrico Betti (1823–1892) also took part in the discussion about the validity of Dirichlet's principle and showed different attitudes towards it in his works on potential theory, namely in his papers published in *Nuovo Cimento* between 1863 and 1864 [Betti 1863–1864], and in his treatise published in 1879 [Betti 1879]. In his 1863–1864 papers, Betti assumed Dirichlet's principle as valid, while in his book, published 15 years later, he made some changes, of which the most important concerns the use of Dirichlet's principle. Indeed in the *Preface* of his treatise Betti [1879, V–VI] expressly questioned the validity of the Dirichlet principle: "In the theory published on *Nuovo Cimento* I had founded the methods of electrostatics on Dirichlet–Riemann's theorem on the existence of a bounded and continuous function together with its first derivatives, satisfying the Laplace equation in a space of any shape and taking some given values on the boundary [that is to say, Dirichlet problem has one and only one solution]." And he goes on: "After critical remarks about the proofs of that theorem in all its generality, I thought that it is better to renounce this kind of method. We shall be able to build again on that theorem when we rigorously state when and where it is valid." Indeed in his book he never used Dirichlet's principle and did not propose any alternative proof of the existence of the solution for the general Dirichlet problem. Therefore, he exhibited the solution of the given Dirichlet problem case by case and used direct methods only.

If the Dirichlet principle is not valid, the solution for a given Dirichlet problem is not generally ensured. Therefore, mathematicians tried to find other alternative methods, which do not use any existence principle. In his 1828 paper, George Green (1793–1841) [Green 1828] developed a direct procedure for solving the Dirichlet problem involving the search for a particular function, named Green's function (see Section 2). Many 19th-century mathematicians aimed to find the solution of a given Dirichlet's problem *directly*, by following Green's method and by showing the existence of the appropriate Green's function for the problem at issue. They deduced functions similar to Green's function for solving problems in acoustics, theory of heat, magnetism, electrodynamics, and elasticity, which involve differential equations of elliptic, hyperbolic, and parabolic type. Such a method—of finding the suitable Green's function—is a very modern approach and leads to the solution of various types of differential equations.

2. THE *ESSAY* BY GREEN

Green was an audodidact and an outsider in his contemporary academic world. He made remarkable contributions to analysis and mathematical physics, in particular to electrostatics, elasticity, and potential theory. In 1828 Green published *An Essay on the Application of Mathematical Analysis to the Theories of Electricity and Magnetism* [Green 1828], a memoir full of new ideas and results in potential theory. He proposed to draw upon the historical development of the theories of electricity and magnetism, but only quoted some works of Cavendish and Poisson. Green noticed that general mathematical methods could be very useful for treating problems in electricity and magnetism, and to this aim he usually employed the "potential function," whose partial derivatives equal the components of the force. "The consideration of this function"—he [Green 1828, 9] wrote—"is of great importance in many inquiries, and probably, there are none in which its utility is more marked than in those about to engage our attention." Indeed, in the *General Preliminary Results* of his *Essay*, he deduced important theorems and formulae in potential theory involving the potential function.

Green proved the well-known Green's formulae and other results valid for harmonic functions. If U, V are function of x, y, z, which are tacitly supposed by Green to be finite and continuous, together with their derivatives, on a regular region R with boundary σ, the following theorem (called *Green's second identity*),[2]

$$\int_\sigma V\frac{\partial U}{\partial \nu}\,d\sigma + \int_R V\nabla^2 U\,dv = \int_\sigma U\frac{\partial V}{\partial \nu}\,d\sigma + \int_R U\nabla^2 V\,dv, \tag{2.1}$$

is valid if ν is the internal normal to the surface. As a corollary, if U and V are harmonic and continuously differentiable in the closed regular domain R, then

$$\int_\sigma \left(U\frac{\partial V}{\partial \nu} - V\frac{\partial U}{\partial \nu}\right) d\sigma = 0. \tag{2.1a}$$

Green assumed that if U takes given continuous values on σ, then one and only one solution of the Laplace equation $\nabla^2 U = 0$ will exist; then as a consequence, the Dirichlet problem has a unique solution.

In Green's second identity (2.1), let P be a fixed interior point of R, Q any point of R, r the distance (variable with Q) between P and Q, and $V = 1/r$. Since $1/r$ tends to infinity

[2] If U and V are continuous in R together with their partial derivatives of the first order, and U have also continuous derivatives of the second order in R, then the following formula, referred to as *Green's first identity*,

$$\int_R V\cdot\nabla^2 U\,dv + \int_R (\text{grad}\,U\cdot\text{grad}\,V)\,dv = \int_\sigma V\frac{\partial U}{\partial \nu}\,d\sigma$$

is valid, where grad $U = (\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z})$ and grad $U\cdot$ grad V means the scalar product of the gradients of U and V.

as $r \to 0$, formula (2.1) cannot be valid for *all* the region R. So we consider a small sphere with center at P and radius r, and remove from R the interior of the sphere. The second identity (2.1) can then be applied to the resulting region, and by calculating the limit for $r \to 0$ one obtains the so-called *Green's third identity*:

$$U(P) = -\frac{1}{4\pi}\int_R \frac{\nabla^2 U}{r}\,dv + \int_\sigma \left(U\frac{\partial}{\partial\nu}\frac{1}{r} - \frac{1}{r}\frac{\partial U}{\partial\nu}\right)d\sigma. \tag{2.2}$$

If U is harmonic in R, then

$$U(P) = -\frac{1}{4\pi}\int_\sigma \left(-\frac{1}{r}\frac{\partial U}{\partial\nu} + U\frac{\partial}{\partial\nu}\frac{1}{r}\right)d\sigma. \tag{2.3}$$

Generally speaking, Green wished to express any harmonic function in terms of its boundary values and then to eliminate the normal derivative of U, $\frac{\partial U}{\partial\nu}$ in the formula (2.3). If a harmonic function V can be found such that $\frac{1}{r} + V$ vanishes at all points of σ, then the normal derivative of U can be eliminated by adding the equations (2.3) and (2.1a) together:

$$U(P) = -\frac{1}{4\pi}\int_\sigma \left(U\frac{\partial}{\partial\nu}\frac{1}{r} + U\frac{\partial V}{\partial\nu}\right)d\sigma. \tag{2.4}$$

Such a function V represents the potential of the charge, which is induced on a grounded sheet conductor with the shape of σ by a unit charge at P. The function $G = \frac{1}{r} + V$ is the value at Q of the potential of the inducing charge at P and the induced charge together. This function is known as *Green's function* for the region R and the pole P. In terms of Green's function, (2.4) becomes

$$U(P) = -\frac{1}{4\pi}\int_\sigma U(Q)\frac{\partial}{\partial\nu}G(P,Q)\,d\sigma. \tag{2.5}$$

Some authors later in the century—such as Betti, Rudolph Lipschitz (1832–1903), and Carl and Franz Ernst Neumann (1798–1895)—designated V, instead of G, as Green's function.

Incidentally, Green claimed that Green's function exists for physical reasons, because the static charge on the surface σ always exists. Nevertheless, there are regions where Green's function does not exist, as Lebesgue [1913] noticed, by showing a particular region whose boundary contains one "irregular" point—that is to say a point in which the boundary behaviour of the harmonic function fails. In another paper, Green [1835] changed his approach. He deduced some results on potential theory by following a procedure similar to that used by Gauss in his [1839] memoir. Their methods were based on the validity of the Dirichlet principle and therefore were easily attacked when this principle was put in doubt.

Though Green's 1828 paper was fundamental for the development of potential theory, it remained unknown to most mathematicians for many years. Thomson rediscovered it and mentioned it in a note to an article published in 1842 [Thomson 1842]. Green's memoir was then published in the renowned *Journal für die reine und angewandte Mathematik* in 1850.

Dirichlet's problem and Green's function are closely connected. In fact, in order to establish the existence of Green's function G, one has to solve a special Dirichlet's problem, namely to deduce a harmonic function taking on the boundary values as $-\frac{1}{r}$ and then to verify that G gives the right solution. Conversely, a given Dirichlet's problem has a solution if the appropriate Green's function exists. Dirichlet's principle also plays an important role in both questions; indeed, its validity implies that any Dirichlet problem is solvable and, consequently, the suitable Green's function can be exhibited, at least in principle. In reality, it is often difficult to explicitly deduce Green's function from the mathematical point of view, and mathematical difficulties arising from the original problem are just shifted to the physical setting.

3. ACOUSTICS

Hermann von Helmholtz (1821–1894) published fundamental works concerning both mathematics and physics. In a paper published in 1860, he [Helmholz 1860] studied the partial differential equation,

$$\nabla^2 U(x, y, z) + k^2 U(x, y, z) = 0 \tag{3.1}$$

(k constant)[3], which regulates stationary propagations in acoustics, elasticity, and electricity. It is the analogue of the Laplace equation for a volume distribution with Newtonian potential function U. After drawing a historical outline of the contributions to acoustics from D. Bernoulli and Euler on, Helmholtz turned to the problem of gas diffusion in a bounded region R with boundary σ, where there are excitation points (*Erregungspunkten*) spread with continuous density.

Helmholtz [1860, 6] remarked that "the extension of Green's theorem to the given question shows its usefulness." If $k^2 \int_R UV dv$ is added to the two sides of Green's theorem (2.1), then

$$\int\limits_\sigma V \frac{\partial U}{\partial \nu}\, d\sigma + \int\limits_R V(\nabla^2 U + k^2 U)\, dv = \int\limits_\sigma U \frac{\partial V}{\partial \nu}\, d\sigma + \int\limits_R U(\nabla^2 V + k^2 v)\, dv. \tag{3.2}$$

Helmholtz proved that the function $V = \frac{\cos kr}{r}$ satisfies the partial differential equation $\nabla^2 V + k^2 V = 0$ in all the space and, in the case that also $\nabla^2 U + k^2 u = 0$, he got

$$U = -\frac{1}{4\pi} \int\limits_\sigma U \frac{\partial}{\partial \nu}\left(\frac{\cos kr}{r}\right) d\sigma + \frac{1}{4\pi} \int\limits_\sigma \frac{\cos kr}{r} \frac{\partial U}{\partial \nu}\, d\sigma. \tag{3.3}$$

This is analogous to Green's formula (2.2), where the role of Green's function is here played by $V = \frac{\cos kr}{r}$. Therefore, as Helmholtz pointed out [1860, 22], the procedure due to Green allows one to deduce the solution for the considered problem, by applying

[3] The problem of integrating equation (3.1) in two dimensions—namely, $U = U(x, y)$—was faced by Weber [1869] and Mathieu [1872, 271].

"the so extremely fruitful Green's theorem [...] to the [potential] functions at issue with the greatest advantage."

In a handwritten note, taken from his lectures on mathematical physics of the academic year 1866/67,[4] Betti followed Helmholtz's approach and introduced Green's function for the acoustic problem by using the same method. "Therefore $\nabla^2 U = 0$ is a particular case of the equation (3.1), since it is deduced from it by posing $k = 0$," Betti remarked. As it will be pointed out (see Sections 6, 7), Betti also used a similar approach in his research on heat theory and theory of elasticity.

Bernhard Riemann (1826–1866), who wrote fundamental papers in several fields of mathematics, also made important contributions to the theory of partial differential equations. In a memoir published in 1860, he applied the method of Green's function in order to integrate the differential equation of hyperbolic type describing the diffusion of acoustic waves [Riemann 1860]. He quoted Helmholtz's [1860] paper, where, however, elliptic differential equations are treated—instead of those of hyperbolic type considered by Riemann. Riemann [1860, 156] solved the problem of gas propagation only for "the case, where the initial motion is the same along each direction, and speed and pressure are constant on each plane orthogonal to these directions." The solutions of more general problems had not been deduced yet.

Riemann posed the following hypotheses, derived from physical and experimental laws due to Boyle and Gay-Lussac and from contributions by Regnault, W. Thomson, and Clausius: the pressure of gas depends on its density ρ only, and the direction of propagation is along a straight line, for example along the x-axis, if x, y are the coordinates. Then, Riemann defined the characteristic lines, r and s:

$$2r = f(\rho) + w, \quad 2s = f(\rho) - w,$$

if w is the speed of propagation of the gas, and $f(\rho)$ is a function of the pressure, which is deduced from the physical structure of the gas. Therefore, the characteristic lines depend on some experimental data, which can be found from physical considerations. In order to solve his problem of wave propagation, Riemann was led to consider the partial differential equation of hyperbolic type

$$\frac{\partial^2 U}{\partial r \, \partial s} - m\left(\frac{\partial U}{\partial r} + \frac{\partial U}{\partial s}\right) = 0, \tag{3.4}$$

where m is a function of the pressure, and U must satisfy the initial conditions. In order to integrate (3.4), Riemann introduced a function V, which plays the same role as Green's function did for the Laplace equation and is today called *Green's function for the hyperbolic problem*. The function V has to be deduced and its existence was not proved by Riemann. Even though Green's results are never quoted in the paper, Riemann was clearly aware of them and applied Green's theorem (2.1) to the functions U, V in the region S bounded by the characteristic lines $x = r$, $y = s$, and by a given curve c cutting the characteristic lines once only.

[4] Betti's notes are contained in "Fondo Betti" at the "Archivio della Biblioteca della Scuola Normale Superiore" in Pise.

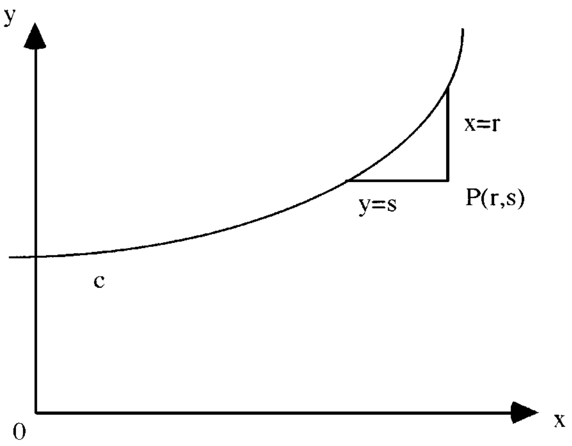

FIGURE 1

By means of Green's theorem, Riemann could reduce the original problem to the calculation of three curvilinear integrals depending on U, V and on their partial derivatives, and extended on the three pieces of the boundary of S. Then, the mathematical development of the proof led Riemann to pose some conditions for the function V.[5]

Riemann deduced the value of the solution U at the point $P = (r, s)$,

$$U(P) = (UV)(P_1) + \int_c V\left(\frac{\partial U}{\partial s} - mU\right) ds + \int_c U\left(\frac{\partial V}{\partial r} + mV\right) dr, \qquad (3.5)$$

where P_1 is the intersection point between the curve c and the line $x = r$ (see Fig. 1).

In the last part of the paper, Riemann showed the relevance of his theory by applying the formula (3.5) to a particular acoustic problem, where $m = \frac{\text{const.}}{r+s}$ follows from Poisson's law. In that case, he introduced a function V, and then proved that it is the suitable Green's function. He indeed could explicitly derive the solution U of the problem for given initial conditions from (3.5), by using Fourier's series.

4. LIPSCHITZ'S PAPER ON ELECTRODYNAMICS

Lipschitz is best known for his contributions to real analysis and number theory; nevertheless, he worked for a long time on topics concerning mathematical physics. Lipschitz extended the principles of mechanics—such as the motion equations of a point and the Euler-Lagrange differential equations—to a Riemannian manifold with constant curvature (see [Tazzioli 1994b]) and also studied questions in electrostatics and electrodynamics. In order to determine the electrodynamic propagation in a homogeneous conductor, he introduced a function similar to Green's function in a paper published in 1861 [Lipschitz 1861].

[5] Such conditions are $\partial^2 V/\partial r \partial s + \partial mV/\partial r + \partial mV/\partial s = 0$ in S; $\partial V/\partial s + mV = 0$ when $x = r$; $\partial V/\partial r + mV = 0$ when $y = s$; $V = 1$ along the lines $x = r$, $y = s$.

In his paper, Lipschitz applied the theory of a double distribution (*Doppelbelegung* or *Doppelschicht*) deeply studied by Kirchhoff [1848] and Helmholtz [1853] several years before. Classical treatises on potential theory usually devote many pages to the theory of double sheets (or double distributions). A double distribution on a connected regular surface σ can be regarded as the limiting form of a set of magnetic particles—or doublets—distributed over σ with their axes normal to the surface and pointing to one side, as the particles are more and more densely distributed and their magnetic moment decreases. Therefore, a double distribution is the natural representation of a magnetic distribution on a surface. But not only that; indeed, since any closed circuit can be replaced by the magnetic double sheet of that surface, whose boundary is the circuit itself, electrodynamic problems can be studied by means of the theory of double sheet.

In order to deduce theorems and results in the theory of double sheet, one is led to define the potential of a double distribution. The following classical procedure also allows one to grasp the physical meaning of the potential. Let σ and σ' be two connected regular surfaces with normal ν, distant λ (λ is the infinitely small thickness of the sheet), having density ζ and ζ' respectively; two "corresponding points" or two "corresponding surface elements" on the surfaces σ and σ' are defined as belonging to the same normal (see Fig. 2). Then σ and σ' are two infinitely close surfaces; an attractive and a repulsive Newtonian force on σ and σ' respectively—having the same intensity—are assumed to originate from any pair of corresponding points. Therefore these surfaces represent the so-called double sheet.

The masses of any pair of corresponding points are then equal; that is to say, the formula

$$\zeta \, d\sigma = \zeta' \, d\sigma' \tag{4.1}$$

is valid, where $d\sigma$ and $d\sigma'$ are two corresponding surface elements. Therefore, the potential U of the double sheet at any point P is expressed by

$$U = \int_\sigma \left(\frac{-\zeta \, d\sigma}{r} + \frac{\zeta' d\sigma'}{r'} \right) \tag{4.2}$$

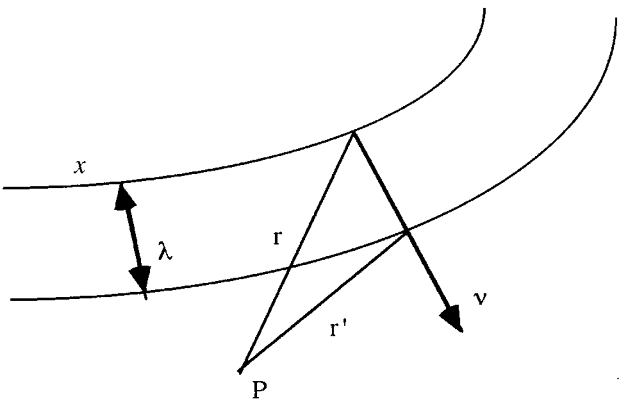

FIGURE 2

(r, r' are the distances between P and $d\sigma$, $d\sigma'$ respectively), and from (4.1):

$$U = \int_\sigma \left(\frac{1}{r} - \frac{1}{r'}\right)\zeta \, d\sigma. \tag{4.3}$$

Since λ is infinitely small, $\frac{1}{r'}$ can be replaced by

$$\frac{1}{r} + \lambda \frac{\partial \frac{1}{r}}{\partial \nu},$$

and (4.3) becomes

$$U = \int_\sigma \frac{\partial \frac{1}{r}}{\partial \nu} \lambda\zeta \, d\sigma = \int_\sigma \frac{\partial \frac{1}{r}}{\partial \nu} \mu \, d\sigma, \tag{4.4}$$

where $\mu = \lambda\zeta$ is called the *moment* of the double sheet.

By using the theory of the double sheet, Lipschitz [1861] aimed to formulate an electrodynamic theory similar to the well-known electrostatic theory. He proved the existence theorem (I):

Let σ be a closed regular surface, and let μ be a finite function on σ; then one and only one function U exists such that $U = U_i$ in the internal region of σ, $U = U_e$ in the external region of σ, and $U_e \to 0$ when $r \to \infty$; moreover, $U_e - U_i = 4\pi\mu$ and $\frac{\partial U_i}{\partial \nu} = \frac{\partial U_e}{\partial \nu}$ on σ. Since Helmholtz [1853] had already proved that such conditions for U completely characterize potential functions of a double sheet, U is the potential function of the double distribution on the surface σ.

Lipschitz reduced the previous problem to the following one (I′):

To find a potential function of a double sheet U—called "electrodynamic potential"—such that $\frac{\partial U}{\partial \nu} = g$, where g is a given continuous function on the surface σ.

Let us remark that g must satisfy $\int_\sigma g \, d\sigma = 0$, since U is harmonic in R. Lipschitz was aware that the problem (I′) for the potential of a double sheet—which we refer to as *Lipschitz's problem*—is analogous to the Dirichlet problem for the usual potential. In particular, Lipschitz's problem is the natural extension to electrodynamics of the Dirichlet problem valid in electrostatics. Lipschitz proved that this problem has a unique solution by using Dirichlet's principle. Nevertheless Dirichlet's principle is only a criterion of existence, which does not give any information about the expression of the solution. Since Lipschitz aimed to find the solution explicitly, he applied Green's theory to his electrodynamic problem, by looking for a function playing the same role as Green's function played in electrostatics.

In order to do this, Lipschitz split the potential function U into two parts—the external potential U_e and the internal potential U_i—and expressed each part of it by means of a suitable Green's function. In particular, in order to deduce the external potential U_e, he looked for the Green's function $G = -\frac{1}{r} + V$, which equals $4\pi\lambda$ on the surface σ (λ is the moment of V). The function V must be found; it must be harmonic in R and such that

$$\frac{\partial V}{\partial \nu} = \frac{\partial \frac{1}{r}}{\partial \nu} \quad \text{on } \sigma, \tag{4.5}$$

where r is the distance between a point on the surface σ and a fixed point outside of σ. Moreover, V satisfies $\int_\sigma \frac{\partial V}{\partial \nu}\, d\sigma = 0$, since $\int_\sigma \frac{\partial \frac{1}{r}}{\partial \nu}\, d\sigma = 0$. Lipschitz then found the following expression for the external potential: $U_e = \int_\sigma (-\frac{1}{r} + V) \frac{\partial U_e}{\partial \nu}\, d\sigma$.

For the internal potential U_i Lipschitz had to find *another* Green's function. In fact, let G' be the *new* Green's function, $G' = V' - \frac{1}{r}$, where V' is the harmonic function to be deduced; since

$$\frac{\partial V'}{\partial \nu} = \frac{\partial \frac{1}{r}}{\partial \nu}$$

(see eq. (4.5)), V' should satisfy the formula

$$\int_\sigma \frac{\partial V'}{\partial \nu}\, d\sigma = \int_\sigma \frac{\partial \frac{1}{r}}{\partial \nu}\, d\sigma = -4\pi$$

by Gauss's theorem. But, on the other hand,

$$\int_\sigma \frac{\partial V'}{\partial \nu}\, d\sigma = 0$$

since V' is harmonic in R.

As a consequence, Green's function for the internal potential U_i must have an expression which is different from the previous expression G'. Lipschitz was led to consider $G'' = V'' - \frac{1}{r}$, which equals $4\pi\lambda$ on σ, and the normal derivative of V'' differs from that of $\frac{1}{r}$ by a constant, κ. Lipschitz derived the following expression for the internal potential:

$$U_i = \int_\sigma \left(-\frac{1}{r} + V'' \right) \frac{\partial U_i}{\partial \nu}\, d\sigma - \int_\sigma \kappa\, U_i\, d\sigma. \tag{4.6}$$

The moment Λ producing the potential U is then deduced by the formula $4\pi\Lambda = U_e - U_i$. The function denoted by G'' is nowadays usually named the *Green's function of the second kind*, and for the so-called Neumann problem it is the analogue of Green's function for a Dirichlet's problem. The Neumann problem is the problem of finding a function harmonic in a region and having normal derivatives equal to the function given on the boundary. It is evident that the Lipschitz problem (I′) is equivalent to the Neumann problem. Therefore, the Neumann problem represents the natural extension to electrodynamics of the Dirichlet problem which is valid in electrostatics; and it is then legitimate to ask whether or not there is a function playing the same role as Green's function. Such a function indeed exists and is the so-called Green's function of the second kind.

5. FRANZ NEUMANN'S LECTURES ON MAGNETISM

Franz Neumann is considered a physicist more than a mathematician; he worked on electricity, magnetism, and electromagnetism. His physical theories are based on the concept of "action-at-a-distance" and are close to the ideas developed by Helmholtz and Weber.

Franz Neumann introduced a particular function analogous to Green's function in his lectures on the theory of magnetism *Vorlesungen über die Theorie des Magnetismus namentlich über die Theorie der magnetischen Induktion*, held during the summer semester 1857 at the University of Königsberg, and published by his son Carl Neumann only in 1881 [F. Neumann 1881].

Franz Neumann proposed to study the magnetic induction in a magnet due to an external magnetic force. From results connected with Green's theorem, he proved that the magnetic induction in the magnet could be described by means of a function similar to Green's function in electrostatics. Neumann defined the magnetic moment induced by the external force as a vector $\mathbf{m} = (\alpha, \beta, \gamma)$, $\alpha = \Sigma_i \mu_i\ x_i$, $\beta = \Sigma_i \mu_i,\ y_i$, $\gamma = \Sigma_i \mu_i\ z_i$, where the sums are extended to all the magnet, and μ_i is the *magnetic mass* concentrated at the point $P_i = (x_i, y_i, z_i)$. If φ is the potential of the induced magnetic moment $\mathbf{m}$ ($\mathbf{m} = \text{grad}(\varphi)$), then Neumann deduced that

$$\varphi = -W - \kappa \int_\sigma W \frac{\partial \psi}{\partial \nu}\, d\sigma, \tag{5.1}$$

where W is the potential of the external magnetic force, κ a constant, σ the boundary of the magnet, ν the internal normal to the surface, and ψ a function which has to be determined. Such a function ψ represents the potential of the induced magnetic moment in a magnet having the shape σ, which is produced by a unit magnetic mass; ψ has here a role similar to that played in electrostatics by the potential function we had denominated by V in Section 2 and that Neumann called Green's function. As we read in Neumann's lectures on magnetism:

"*By means of formula (5.1) the problem of magnetic induction for a given homogeneous body is reduced to finding a function ψ depending only on the surface of the body.* Therefore Neumann named this function ψ the *characteristic function* of the surface or of the given body respectively. For *magnetic* induction that function plays, as it is evident, a role similar to the one played by the well-known *Green's function* in *electric* induction or in problems of stationary heat respectively" [F. Neumann 1881, 112].

6. THE THEORY OF HEAT IN BETTI'S WORK

Betti played an important role in the development of Italian mathematics in the period just after national unification. In 1857 he became professor at the Scuola Normale of Pise, whose he was the director from 1864 until his death—save two years (1874–1876) (see [Bottazzini 1982]). Betti's scientific research concerned the solution of algebraic equations of fifth degree, the theory of functions of a complex variable, the theory of elliptic functions, and several fields in mathematical physics. He investigated topics in mathematical physics for more than 20 years and his works were also connected with the validity of the Dirichlet principle. The role of physics in his mathematical work is very important; in this regard, Vito Volterra (1860–1940) remarks: "Ceux qui ont connu Betti, non seulement par ses travaux, mais aussi par sa conversation, savent que s'il parlait Mathématiques, bien souvent il pensait Physique" [Volterra 1902, 8].

Betti and some of his students tried to solve problems in mathematical physics leading to elliptic or parabolic differential equations by defining functions similar to the ordinary Green's function. In potential theory Betti expressed the solution of a given Dirichlet's problem by means of the suitable Green's function and, in his lectures on mathematical physics, he approached the problem of gas diffusion by finding a function similar to Green's function, as Helmholtz had already done (see Section 3). Betti also used Green's fruitful method in order to solve other questions in mathematical physics, in particular in heat theory[6] and elasticity (see Section 7). In heat theory, Betti solved a more general problem than the problem considered by Carl Neumann.

C. Neumann's earliest works on potential theory were devoted to the Dirichlet problem, which he considered closely connected with electrostatics and theory of heat (see Section 1). In some papers published from 1861 to 1863 [C. Neumann 1861, 1862, 1863], Neumann had applied Green's method to some problems in the theory of stationary heat. In his [1862] paper, Neumann had explicity noticed that the two following questions are equivalent from a mathematical point of view:

I. "*To determine the state of stationary heat in a body, whose surface is everywhere in contact with given and constant sources of heat*";

II. "*To determine the electric distribution in a body, which is subjected to given and constant electric forces*" [C. Neumann 1862, V].

It is sufficient to find a potential function harmonic in R taking continous values on σ, that is to say to solve a particular Dirichlet's problem. To this aim, Neumann used the method of Green's function and deduced a function G which is harmonic in R and such that $G = 1/r$ on the boundary, where r is the distance between (x, y, z) and a fixed point in R. "The function G is such that it only depends on geometric relations," he remarked [C. Neumann 1862, 82]. Indeed, the potential functions of problems (I) and (II) satisfy the same differential equation; that is to say, they are both harmonic functions and their Green's functions are the same.

As a consequence of his results, C. Neumann determined the Green's function for the Dirichlet problem of two nonconcentric spheres. In his [1863] paper, he deduced results very similar to those contained in his 1862 work. In a previous note, C. Neumann [1861] had introduced the suitable Green's function in order to solve the Dirichlet problem for a sphere, and had deduced the so-called Poisson integral. As already mentioned in Section 1, about 1870 he realized that many difficulties arose from Dirichlet's principle and felt some urgency in avoiding its use in mathematics.

Betti also devoted himself to the theory of heat and examined problems more general than those considered by C. Neumann. In particular, in a paper published in 1868, Betti [1868a] tried to solve the following general problem in heat theory, which he reduced to a question of analysis:

[6] For parabolic differential equations regulating heat propagation, see [Sommerfeld 1894].

To determine a function—the temperature, $U = U(x, y, z, t)$ (x, y, z being points of a connected space R and t time)—which satisfies the equation of heat propagation in a body,

$$\frac{\partial U}{\partial t} - k\nabla^2 U = 0 \tag{6.1}$$

($k = \frac{\text{internal conductivity}}{\text{caloric} \times \text{density}}$), is equal to a given function $V = V(x', y', z', t)$ on a portion σ' of σ (σ is the boundary of R), and satisfies the equation

$$\frac{\partial U}{\partial \nu} = h(U - \zeta) \tag{6.2}$$

on $\sigma'' = \sigma - \sigma'$ (ν is the internal normal to σ'', $h = \frac{\text{external conductivity}}{\text{internal conductivity}}$, and $\zeta = \zeta(x'', y'', z'', t)$ is an arbitrary function of x'', y'', z'' in σ''). Moreover, U must verify the initial condition $U = U_0$ for $t = 0$, with $U_0 = U_0(x, y, z),\ x, y, z$ in R.

If U and ζ are independent of t, then $U = U_0$. In fact, such a function U satisfies the foregoing conditions and is called the function of stationary heat. This is the case of stationary heat already considered by C. Neumann. In his proof, Betti used an argument similar to Dirichlet's principle; indeed, he constructed a functional Ω, which depends on U_0 and on its first derivatives and is equal to or greater than zero. "It is evident," Betti [1868a, 219] wrote, "that [...] there exists a function which makes Ω minimum." This function represents the temperature of the body and solves the above problem in heat propagation; by applying variational methods, he then proved that such a function is unique, is harmonic in R, and satisfies the required conditions. (It is interesting to remark that in his later treatise on potential theory, Betti [1879] changed his mind. Here, he clearly expressed his opinion against the use of Dirichlet's principle—or of similar arguments—in mathematical proofs (see Section 1)).

In the more general case where U_0 does not satisfy the equation $\nabla^2 U_0 = 0$, but only (6.2), the solution of the problem U will depend on time t. By means of physical considerations and using procedures depending on the validity of Dirichlet's principle, Betti proved that the function U exists and is unique and then sought a method to explicitly describe it. This method consists of finding a function G similar to Green's function: he aimed to express the solution U by means of the function G, which, however, he could explicitly deduce only in particular cases.

Betti found the formula

$$U(x', y', z') = -\frac{1}{4\pi}\int_{\sigma'} U\frac{\partial G}{\partial \nu}\,d\sigma' - \frac{1}{4\pi}\int_{\sigma''} G\left(G\frac{\partial U}{\partial \nu} - hU\right)d\sigma'', \tag{6.3}$$

where $G = \frac{1}{r} + V$, r is the distance between (x', y', z') and (x, y, z) on σ, G is harmonic, $G = 0$ on σ', $\frac{\partial G}{\partial \nu} = hG$ on σ'', and V is a harmonic function to be determined.

As a consequence of (6.3), Betti [1868a, 225–226] wrote: "G represents the stationary temperatures in a homogeneous solid body having constant temperature on an infinitely small sphere with center at (x', y', z'), that is to say [...] having a constant source of heat [at (x', y', z')]. [Moreover] the temperature of the body vanishes on a portion of [σ, denoted by]

σ', and the portion of the body $\sigma''[= \sigma - \sigma']$ is in contact with free air whose temperature is also zero.” The physical interpretation of Green’s function given by Betti in heat theory is in line with what happens in electrostatics and magnetism, as Green and Franz Neumann had already pointed out (see Section 2, Section 5).

The problem of heat propagation in a body is then reduced to the deduction of the suitable Green’s function, in order to express the solution U. Nevertheless, the latter problem is not easy at all; Betti could in fact deduce Green’s functions for particular cases only, such as if R is a sphere (or a half-sphere) and its surface σ has a constant temperature, and if R is a parallelepiped and the temperatures are given on each face. In a paper published the same year, Betti [1868b] used an analogous method in order to solve the problem of heat propagation in a cylinder.

7. BETTI’S RECIPROCAL THEOREM

For several years Betti investigated the theory of elasticity and made some remarkable discoveries. In a series of papers on elasticity, published in *Nuovo Cimento* in 1872 and 1873, he deduced his well-known reciprocal theorem, whose proof and applications to physics can be found in another paper published a year later [Betti 1874]. The relevance of the reciprocal theorem was openly noticed by Marcolongo [1907] in his compendium on the development of mathematical physics in Italy from 1870 to 1907: “In potential theory the importance of a certain identity—called Green’s formula—is well known. Betti discovered a famous theorem, the reciprocal theorem, which plays the same role as Green’s formula [see (2.1)] for the equations of elasticity.”

Betti considered a solid, elastic, and homogeneous body R with boundary σ and density ρ, and proved that if (u, v, w), (u', v', w') are two systems of displacements produced by the volume forces (X, Y, Z), (X', Y', Z') and by the pressures (L, M, N), (L', M', N') respectively, the formula

$$\int_\sigma (L', M', N') \cdot (u, v, w)\, d\sigma + \rho \int_R (X', Y', Z') \cdot (u, v, w)\, dv = \int_\sigma (L, M, N) \cdot (u', v', w')\, d\sigma + \rho \int_R (X, Y, Z) \cdot (u', v', w')\, dv \qquad (7.1)$$

is valid. This formula is actually referred to as the reciprocal theorem; it shows that the two systems of forces and displacements cannot be independent one of the other, but change according to the relation (7.1). As Gustavo Colonnetti (1886–1968) pointed out in his classical treatise [1928], the reciprocal theorem is usually used in construction engineering. “There is no problem in all construction engineering,” Colonnetti remarked [1928, 261], “to which this very elegant principle of mathematical physics cannot be successfully applied.” He indeed showed that Betti’s theorem includes, as a particular case, the celebrated Maxwell reciprocal theorem [Maxwell 1864] for frames: “The extension in BC due to the tension along DE is always equal to the extension in DE due to the tension in BC” if all the members of the frame are extensible (see Fig. 3).

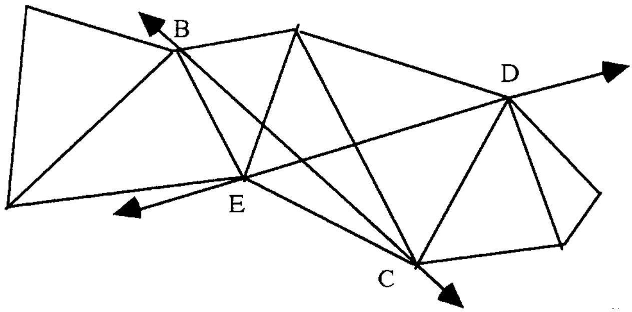

FIGURE 3

From the reciprocal theorem, Betti deduced some functions, similar to Green's functions, which allowed him to describe fundamental properties of elastic bodies. Indeed, if an elastic and isotropic body is considered and (u, v, w) are the components of its displacement, then Betti expressed its rotation (p, q, r) and dilatation Θ by means of the given forces acting on the body—where

$$p = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad q = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad r = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right),$$

$$\text{and} \quad \Theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}.$$

To this aim, he considered the particular displacement (u, v, w) given by

$$u = \frac{\partial \frac{1}{r}}{\partial x} + \xi; \quad v = \frac{\partial \frac{1}{r}}{\partial y} + \eta; \quad w = \frac{\partial \frac{1}{r}}{\partial z} + \zeta, \tag{7.2}$$

where r is the distance between (x, y, z) and a point (x', y', z') of R, and the functions ξ, η, ζ have to be determined; they must be continuous and single-valued, together with their derivatives, and satisfy the equations of elastic equilibrium for the body. It follows that u, v, w satisfy the same equations on R and are single-valued and continuous, together with their derivatives on R', which is the portion of the body R obtained by subtracting from R a space as small as one likes containing (x', y', z') inside, where the function $1/r$ and its derivatives become infinite. Moreover, the displacement

$$\left(\frac{\partial \frac{1}{r}}{\partial x}, \frac{\partial \frac{1}{r}}{\partial y}, \frac{\partial \frac{1}{r}}{\partial z}\right)$$

defines a system of forces which acts on R' and satisfies the reciprocal theorem together with the forces produced by the displacement (ξ, η, ζ). By applying the reciprocal theorem to two appropriate systems of forces and displacements on R', and by calculating the limit for $r \to$ O, Betti found certain conditions which allowed him to explicitly deduce ξ, η, ζ. Therefore, from (7.2) Betti could easily deduce the displacement (u, v, w). As a consequence of these results, he could express the elastic dilation Θ of the body by means of known quantities, the displacement (u, v, w) and the given forces (X, Y, Z) and (L, M, N) acting on the body R with boundary σ,

$$\Theta = c_0 \int_\sigma [(L, M, N) \cdot (u, v, w)]\, d\sigma + c_1 \int_R [(X, Y, Z) \cdot (u, v, w)]\, dv \tag{7.3}$$

(c_0, c_1 constants). Betti [1874, 386] noticed that "the three functions ξ, η, ζ [see (7.2)], which play here the same role as Green's function in the expressions of potential functions, have a physical meaning, just as Green's function does in electrostatics." In fact, "*they express the components of the displacements of an elastic and isotropic body, which equilibrates forces acting only on the surface of the body. The action of the forces is* [...] *equal to that produced by a magnetic element placed at the point* (x', y', z'), *having* [...] *axis parallel to the normal to the surface* [...], *and acting on a magnetic pole at that point*." Green's function and the functions ξ, η, ζ are all unknown and are useful in order to express the solution of the given Dirichlet problem. Moreover, at least in principle, they can be deduced from experimental data, since they have a definite physical meaning.

Betti then considered another particular displacement and, as in the previous case, from the reciprocal theorem he deduced some conditions on ξ, η, ζ which led him to express the components of the rotation by means of the given forces and other known quantities.

Betti's method, leading to the expressions of elastic dilatation and rotation from the reciprocal theorem, is analogous to the procedure already employed by Green in his [1828] paper. As Cataldo Agostinelli [1961–1962, 43–44] pointed out, "Betti thought to adapt these [Green's] methods to the theory of elastic equilibrium, then to the theory of heat and, as a foundation of his calculations, formulated the reciprocal theorem, which plays the same role as Green's formula and is now named Betti's theorem."

Valentino Cerruti (1850–1910) and Carlo Somigliana (1860–1956) closely followed Betti's approach to the theory of elasticity. Cerruti [1882] simplified Betti's procedure for integrating the elastic equations of equilibrium and elaborated a method, which is nowadays well known as the *Betti-Cerruti method*. Somigliana [1890] deduced some equations describing the displacements of an elastic body by means of the given forces and the displacements on the surface of the body. These equations, actually referred to as *Somigliana's formulae*, were found by applying Betti's reciprocal theorem together with arguments usually employed in potential theory, and are analogous to formulae and theorems proved by Green for harmonic functions.

8. CONCLUSIONS

Many 19th-century mathematicians employed the method of Green's function to express the solution of a given Dirichlet problem in an explicit way. Moreover, many of them believed that if the Green's function is exhibited, then the solution of the Dirichlet problem exists. Such a procedure allowed them to avoid the use of any existence criterion and, in particular, to overcome the difficulties arising from the Dirichlet principle. But the existence of a Green's function is a statement as well, which has to be proved. According to some authors, among them Green himself, the Green's function exists *a priori* from physical evidence; other mathematicians, for example Betti [1879] in his treatise on potential theory, constructed such a function case by case. Indeed, one cannot develop a general method leading to the derivation of a suitable Green's function for any Dirichlet problem.

The introduction of Green's function for the integration of differential equations was a procedure also followed by some of Betti's students. In a paper published in 1872 Ulisse Dini (1845–1918) [1871–1872], who studied with Betti in Pise, considered the so-called *generalized* Dirichlet problem,

$$\nabla^2 u = f \text{ in the region } R; \quad u = v \text{ on the boundary of } R, \text{ named } \sigma, \tag{8.1}$$

where u is continuous together with its first and second derivatives, and f, v are continuous functions. For such a problem, Dini deduced the appropriate Green's function, by means of which he expressed the solution of (8.1). Nevertheless, he believed that the existence of u is not generally guaranteed by Green's function and checked every time that u is the *right* solution. In another paper, by following a similar method, Dini [1876] formulated and solved the *generalized* Neumann problem

$$\nabla^2 u = f \text{ in the region } R; \quad \frac{\partial u}{\partial \nu} = v \text{ on } \sigma \text{ } (\nu \text{ is the normal to } \sigma \text{ drawn inwards}). \tag{8.2}$$

He found a function similar to Green's function, which he explicitly deduced for some particular cases—circle, sphere, and two circles.

Gregorio Ricci Curbastro (1853–1925), who was one of Betti's students, studied the generalized Dirichlet's problem and published a paper on it [Ricci Curbastro 1885]. Ricci is well-known for his theory of tensor calculus, but he was interested in mathematical physics too. The subjects of his first works on mathematical physics were suggested by Betti and dealt with electrostatics, electrodynamics, mechanical interpretation of Mawell's equations, and potential theory.

In his [1885] paper, Ricci considered the problem (8.1) and remarked that "the existence of the solution u has not yet been proved *a priori*." Indeed, from the explicit expression of u by means of Green's function one cannot conclude that such a solution really *exists*. Dini [1871–1872] had already pointed out that the existence of u must always be verified. Ricci went further; he posed certain conditions on u and its derivatives which ensure the solvability of the problem (8.1) *a priori* and the consequent existence of the function u. The Dirichlet principle is then simply replaced by another existence criterion.

The method of Green's function was not only connected with Dirichlet's problem and Dirichlet's principle; it also helped in the development of the theory of partial differential equations. In fact, 19th-century mathematicians often tried to find functions similar to Green's function, in order to solve problems involving partial differential equations—elliptic, hyperbolic, or parabolic. Riemann and Helmholtz in acoustics, Lipschitz in electrodynamics, Franz Neumann in magnetic induction, Betti in heat theory and elasticity, all tried to extend the method of Green's function to solve their particular problems and showed that this procedure is fruitful in many fields of mathematical physics and in the solution of partial differential equations.

Green's method is often hard to follow. In principle, the identification of a suitable Green's function—or a function generalizing Green's function—is sufficient in order to express the solution of the Dirichlet problem, if such a solution exists; no existence principle, but just a constructive procedure is employed. Nevertheless, Green's function is often difficult to find from a mathematical point of view, without appealing to physical considerations. The original difficulties are then removed only partly; generally, they are shifted to the derivation of Green's function.

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