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Lower bounds of the Laplacian spectrum of graphs based on diameter

Mei Lu [a,*,1], Lian-zhu Zhang [b,1], Feng Tian [c,2]

[a] *Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China*
[b] *School of Mathematical Science, Xiamen University, Xiamen, Fujian 361005, China*
[c] *Institute of Systems Science, Academy of Mathematics and Systems Sciences, Chinese Academy of Sciences, Beijing 100080, China*

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Abstract

Let G be a connected graph of order n. The diameter of G is the maximum distance between any two vertices of G. In the paper, we will give some lower bounds for the algebraic connectivity and the Laplacian spectral radius of G in terms of the diameter of G.

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1. Introduction

Let $G = (V, E)$ be a simple undirected graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$. If G is a path, then G is denoted P_n. For $v_i \in V$, the degree of v_i, written as $d(v_i)$, is the number of edges incident with v_i. For two vertices v_i and $v_j (i \neq j)$, the distance between v_i and v_j is the

* Corresponding author.
E-mail addresses: mlu@math.tsinghua.edu.cn (M. Liu), lz_zhang@126.com (L.-z. Zhang), ftian@mail.iss.ac.cn (F. Tian).
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number of edges in a shortest path joining v_i and v_j. The diameter of a graph is the maximum distance between any two vertices of G.

Let $A(G)$ be the adjacency matrix of G and $D(G) = \mathrm{diag}(d(v_1), d(v_2), \ldots, d(v_n))$ be the diagonal matrix of vertex degrees. The Laplacian matrix of G is $L(G) = D(G) - A(G)$. Clearly, $L(G)$ is a real symmetric matrix. From this fact and Geršgorin's Theorem, it follows that its eigenvalues are non-negative real numbers. The eigenvalues of an $n \times n$ matrix M are denoted by $\lambda_1(M), \lambda_2(M), \ldots, \lambda_n(M)$, while for a graph G, we will use $\lambda_i(G) = \lambda_i$ to denote $\lambda_i(L(G))$, $i = 1, 2, \ldots, n$ and assume that $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_{n-1}(G) \geqslant \lambda_n(G)$. It is well known that $\lambda_n(G) = 0$ and the algebraic multiplicity of zero as an eigenvalue of $L(G)$ is exactly the number of connected components of G [10], i.e., the second smallest eigenvalue $\lambda_{n-1}(G) > 0$ if and only if G is connected. This led Fiedler [2] to define it as the algebraic connectivity of G, which has a relation to the classical connectivity parameters of a graph G—the vertex connectivity and the edge connectivity. The eigenvalues, $\lambda_1(G)$ (also called the Laplacian spectral radius of G) and $\lambda_{n-1}(G)$, have received a great deal of attention (see, for example [5,7–13]) and some results involved the diameter of a graph. For example, Mohar [12] showed that $\lambda_{n-1} \geqslant \frac{4}{nd}$, Alon and Milman [1] showed $d \leqslant 2\left[\sqrt{(2\Delta/\lambda_{n-1})}\log_2 n\right]$ and Chung [4] gave $d \leqslant \left\lceil \frac{\log(n-1)}{\log(1/(1-\lambda_{n-1})} \right\rceil$, where n, d and Δ are the order, diameter and maximum degree of the graph G, respectively. In Section 2, we also consider the diameter of G and give lower bounds of the Laplacian spectral radius and algebraic connectivity of G involving the diameter.

2. Lower bounds for the Laplacian eigenvalues

Let G be a simple connected graph and $L(G) = D(G) - A(G)$ be the Laplacian matrix of G. It is well known that $\lambda_n(G) = 0$ with eigenvector $e = (1, 1, \ldots, 1)^{\mathrm{T}}$ and $\lambda_{n-1}(G) > 0$ by G being connected. Since $L(G)$ is symmetric, by the Rayleigh–Ritz Theorem (see for example [6]), we have

$$\lambda_{n-1}(G) = \min_{x \perp e, x \neq 0} \frac{x^{\mathrm{T}} L(G) x}{x^{\mathrm{T}} x} \tag{1}$$

and

$$\lambda_1(G) = \max_{x \neq 0} \frac{x^{\mathrm{T}} L(G) x}{x^{\mathrm{T}} x}. \tag{2}$$

Now, we will give a sharp lower bound for $\lambda_{n-1}(G)$ by using some ideas of Fiedler [3].

Theorem 1. *Let G be a connected simple graph of order n, size m and diameter d. Then*

$$\lambda_{n-1}(G) \geqslant \frac{2n}{2 + n(n-1)d - 2md}. \tag{3}$$

Equality holds if and only if $G = P_3$ or G is a complete graph.

Proof. Let $l^2(V)$ be a vector space and $x = (x_1, x_2, \ldots, x_n) \in \Phi$ be an eigenvector for $\lambda_{n-1}(G)$, where Φ is the set of all non-constant vectors $x \in l^2(V)$. Using the facts that $\sum_{v \in V(G)} x_v = 0$ (orthogonality to the eigenvector of λ_n) and that $\sum_{uv \in E(G)} (x_u - x_v)^2 = \lambda_{n-1}(G) \sum_{v \in V(G)} x_v^2$, we can show easily that

$$\lambda_{n-1}(G) = 2n \frac{\sum_{uv \in E(G)} (x_u - x_v)^2}{\sum_{u \in V(G)} \sum_{v \in V(G)} (x_u - x_v)^2}. \tag{4}$$

Since $\sum_{v\in V(G)} x_v = 0$ and G is connected, we have $\sum_{uv\in E(G)}(x_u - x_v)^2 \neq 0$. Then, from (4), we have

$$\begin{aligned}\lambda_{n-1}(G) &= 2n\frac{\sum_{v_iv_j\in E(G),i<j}(x_{v_i}-x_{v_j})^2}{\sum_{i=1}^{n}\sum_{j=1}^{n}(x_{v_i}-x_{v_j})^2}\\ &= 2n\frac{\sum_{v_iv_j\in E(G),i<j}(x_{v_i}-x_{v_j})^2}{2\sum_{i<j}(x_{v_i}-x_{v_j})^2}\\ &= \frac{n}{1+g},\end{aligned}\tag{5}$$

where

$$g = \frac{\sum_{v_iv_j\notin E(G),i<j}(x_{v_i}-x_{v_j})^2}{\sum_{v_iv_j\in E(G),i<j}(x_{v_i}-x_{v_j})^2}.\tag{6}$$

Assume that u_0 and v_0 are the vertices such that $(x_{u_0} - x_{v_0})^2 = \max_{u,v\in V(G)}(x_u - x_v)^2$. Then $x_{u_0} \neq x_{v_0}$. Let $P = v_1v_2\cdots v_{r+1}$ be the shortest path in G joining u_0 and v_0 (where $v_1 = u_0$ and $v_{r+1} = v_0$). Note that

$$\sum_{v_iv_j\in E(G),i<j}(x_{v_i}-x_{v_j})^2 \geqslant \sum_{v_iv_j\in E(P),i<j}(x_{v_i}-x_{v_j})^2 \tag{7}$$

$$\geqslant \frac{1}{r}(x_{v_1}-x_{v_{r+1}})^2 \tag{8}$$

$$\geqslant \frac{1}{d}(x_{u_0}-x_{v_0})^2$$

by using the Cauchy–Schwarz inequality. On the other hand, we have that

$$\sum_{v_iv_j\notin E(G),i<j}(x_{v_i}-x_{v_j})^2 \leqslant \left(\frac{n(n-1)}{2}-m\right)(x_{u_0}-x_{v_0})^2.\tag{9}$$

Therefore, from (6) up to (9), we have

$$g \leqslant \left(\frac{n(n-1)}{2}-m\right)d.$$

Thus, by (5), (3) holds.

In order for the equality to hold, the inequalities from (7) up to (9) should be equalities. Then $r = d$. By (7), we have that

$$x_u - x_v = 0 \tag{10}$$

for any $uv \in E(G) - E(P)$. By (8), we have

$$x_{v_i} - x_{v_{i+1}} = a \neq 0 \tag{11}$$

for all $v_iv_{i+1} \in E(P)$, where a is a constant (since $(x_{v_1} - x_{v_{d+1}})^2 = (x_{u_0} - x_{v_0})^2 \neq 0$, $a \neq 0$ is obvious).

If the set $\{uv \notin E(G)|u, v \in V(G)\} = \emptyset$, then G is a complete graph. Hence, we assume that $\{uv \notin E(G)|u, v \in V(G)\} \neq \emptyset$. Then $r = d \geqslant 2$. Note that $v_1v_{r+1} \notin E(G)$ and $x_{v_1} \neq x_{v_{r+1}}$, we have $g \neq 0$. From (9), for any $uv \notin E(G)$, we have

$$(x_u - x_v)^2 = (x_{u_0} - x_{v_0})^2.\tag{12}$$

Now we show that $d \leqslant 2$. If $d \geqslant 3$, then we have $v_1v_d, v_1v_{d+1}, v_{d-1}v_{d+1} \notin E(G)$ by P being the shortest path connecting v_1 and v_{d+1}. From (12), we have

$$(x_{v_1} - x_{v_d})^2 = (x_{v_1} - x_{v_{d+1}})^2 = (x_{v_{d-1}} - x_{v_{d+1}})^2. \tag{13}$$

First we prove that $x_{v_1} \neq x_{v_{d-1}}$. If $d = 3$, then from (11), we have $x_{v_1} \neq x_{v_{d-1}}$. Thus we assume that $d \geqslant 4$. Then $v_1v_{d-1} \notin E(G)$ by P being the shortest path. From (12), we have

$$(x_{v_1} - x_{v_d})^2 = (x_{v_1} - x_{v_{d+1}})^2 = (x_{v_1} - x_{v_{d-1}})^2. \tag{14}$$

If $x_{v_1} = x_{v_{d-1}}$, then $(x_{v_1} - x_{v_d})^2 = (x_{v_1} - x_{v_{d+1}})^2 = 0$ by (14), i.e., $x_{v_d} = x_{v_{d+1}}$, a contradiction with (11). Hence $x_{v_1} \neq x_{v_{d-1}}$.

Now, by (13), we have

$$2x_{v_1} = x_{v_d} + x_{v_{d+1}} \quad \text{and} \quad 2x_{v_{d+1}} = x_{v_1} + x_{v_{d-1}}.$$

Therefore, we have $3(x_{v_1} - x_{v_{d+1}}) = x_{v_d} - x_{v_{d-1}}$. Note that

$$(x_{v_d} - x_{v_{d-1}})^2 \leqslant (x_{v_1} - x_{v_{d+1}})^2 = (x_{u_0} - x_{v_0})^2 \neq 0.$$

But we have

$$9(x_{v_1} - x_{v_{d+1}})^2 = (x_{v_d} - x_{v_{d-1}})^2 \leqslant (x_{v_1} - x_{v_{d+1}})^2$$

a contradiction. Hence, $d \leqslant 2$.

Suppose that $V(G) - V(P) \neq \emptyset$. Thus, by the connectedness of G, there exists $u \in V(G) - V(P)$ and some i, $1 \leqslant i \leqslant d+1$, such that $uv_i \in E(G)$. Then $x_u = x_{v_i}$ by (10). If $i = 1$, then $uv_2 \notin E(G)$ (otherwise $x_u = x_{v_2}$ by (10) which implies that $x_{v_1} = x_{v_2}$, a contradiction with (11)) and $uv_3 \notin E(G)$ when $d = 2$ (otherwise $x_u = x_{v_3}$ which implies that $x_{v_1} = x_{v_3}$, a contradiction with $(x_{v_1} - x_{v_3})^2 = (x_{u_0} - x_{v_0})^2 \neq 0$). Thus uv_1Pv_{d+1} is the shortest path of length $d+1$ joining u and v_{d+1}, a contradiction. Thus, we may assume that $i \neq 1$, and $i \neq d+1$ by the same argument. So, we have that $d = 2$ and $uv_2 \in E(G)$. Then $x_u = x_{v_2}$. By (10) and (11), $uv_1, uv_3 \notin E(G)$. But by (12), we have $x_{u_0} = x_{v_0}$, a contradiction. Thus $V(G) - V(P) = \emptyset$, and hence $G = P_3$.

Conversely, let G be a complete graph or $G = P_3$. Then the equality holds by an elementary calculation.

This completes the proof of Theorem 1. □

Remark 2. In [12], Mohar showed that

$$\lambda_{n-1} \geqslant \frac{4}{nd}. \tag{15}$$

The bounds of (3) and (15) are incomparable. However we can give some graphs to show that the lower bound (3) is better than (15) in some cases. For example, it is easy to check that the lower bound (3) is better than (15) when G is a complete graph or $G = P_3$. In fact, if the size m of a graph G satisfies $m \geqslant \frac{n(n-2)}{4} + 1$, then, from (3), we have

$$\lambda_{n-1} \geqslant \frac{4}{nd + \frac{4-4d}{n}} \geqslant \frac{4}{nd}.$$

Thus the lower bound (3) is better than (15). On the other hand, if $m \leqslant \frac{n(n-2)}{4}$, then from (15), we have

$$\lambda_{n-1} \geqslant \frac{4n}{n^2d} \geqslant \frac{4n}{2n(n-1) - 4md} > \frac{2n}{2 + n(n-1)d - 2md}.$$

Thus the lower bound (15) is better than (3).

Next, we will give some lower bounds of $\lambda_1(G)$ involving the diameter. Let P be a path of G. We call P an induced path if the subgraph induced by $V(P)$ in G is P itself, i.e., $G[V(P)] = P$. Obviously, the shortest path between any two distinct vertices of G is an induced path.

Theorem 3. *Let $P = v_1v_2\cdots v_{s+1}$ be an induced path of G. Then*

$$\lambda_1(G) \geqslant \frac{\sum_{i=1}^{s+1} d_i + 2s}{s+1}. \tag{16}$$

Proof. Let the vertices of $V(G) - V(P)$ be labelled by $v_{s+2}, v_{s+3}, \ldots, v_n$ if $s < n-1$. Denote $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$ such that

$$y_i = \begin{cases} 1 & \text{if } i = 1, 3, \ldots, s, \\ -1 & \text{if } i = 2, 4, \ldots, s+1, \\ 0 & \text{otherwise,} \end{cases}$$

if s is an odd number, and

$$y_i = \begin{cases} 1 & \text{if } i = 1, 3, \ldots, s+1, \\ -1 & \text{if } i = 2, 4, \ldots, s, \\ 0 & \text{otherwise,} \end{cases}$$

if s is an even number. Obviously, $y \neq 0$ and $y^{\mathrm{T}}y = s+1$. Since $P = v_1v_2\cdots v_{s+1}$ is an induced path, we have

$$\begin{aligned} y^{\mathrm{T}}L(G)y &= (d(v_1)+1) + \sum_{i=2}^{s}(d(v_i)+2) + (d(v_{s+1})+1) \\ &= \sum_{i=1}^{s+1} d(v_i) + 2(s-1) + 2 \\ &= \sum_{i=1}^{s+1} d_i + 2s. \end{aligned}$$

Thus by (2), (16) holds. □

Remark 4. In [5], Grone and Merris had showed that

$$\lambda_1 \geqslant \Delta + 1, \tag{17}$$

where Δ is the maximum degree of the graph G. We can easily find some graphs, for example, G is a path with at least 5 vertices or G is a regular graph, to show that the lower bound (16) is better than (17). In fact, if G exists an edge uv such that $d(u) = d(v) = \Delta$, then the bound (17) can be derived by Theorem 3.

Let P be a path of G. Set $d(P) = \sum_{v \in P} d_G(v)$, where $d_G(v)$ is the degree of v. Denote by $\mathbf{P}_k$ the set of the induced paths of length k. Then, by Theorem 3, we can easily show the following result.

Corollary 5. *Let G be a connected graph of order n with diameter d. Then*

$$\lambda_1(G) \geqslant \max_{1 \leqslant k \leqslant d} \max_{P \in \mathbf{P}_k} \left\{ \frac{d(P) + 2k}{k+1} \right\}.$$

Let G be a graph of order n with vertices of degrees $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_n$. *Set*

$$e_r = \frac{1}{r}(d_1 + d_2 + \cdots + d_r), \quad 1 \leqslant r \leqslant n.$$

Then by Corollary 5, *the following result holds immediately*.

Corollary 6. *Let G be a connected graph of order n with diameter d. Then*

$$\lambda_1(G) \geqslant \frac{(d+1)e_{d+1} + 2d}{d+1}. \tag{18}$$

Remark 7. The inequality (18) is sharp. Equality holds, for example, G is a complete graph.

Since $d(v_i) \geqslant 2$ for $i = 2, \ldots, d$ and $d(v_1), d(v_{d+1}) \geqslant 1$, we have the following result by Corollary 6.

Corollary 8. *Let G be a connected graph of order n with diameter d. Then*

$$\lambda_1(G) \geqslant \frac{4d}{d+1}.$$

Note that e_r is a monotonously non-decreasing function of r, i.e., $e_r \leqslant e_{r+1}$, we have the following result by Corollary 6.

Corollary 9. *Let G be a connected graph of order n with diameter d, minimum degree* $\delta(G)$. *Then*

$$\lambda_1(G) \geqslant \frac{(d+1)\delta(G) + 2d}{d+1}.$$

Particularly, when G is k-regular, we have

$$\lambda_1(G) \geqslant \frac{(d+1)k + 2d}{d+1}.$$

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References

[1] N. Alon, V.D. Milman, λ_1, isoperimetric inequalities for graphs and superconcentrators, J. Combin. Theory Ser. B 38 (1985) 73–88.
[2] M. Fiedler, Algebraic connectivity of graphs, Czechoslovak Math. J. 23 (1973) 298–305.
[3] M. Fiedler, A property of eigenvectors of nonnegative symmetric matrices and its application to graph theory, Czechoslovak Math. J. 25 (1975) 619–633.
[4] F.R.K. Chung, Diameters and eigenvalues, J. Amer. Math. Soc. 2 (1989) 187–196.
[5] R. Grone, R. Merris, Algebraic connectivity of trees, Czechoslovak Math. J. 37 (1987) 660–670.
[6] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.
[7] S. Kirkland, A bound on algebraic connectivity of a graph in terms of the number of cutpoints, Linear and Multilinear Algebra 47 (2000) 93–103.
[8] J.S. Li, X.D. Zhang, On the Laplacian eigenvalues of a graph, Linear Algebra Appl. 285 (1998) 305–307.

[9] R. Merris, Characteristic vertices of trees, Linear and Multilinear Algebra 22 (1987) 115–131.
[10] R. Merris, Laplacian matrices of graphs: a survey, Linear Algebra Appl. 197–198 (1994) 143–176.
[11] R. Merris, A note on Laplacian graph eigenvalues, Linear Algebra Appl. 285 (1998) 33–35.
[12] B. Mohar, Eigenvalues, diameter, and mean distance in graphs, Graphs Combin. 7 (1991) 53–64.
[13] O. Rojo, R. Soto, H. Rojo, An always non-trivial upper bound for Laplacian graph eigenvalues, Linear Algebra Appl. 312 (2000) 155–159.