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Cut elimination for a simple formulation of epsilon calculus

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Abstract

A simple cut elimination proof for arithmetic with the epsilon symbol is used to establish the termination of a modified epsilon substitution process. This opens a possibility of extension to much stronger systems.
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1. Introduction

The substitution method for first-order arithmetic introduced by Hilbert [6] employs a formulation where quantifiers are defined

$$\exists x F[x] := F[\epsilon X F[x]]; \forall x F[x] := F[\epsilon x \neg F[x]]$$

in terms of the ϵ-symbol $\epsilon x F[x]$ read as "the least x satisfying $F[x]$". The corresponding axioms are *critical formulas*

$$F[t] \to F[\epsilon x F[x]]. \tag{1.1}$$

The essential part of an arithmetical proof is a finite sequence

$$\mathrm{Cr} = Cr_0, \ldots, Cr_N \tag{1.2}$$

of critical formulas. The goal of the substitution process (or H-process of D. Hilbert) is to find a *solving* ϵ-substitution of numbers $n_1, \ldots, n_k$ for ϵ-terms $e_1, \ldots, e_k$

$$S \equiv (e_1, n_1), \ldots, (e_k, n_k)$$

making Cr true: $|\mathrm{Cr}|_S \hookrightarrow \mathsf{TRUE}$, where $|\mathrm{Cr}|_S$ is the result of iteratively replacing e_i by n_i. The substitution method generates successive substitutions

$$\emptyset \equiv S_0, S_1, \ldots, S_n, \ldots. \tag{1.3}$$

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If $S_n := S$ is non-solving then one of the critical formulas (1.1) is false, that is

$$|F[\epsilon x F[x]]|_S \hookrightarrow \mathsf{FALSE} \text{ and } |F[t]|_S \hookrightarrow \mathsf{TRUE}.$$

Then S_n is corrected by adding a component (up to more technical details below)

$$(\epsilon x F[x], n) \text{ with the least } n \leq |t|_S \text{ such that } F[n] \hookrightarrow_S \mathsf{TRUE}, \tag{1.4}$$

which produces $S_{n+1} := \mathsf{H}(S_n)$. The ϵ-term e added at the H-*step* from S_n to S_{n+1} and the number v are the H-term and H-value of the substitution S_n.

The first termination proof for the substitution method for the first-order arithmetic PA was given by W. Ackermann [2] after the failed attempt [1]. The definition and termination proof were extended to stronger systems in [11,12,3, 10,4,14]. The goal of the present paper is to simplify the termination proof for the epsilon substitution method (cf. [6, 11]) to make possible its extension to much stronger systems.

Simplification is achieved due to a slight modification of the substitution method. The standard reduction step introduced by W. Ackermann not only replaces the default value 0 of the term $\epsilon x F[x]$ by the "correct" value n in (1.4), but also deletes from S all values of complexity (rank) greater than $r = \mathrm{rk}(\epsilon x F)$. This corresponds to the rule

$$\frac{(e, n),\ \Theta_{\leq r}}{(e, ?),\ \Theta}\ \mathsf{H}_{e,n}$$

which is replaced here by the rule $\mathsf{H}'_{e,n}$, cf. Section 4:

$$\frac{(e, n),\ \Theta'}{(e, ?),\ \Theta}\ \mathsf{H}'_{e,n},$$

where Θ' is obtained from $\Theta_{\leq r}$ by dropping several (possibly zero) components of rank r.

In fact these are components that depend on the default value $(e, ?)$ of e, but the exact formulation of dependence is too complicated and too difficult to trace through all transformations we need, as the referee pointed out and convincingly illustrated. Instead we define an *indeterministic* H'-process where an arbitrary number of values of rank r can be dropped at each step. It is turned into a *deterministic* H'-process in a standard way by an exhaustive breadth-first search. The proof that the H'-process terminates in a solving substitution uses the general schema from [9,11].

(1) A Tait-style ϵ-calculus $PA\epsilon$ is defined and for every system Cr of critical formulas a derivation of the empty sequent $\emptyset$ (original derivation, Section 5) is constructed. It depends on Cr and proves the existence of a solution for Cr.
(2) Cut elimination (normalization) theorem for $PA\epsilon$ is proved in a standard way.
(3) An additional structure is introduced into the same cut elimination process in such a way that a cut-free proof of $\emptyset$ becomes (after deleting redundant steps) a non-deterministic H'-process terminating in a solving substitution.

The main difference with [9,11] is the simplification of the cut elimination procedure in the stage 2 of the proof. It becomes very close to a standard method for the first-order arithmetic with the ω-rule. Since the statement of the existence of a solving substitution is Σ^0_1, say

$$\exists e_1 \ldots \exists e_k \exists n_1 \ldots \exists n_k SOLUTION[e_1, \ldots, e_k, n_1, \ldots, e_k],$$

its cut-free proof in a standard formulation of PA (with quantifiers) consists of a finite number of $\exists$-instantiations. Moreover, since *SOLUTION* is primitive recursive, it is possible to choose a solution among them, so in fact one instantiation is enough. This explains the fact that any cut-free proof of $\emptyset$ in the system $PA\epsilon$ of Section 3 is an axiom providing a substitution solving Cr.

Richer structure introduced into the same cut elimination procedure in Section 4 uses many technical aspects of the earlier work (cf. [11]) simplified and provided with a new interpretation. The CutFr rule used in the earlier work is avoided. The new system $PA\epsilon'$ has in addition to cut only two rules: H' for making H'-steps and Fr for marking the beginning of a "history". Fr-inference replaces a cut which is eliminated. In this role it is similar to the repetition rule Rep in continuous cut elimination see [7,5].

The termination of the non-deterministic process is a trivial consequence of the termination of the H-process [2,9, 11]. The non-deterministic H'-process is introduced since it is connected more closely (than H-process) to the standard cut elimination procedure. To model the H-process we introduced [9,11] additional cut elimination steps that may look unnatural for a Gentzen-style calculus and are not guaranteed to apply to second-order systems where the role of rank is not so distinct. It may turn out that for such systems a non-deterministic version is preferable or even constitutes the only manageable approach.

As an adaptation of an example given by the referee shows, making the H′-process too deterministic may lead to divergence. Consider two critical formulas

$$E[1] \to E[e],\ F[1] \to F[f],\ \text{with } e = \epsilon x E,\ f = \epsilon x F$$

where $E[1]$, $F[1]$ are quantifier-free sentences such that

$$1 = \min\{n \geq 0 : E[n]\} = \min\{n \geq 0 : F[n]\}$$

and an H′-process when *all* existing non-default components of rank r are deleted at the H′-step of rank r. Then the H′-process oscillates:

$$S_0 = \emptyset, S_1 = (e, 1), S_2 = (f, 1), \ldots, \qquad S_{2k} = (e, 1), S_{2k+1} = (f, 1).$$

In Section 2 we recall standard notions and elementary results concerning ϵ-calculus, cf. [9,11]. The only new points are Definitions 2.14 and 2.15 of H′-process and obviously Lemma 2.4. Section 3 contains cut elimination proof for the arithmetic $PA\epsilon$ given in detail for future reference. Together with the construction of the original derivation in Section 5 it establishes the existence of a solving substitution by induction on ϵ_0. Sections 4 and 5 provide realization of the steps 1–3 above leading to a termination proof for the ϵ-substitution process (Theorem 5.2) by induction on ϵ_0.

The results of this paper were presented at the workshop WOLLIC'06, Florianopolis, Brazil. The preliminary draft was published as [13].

2. Language of ϵ-calculus; H-process

2.1. The Language of ϵ-substitutions

Definition 2.1. *Variables* $x, y, z, \ldots$ represent natural numbers. *Numerical terms* are variables, 0, St and $\epsilon x F$ for all formulas F. There are many (as much as needed) primitive recursive *predicates* including =. *Formulas* are constructed from atomic formulas by propositional connectives: $\wedge FG, \to FG, \neg F, \ldots$ written as $(F \wedge G), (F \to G), \ldots$.

Quantifiers can be defined from ϵ in a standard way.

Critical formulas:

$$(pred) \quad s \neq 0 \to s = S\epsilon x(s = Sx)$$
$$(\epsilon) \quad F[t] \to F[\epsilon v F].$$

Definition 2.2. An ϵ-term is *canonical* if it is closed and contains no proper closed ϵ-subterms. An expression e is *simple* if it is closed and contains no ϵ. TRUE (FALSE) denotes the set of all true (false) simple formulas. [A simple formula contains no variables and is constructed from computable atomic formulas by Boolean connectives. Every simple term is a numeral].

$\mathbb{N}$ is the set of natural numbers.

Definition 2.3. An ϵ-*substitution* is a function from canonical ϵ-terms into the set $\{?\} \cup \mathbb{N}$.

A finite ϵ-substitution will be written as a finite list consisting of components of the form

$$(e, ?), (e, n).$$

Definition 2.4. Two ϵ-substitutions Σ, Θ are *multiplicable* if $\Theta \cup \Sigma$ is a function. In this case we write $\Theta * \Sigma$ for $\Theta \cup \Sigma$, and say that $\Theta * \Sigma$ is defined.

The set $FV(e)$ of free variables of an expression e is defined in the standard way: ϵx binds x. An expression e is *closed* iff $FV(e) = \emptyset$.

We identify expressions which are equivalent modulo renaming of bound variables; $e[x/u]$ denotes the result of substituting u for each free occurrence of x in e, where bound variables in e are renamed if necessary. If x is known from the context we write $e[u]$ for $e[x/u]$.

We assume as always a fixed system

$$\mathrm{Cr} = \{Cr_0, \ldots, Cr_N\}$$

of closed critical formulas.

2.2. Computations with the ϵ-substitutions

Definition 2.5. An ϵ-substitution S is *total* if $\mathrm{dom}(S)$ is the set of all canonical ϵ-terms.
$\overline{S} := S \cup \{(e, ?) : e$ is a canonical ϵ-term $\notin \mathrm{dom}(S)\}$ is called the *standard extension of S*.

(1) If $(e, u) \in S$ and $u \neq ?$, then $e \hookrightarrow^1_S u$
(2) If $(e, ?) \in S$, then $e \hookrightarrow^1_S 0$
(3) If $0 \leq i \leq n$, $e_i \hookrightarrow^1_S e'_i$ then $e_0 e_1 \ldots e_n \hookrightarrow^1_S e_0 \ldots e_{i-1} e'_i e_{i+1} \ldots e_n$
(4) If $F \hookrightarrow^1_S F'$ then $\epsilon x F \hookrightarrow^1_S \epsilon x F'$.

Definition 2.6. e is *S-reducible* if there exists an e' with $e \hookrightarrow^1_S e'$. Otherwise e is *S-irreducible* or *in S-normal form*. $\hookrightarrow_S$ denotes the transitive and reflexive closure of $\hookrightarrow^1_S$.
The unique S-irreducible expression e^* with $e \hookrightarrow_S e^*$ is called *the S-normal form of e* and denoted by $|e|_S$.

Definition 2.7. Let S be an ϵ-substitution.
An expression e is *S-computable* if $|e|_S$ does not contain closed ϵ-terms.
S computes a set Φ of closed formulas iff all formulas in Φ are S-computable.
For a pair $(\epsilon x F[x], n) \in S$ define

$$Cr(e, S) := F[\![n]\!] := F[x/n] \wedge \neg F[x/0] \wedge \cdots \wedge \neg F[x/(n-1)].$$

If t is not a numeral, then $F[\![t]\!] := F[t]$. Let

$$\mathcal{F}(S) := \{Cr(e, S) : (e, n) \in S \text{ for } n \in \mathbb{N}\}.$$

S is *computationally inconsistent (ci)* if $A \hookrightarrow_S \mathsf{FALSE}$ for some $A \in \mathcal{F}(S)$. Otherwise S is *computationally consistent (cc)*.
S is *correct* if $\wedge\mathcal{F}(S) \hookrightarrow_S \mathsf{TRUE}$.
Let

$$\mathcal{CR}(S) := \{F[\![|t|_S]\!] : \text{critical formula } F[t] \to F[\epsilon x F[x]] \text{ is in Cr.}\}$$

S is *computing* iff all formulas $A \in \mathcal{F}(S)$ are S-computable.
S is *deciding* iff S is computing and the critical formulas $Cr_0, \ldots, Cr_N$ are S-computable.
S is *solving* iff S is correct, deciding and $\mathrm{Cr} \hookrightarrow_S \top$.

Lemma 2.1. *Let $S' \supseteq S$ be substitutions. Then*

$$e \hookrightarrow_S e' \textit{ implies } e \hookrightarrow_{S'} e'.$$

In particular $e \hookrightarrow_S v$ for $v \in \omega \cup \{\mathsf{TRUE}, \mathsf{FALSE}\}$ implies $e \hookrightarrow_{S'} v$;
If S is c.i., then S' is c.i.

Proof. Every S-computation is an S'-computation. $\dashv$

2.3. The rank function

The rank is a measure of nesting of bound variables. For closed expressions it will be the same as in [6] and [11]. Note that an arbitrary closed ϵ-term $\epsilon x F$ can be written as

$$\epsilon x F \equiv \epsilon x F'[x_1/t_1, \ldots, x_n/t_n], \quad n \geq 0 \tag{2.1}$$

where $\epsilon x F'$ is canonical, and $t_1, \ldots, t_n$ are closed ϵ-terms.

Definition 2.8. If e does not contain ϵ, then $\mathrm{rk}(e) := 0$.
If $\epsilon x F$ is canonical, then

$$\mathrm{rk}(\epsilon x F) := \max\{\mathrm{rk}(f) : f \text{ is a closed } \epsilon\text{-subterm of } F[x/0]\} + 1.$$

In particular, if F does not contain ϵ, then $\mathrm{rk}(\epsilon x F) = 1$.
If (2.1) holds with a canonical $\epsilon x F'$, then

$$\mathrm{rk}(\epsilon x F) := \max\{\mathrm{rk}(\epsilon x F'[x_1/0, \dots, x_n/0], \mathrm{rk}(t_1), \dots, \mathrm{rk}(t_n)\}.$$

For an arbitrary closed expression e,

$$\mathrm{rk}(e) := \max\{\mathrm{rk}(t) : t \text{ is a closed } \epsilon\text{-subterm of } e\}.$$

Definition 2.9 (*Truncation to a Given Rank*). For each ϵ-substitution S and $r < \omega$ we set

$$S_{\leq r} := \{(e, u) \in S : \mathrm{rk}(e) \leq r\}.$$

Analogously we define $S_{\geq r}$, $S_{<r}$, $S_{>r}$.

Lemma 2.2. *If S, S' are ϵ-substitutions with $S_{\leq r} = S'_{\leq r}$ then $|e|_S = |e|_{S'}$ holds for all closed expressions e of rank $\leq r$.* ⊣

2.4. H*-term and* H*-value*

Definition 2.10. Let S be an ϵ-substitution such that $\overline{S}$ is non-solving. (Then $|Cr_I|_{\overline{S}} \in \mathsf{FALSE}$ for some $I \leq N$.)
Set $r_I := \mathrm{rk}(\epsilon x |F|_{\overline{S}})$, where $Cr_I = F_0 \to F[\epsilon x F]$.
$\mathrm{Cr}(S) := Cr_I$, where $I \leq N$ is such that
$|Cr_I|_{\overline{S}} \in \mathsf{FALSE} \,\&\, \forall J \leq N[\, |Cr_J|_{\overline{S}} \in \mathsf{FALSE} \Rightarrow r_I < r_J \vee (r_I = r_J \wedge I \leq J)]$.
Let $\mathrm{Cr}(S) = F_0 \to F[\epsilon x F]$:
$\epsilon x |F|_{\overline{S}}$ is called the *H-term of S*.
The *H-value* v of S is defined as follows
a) if $F_0 = (s \neq 0)$, and $F = (s = \mathsf{S}x)$ then $v := |s|_{\overline{S}} - 1$,
c) if $F_0 = F[t]$ then $v :=$ *the unique* $n \in \mathbb{N}$ *with* $|F|_{\overline{S}}[\![n]\!] \hookrightarrow_{\overline{S}} \mathsf{TRUE}$.

Definition 2.11. The H-rule applies to an ϵ-substitution S if S is cc, deciding, non-solving and computes $|F[\![|t|_S]\!]|_S$, where $\mathrm{Cr}(S)$ is $F[t] \to F[\epsilon x F]$.

2.5. H*-process and* H$'$*-process*

Definition 2.12 (*The Step of the ϵ-Substitution Process*). If $\overline{S}$ is nonsolving then

$$\mathrm{H}(S) := (S \setminus \{(e, ?)\})_{\leq \mathrm{rk}(e)} \cup \{(e, v)\},$$

where e is the H-term and v the H-value of S.

The following properties of H(S) are well known (cf. [6,11]).

Lemma 2.3 (*Properties of* H(S)). *Let S be an ϵ-substitution such that $\overline{S}$ is correct and non-solving, and let e be the H-term, v the H-value of S. Then the following holds:*
(a) $(e, ?) \in \overline{S}$,
(b) $|e|_{\mathrm{H}(S)} = v \neq 0$,
(c) $\overline{\mathrm{H}(S)}$ *is correct.* ⊣

Let us recall the definition of the ϵ-substitution process used by Ackermann and in almost all previous work on the ϵ-substitution.

Definition 2.13. The H-*process* for the system Cr of critical formulas $Cr_0, \ldots, Cr_N$ *with an initial substitution* S_0 is defined as follows:

$$S_{n+1} := \begin{cases} \mathrm{H}(S_n) & \text{if } \overline{S_n} \text{ is non-solving,} \\ \emptyset & \text{otherwise.} \end{cases}$$

The H-process *terminates* iff there exists an $n \in \mathbb{N}$ such that $\overline{S_n}$ is solving.

If the initial substitution is not mentioned (as mostly the case will be), it is assumed that $S_0 \equiv \emptyset$.

Now we adjust the previous definition.

Definition 2.14. Let the H-rule apply to S, e be the H-term, v the H-value of S. Then $\mathsf{H}'(S, S')$ means that $\{(e, v)\} \cup S_{<\mathrm{rk}(e)} \subseteq S' \subseteq \mathsf{H}(S)$,

An *indeterministic* H'*-process* for Cr with an initial substitution S_0 is an arbitrary sequence $(S_n)_{n<n_0\leq\omega}$ of substitutions such that for every $n < n_0 - 1$, if S_n is non-solving then $\mathsf{H}'(S_n, S_{n+1})$.

We define below an H'-step as an application of all possible versions of the H'-rule. After the very first step the H'-process works in parallel with a finite number of ϵ-substitutions.

Definition 2.15. Let $S^1, \ldots, S^m$ be correct substitutions such that $\overline{S^1}, \ldots, \overline{S^m}$ are all non-solving. For every $i \leq m$ let $S^{i1}, \ldots, S^{ip_i}$ be a complete list of substitutions S' such that $\mathsf{H}'(S, S')$. Define

$$\mathsf{H}'(S^1; \ldots; S^m;) := S^{11}; \ldots; S^{1p_1}; \ldots; S^{m1}; \ldots; S^{mp_m}.$$

The *deterministic* H'*-process* for Cr with an initial substitution S_0 is defined by

$$\mathbf{S}_0 := S_0; \qquad \mathbf{S}_{n+1} := \mathsf{H}'(\mathbf{S}_n),$$

if $\mathbf{S}_n$ is not a solution, that is it does not contain a solving component.

Lemma 2.4. *If an indeterministic* H'*-process of length n ends in a solution, then the deterministic* H'*-process terminates after at most n steps.*

Proof. Obvious induction on n. ⊣

3. The system PAϵ

3.1. Axioms and the inference rule of PAϵ

The system PAϵ is the arithmetical part of the infinitary system ϵEA from [11] with the only inference rule, Cut. Derivable objects or *sequents* of PAϵ are finite ϵ-substitutions.

Definition 3.1. Let Θ and Θ' be sequents that agree on their domain:
if $(e, v) \in \Theta$ and $(e, w) \in \Theta'$ then $v = w$.
Then $\Theta * \Theta' = \Theta \cup \Theta'$, and we say that $\Theta * \Theta'$ is defined.

In other words, $*$ amounts to contracting repetitions.

Definition 3.2. Let Θ, Ξ be a correct and deciding sequent, $\mathrm{rk}(A) \leq \mathrm{rk}(F)$ for all formulas $A \in \Theta$, $F \in \Xi$.

(1) Ξ is an S-completion of Θ iff Θ, Ξ is solving
(2) Ξ is an $\mathsf{H}_{e,v}$-completion of Θ iff the H-rule applies to Θ, Ξ, $(e, ?) \in \Theta$, e is the H-term, and v is the H-value of Θ, Ξ.

The *completion rank* is the minimal rank of formulas in Ξ.

Axioms:

$\mathsf{AxF}(\Theta)$	Θ is c.i.
$\mathsf{AxS}(\Theta)$	S-completion of Θ is given.
$\mathsf{AxH}_{e,v}(\Theta)$	$\mathsf{H}_{e,v}$-completion of Θ is given.

Rule of inference:

$$\frac{(e,?),\Theta\ldots\quad(e,n),\Theta\ldots\quad(n\in\mathbb{N})}{\Theta}\ \mathsf{Cut}_e.$$

We call e the *main term*, (e, v) the *side components*, $\mathrm{rk}(e)$ the *rank* of the Cut_e.

Definition 3.3. A *derivation* $d \in PA\epsilon$ is defined in a standard way with an additional
Proviso:
the completion rank of every axiom is $\geq$ maximal rank of cuts in d.

3.2. Simple properties of derivations

Definition 3.4. The *cut rank* $\mathrm{rk}(d)$ of a derivation d is the maximal rank of cut formulas in d.

The following obvious observation is used repeatedly.

Lemma 3.1. *Let d be a derivation of a sequent Θ. Then every sequent in d has a form*

$$\Theta, \Sigma$$

where Σ consists of the side components of all Cut *rules in d below this sequent.*
In particular, if $\mathrm{rk}(\Theta), \mathrm{rk}(d) < r$ *then all components in d have ranks $< r$.*

Proof. Bottom-up induction on the derivation d. Induction base is the endsequent Θ.
The induction step assumes $\Theta' \equiv \Theta, \Sigma$, $\mathrm{rk}(\Sigma) < r$ for the conclusion Θ' of a rule Cut_e, then concludes $\Theta'' \equiv \Theta, \Sigma, (e, v)$ for the premises with $\mathrm{rk}(\Sigma, (e, v)) < r$, since $\mathrm{rk}(e) < r$. ⊣

The following statements are used in the cut elimination proof below.

Lemma 3.2. *Let $(e, ?), \Pi$ with a completion Ξ be an* AxS*or* $\mathsf{AxH}_{f,m}$ *with $f \neq e$ and* $\mathrm{rk}(e) \geq \mathrm{rk}(\Pi)$.
Then Π is an axiom of the same kind with the completion $(e, ?), \Xi$.

Proof. The Definition 3.2 is to be checked for one and the same sequent $\Pi, (e, ?), \Xi$. The rank condition for completion follows from $\mathrm{rk}(e) \geq \mathrm{rk}(\Pi)$. In the case of $\mathsf{AxH}_{f,n}$ we have $(f, ?) \in \Pi$, since $(f, ?) \in (e, ?), \Pi$ and $f \neq e$. ⊣

Lemma 3.3. *Let*

$$(e, n), \Theta, \Delta \text{ with a completion } \Xi$$

be an AxH *or* AxS *and let Γ be a sequent such that*

$$\mathrm{rk}(\Theta, \Delta, \Gamma) \leq r = \mathrm{rk}(e), \tag{3.1}$$

$$(\mathrm{dom}\,\Gamma)_{=r} \cap (\mathrm{dom}\,\Xi)_{=r} = \emptyset, \tag{3.2}$$

$$((e, n), \Theta, \Delta) * \Gamma \text{ is a sequent }, e \notin \mathrm{dom}\,\Gamma \tag{3.3}$$

and

$$\mathcal{F}((e, n), \Theta, \Gamma) \hookrightarrow_{\Theta,\Gamma} \mathsf{TRUE}. \tag{3.4}$$

Then

$$(\Theta, \Delta) * \Gamma \text{ is } \mathsf{AxH} \text{ or } \mathsf{AxS} \text{ with a completion } (e, n), \Xi \tag{3.5}$$

or $(\Theta, \Delta) * \Gamma$ *is* AxF.

Proof. Assume that $(\Theta, \Delta) * \Gamma$ is c.c., since otherwise this sequent is AxF.

Case 1. Given sequent $(e, n), \Theta, \Delta$ is $\mathsf{AxH}_{f,m}$. Then the sequent

$$S \equiv (e, n), \Theta, \Delta, \Xi$$

is cc, deciding, non-solving and computes $|F[\![|t|_{S_i}]\!]|_S$, where $\mathrm{Cr}(S)$ is $F[t] \to F[\epsilon x F]$. The H-term and H-value of S are f and m and Θ, Δ contains $(f, ?)$. We have to verify similar relations for (3.5). Consider

$$S' \equiv (\Theta, \Delta) * \Gamma, (e, n), \Xi.$$

By (3.3), S' is a sequent unless $\mathrm{dom}\,\Gamma \cap \mathrm{dom}\,\Xi \neq \emptyset$, which is excluded by (3.1), $\mathrm{rk}(\Xi) \geq r$ and (3.2).

Since $\mathrm{Cr}_i \hookrightarrow_S$ FALSE for the critical formula Cr_i corresponding to the term f, and $S' \supseteq S$, we have $\mathrm{Cr}_i \hookrightarrow_{S'}$ FALSE, so S' is non-solving. Similarly $S' \supseteq S$ implies that S' computes all necessary formulas and by (3.4), S' computes $\mathrm{Cr}(S')$. Moreover all numerical and truth values computed by S are preserved by S'. In particular f and m are still the H-term and H-value of S'. The rank condition for completion follows from (3.1).

Case 2. Given that the sequent is AxS: Since $S' \supseteq S$, we only have to check the correctness of S'. This follows from the correctness of S and (3.4). $\dashv$

3.3. Cut reduction

Recall that $\mathrm{rk}(\Theta) = \max\{\mathrm{rk}(F) : F \in \Theta\}$. We describe a cut reduction of one Cut:

$$\frac{(e, ?), \Theta \quad \ldots (e, n), \Theta \ldots}{\Theta}\ \mathsf{Cut}_e. \tag{3.6}$$

Consider a derivation d ending in a cut of maximal rank r with $\mathrm{rk}(\Theta) \leq r$ and containing no other cuts of the rank $\geq r$.

$$\begin{array}{ll} (e, ?), \Theta, \Gamma & (e, n), \Theta, \Delta \\ \vdots & \vdots \\ (e, ?), \Theta, \Sigma & (e, n), \Theta, \Pi \\ \vdots & \vdots \end{array}$$
$$\frac{d_0 : (e, ?), \Theta \quad \ldots\ d_n : (e, n), \Theta \ldots}{d : \Theta}\ \mathsf{Cut}_e.$$

d is transformed as follows:

$$\begin{array}{ll} \Theta, \Delta * \Gamma & \\ \vdots & \\ \Theta, \Pi * \Gamma & (d_n - \{(e, n)\}) * \Gamma \\ \vdots & \\ \Theta, \Gamma & \\ \vdots & \\ \Theta, \Sigma & d_0 - \{(e, ?)\} \\ \vdots & \\ \Theta. & \end{array} \tag{3.7}$$

The lower part $d_0 - \{(e, ?)\}$ is obtained by deleting the component $(e, ?)$ from all sequents in d_0. If the upper sequent of some branch in d_0 is not of the form $\mathsf{AxH}_{e,n}$ nothing further is done with that sequent except adding $(e, ?)$ to the completion.

If some upper sequent $(e, ?), \Theta, \Gamma$ is $\mathsf{AxH}_{e,n}$, then the component $(e, ?)$ is deleted and the figure

$$(d_n - \{(e, n)\}) * \Gamma$$

obtained from d_n by deleting the component (e, n) and "multiplying" by Γ (Definition 4.1) is superimposed. In case when a Cut_f with $(f, v) \in \Gamma$ is encountered, all premises except the vth are deleted. Both the premise and conclusion of the cut

$$\frac{\ldots (f, v), \Theta, \Pi}{\Theta, \Pi}\ \mathsf{Cut}_f$$

become the sequent $(f, v), \Theta, \Pi * \Gamma$ and the cut is deleted.

Lemma 3.4. *The figure* (3.7) *is a derivation.*

Proof. First, every line in (3.7) is a sequent. This is evident for lines in $d_0 - \{(e, ?)\}$, and is proved by the bottom-up induction for lines in $(d_n - \{(e, n)\}) * \Gamma$. Indeed, by Lemma 3.1 $\mathrm{rk}(\Gamma) < r$ and Π consists of the side formulas (f, v) of cuts Cut_f in d_n. But then v (that is the premise of the Cut_f to be retained in $(d_n - \{(e, n)\}) * \Gamma$ is chosen so that $(f, v) \in \Gamma$ if $f \in \mathrm{dom}\,\Gamma$. Moreover, for every axiom $\mathsf{AxS}(\Pi)$, $\mathsf{AxH}(\Pi)$ in d_n with a completion Ξ the expression $(\Pi, \Xi) * \Gamma$ is a sequent, since $\mathrm{rk}(\Gamma) < r = rk(e) \le \mathrm{rk}(F)$ for every formula $F \in \Xi$.

It remains to check that all non-deleted axioms except $\mathsf{AxH}_{e,n}$ go into axioms after we move (e, n) to completion if necessary. The rank condition for the new completion will be satisfied, since $\mathrm{rk}(e) = r$.

(1) The axioms from d_0.
   (a) AxS, $\mathsf{AxH}_{f,v}$ for $f \neq e$. Use Lemma 3.2.
   (b) $\mathsf{AxF}((e, ?), \Theta, \Gamma)$. We have a computation

$$F[\![m]\!] \hookrightarrow \mathsf{FALSE}$$

for some $(f, m) \in \Theta$ with $f = \epsilon x F[x]$. Since the ranks of all formulas in Θ are $\le r$, the deleted formula $(e, ?)$ of rank r cannot take part in the computation by Lemma 2.2, hence $F[\![n]\!] \hookrightarrow_{\Theta} \mathsf{FALSE}$ as required for $\mathsf{AxF}(\Theta, \Gamma)$.

(2) The axioms from $(d_n - \{(e, n)\}) * \Gamma$.
   (a) AxS,$\mathsf{AxH}_{f,v}$, $f \neq e$. Use Lemma 3.3.
   (b) $\mathsf{AxF}((e, n), \Theta, \Delta)$. Since $\mathrm{rk}(e) \ge \mathrm{rk}((e, n), \Theta, \Delta)$ the component (e, n) is not used in the computation on the contradiction unless $e = \epsilon x F[x]$ and

$$F[\![n]\!] \hookrightarrow_{\Theta,\Delta} \mathsf{FALSE}$$

despite

$$F[\![n]\!] \hookrightarrow_{\Theta,\Gamma} \mathsf{TRUE}.$$

This is impossible. Indeed, consider the shortest term f that reduces differently under Θ, Δ and Θ, Γ. We have $(f, u) \in \Theta, \Delta$ and $(f, v) \in \Theta, \Gamma$ with $u \neq v$, hence $\Theta, \Delta * \Gamma$ is not a sequent.

This concludes the proof. ⊣

Theorem 3.5. *Cut elimination holds for $PA\epsilon$.*

Proof. Standard induction on cut degree. ⊣

Note. A cut-free proof of the empty sequent ∅ in $PA\epsilon$ consists only of a single axiom. Since AxF, AxH are non-empty, it should be AxS. In other words, the cut elimination accumulates a solution.

4. System $PA\epsilon'$

4.1. Axioms and rules of $PA\epsilon'$

Derived objects or *sequents of the system $PA\epsilon'$* are of the form

$$\Gamma; \Xi\ ,$$

where Γ, Ξ is a substitution such that

$$\mathrm{rk}(\Gamma) \le \mathrm{rk}(F) \text{ for every formula } F \in \Xi.$$

Γ is the *fixed part*, Ξ is the *completion*. For $T \equiv \Gamma, \Xi$ we write $T^f := \Gamma,\ T^c := \Xi$.

Definition 4.1. Let $\Theta; \Xi$ and $\Theta'; \Xi'$ be sequents of $PA\epsilon'$ that agree on their domain:
if $(e, v) \in \Theta, \Xi$ and $(e, w) \in \Theta', \Xi'$ then $v = w$.
Then $(\Theta; \Xi) * (\Theta'; \Xi') := \Theta \cup \Theta'; \Xi \cup \Xi'$, and we say that $(\Theta; \Xi) * (\Theta', \Xi')$ is defined.

In other words, $*$ amounts to contracting repetitions.

The cut rule and axioms are changed very little compared to $PA\epsilon$: now completions are shown explicitly. Two new rules are added: Fr and H′. Both change only completion.

Axioms:

$\mathsf{AxF}(\Theta; \Xi)$	Θ is c.i.
$\mathsf{AxS}(\Theta; \Xi)$	Θ, Ξ is solving.
$\mathsf{AxH}_{e,v}(\Theta; \Xi)$	Ξ is an $\mathsf{H}_{e,v}$-completion of Θ.

Rules of inference:

$$\frac{(e, ?), \Theta; \Xi \ldots \quad (e, n), \Theta; \Xi \ldots \quad (n \in \mathbb{N})}{\Theta; \Xi}\ \mathsf{Cut}_e$$

$$\frac{\Theta; (e, ?), \Xi}{\Theta; \Xi}\ \mathsf{Fr}_e \qquad \frac{\Theta; (e, n), \Xi'}{\Theta; (e, ?), \Xi}\ \mathsf{H}'_{e,n}$$

where H-rule applies, e, n are H-term and H-value of cc and deciding sequent $\Theta, (e, ?), \Xi$ and $\Xi_{<r} \subseteq \Xi' \subseteq \Xi_{\leq r}$.

The *main term*, the *side components*, the *rank* of a Cut are defined as before.

Definition 4.2. A *derivation d* in $PA\epsilon'$ is defined in a standard way with the following Proviso:

(1) The completion rank of every H-axiom is $\geq$ maximal rank of cuts in d.
(2) Fr_e, $\mathsf{H}'_{e,n}$ do not occur below a cut of rank $\geq \mathrm{rk}(e)$ or in a derivation containing cuts of rank $> \mathrm{rk}(e)$.

The first proviso is the same as that in $PA\epsilon$ (Definition 3.3) , the second proviso replaces the machinery of r-derivations and $r+$-derivations from [11].

The *rank* of the derivation is as before the maximal rank of Cut.

4.2. Cut elimination in $PA\epsilon'$

Definition 4.3. If d is a derivation in $PA\epsilon'$ let d^{fix} be the result of retaining only fixed parts of sequents. More precisely, replace $\Theta; \Xi$ by Θ retaining completion Ξ only in axioms, and delete all Fr,H-inferences.

The following propositions are similar to the corresponding propositions in the previous section.

Lemma 4.1. *Let d be a derivation in $PA\epsilon'$ of rank $< r$ of a sequent $\Theta; \Xi$. Then*

(1) *d^{fix} is a derivation of Θ in $PA\epsilon$.*
(2) *Every sequent in d has a form $\Theta, \Sigma; \Xi'$ where Σ consists of the side components of* Cut *rules in d. In particular, if* $\mathrm{rk}(\Theta) < r$*,then* $\mathrm{rk}(\Theta, \Sigma) < r$.

Proof. (1) Rules Fr, H become repetitions after completions are deleted. Cut inferences and axioms are preserved.
(2) Immediate by Lemma 3.1 (or by induction on d). $\dashv$

The following two statements are easy consequences of Lemmas 3.2 and 3.3 respectively.

Lemma 4.2. *Let $(e, ?), \Pi; \Xi$ be an* AxS *or* $\mathsf{AxH}_{f,m}$ *with $f \neq e$ and* $\mathrm{rk}(e) \geq \mathrm{rk}(\Pi)$.
Then $\Pi; (e, ?), \Xi$ is an axiom of the same kind with the completion $(e, ?), \Xi$. $\dashv$

Lemma 4.3. *Let $(e, n), \Theta, \Delta; \Xi$ be an* AxH *or* AxS *and let Γ be a sequent such that* $\mathrm{rk}(\Gamma) < r = \mathrm{rk}(e)$, *$((e, n), \Theta, \Delta) * \Gamma$ is a sequent and $\mathcal{F}((e, n), \Theta, \Gamma) \hookrightarrow_{\Theta,\Gamma}$* TRUE. *Then $(\Theta, \Delta) * \Gamma; (e, n), \Xi$ is* AxH*or* AxS*or or $(\Theta, \Delta) * \Gamma$ is* AxF.

Before describing the reduction of the uppermost cut of the maximal rank r let us recall a transformation from [11], Lemma 6.4 that supplements a new H'-rule by inserting Fr and H'-rules copied from the path leading from the endsequent $\emptyset$ to the conclusion of the reduced cut.

Definition 4.4. A *deduction* is a tree that proceeds by inference rules, but may begin with arbitrary sequents (not axioms).

Lemma 4.4. *Let $\pi := (S_0, \ldots, S_n)$ be a path in a correct derivation d of cut rank r leading from the endsequent $S_0 = \emptyset$ to a conclusion $S_n \equiv \Theta; \Xi$ of an uppermost cut Cut_e with $\mathrm{rk}e = r$. Assume that $\mathrm{rk}(\Gamma) < r$ and $\Theta, \Gamma, (e, n)$ is a correct sequent.*

Then there is a correct deduction (called below $\mathcal{FH}(\pi, \Gamma, e, n)$) of $\Theta, \Gamma; (e, n)$ from $\Theta, \Gamma; (e, n), \Xi$ consisting exactly of Fr and H'-inferences in π, all of them of ranks $> r$.

Proof. Let $T := \Theta, \Gamma; (e, n)$. Since the cut rank is r, all Fr, H'-inferences in the given path have ranks $> r$. Hence for all $i \leq n$, S_i^c consists of some side components of such inferences of ranks $> r$, while $S_i^f \leq r$ and consists of all side components of Cut inferences in the path $(S_0, \ldots, S_i)$. So for all $i \leq n$,

$$S_i * T = S_i^f, \Theta, \Gamma; S_i^c, (e, n).$$

We prove that $S_n * T, \ldots, S_0 * T$ with repetitions removed is the required deduction.

Let $1 \leq k \leq n$ and $S_{k-1} = \Theta'; \Xi'$. Consider the inference from S_k to S_{k-1}.

(1) Cut_f. Since $\mathrm{rk}f \leq r$, a side formula (f, u) belongs to Θ, hence to $S_{k-1} * T$, hence the Cut becomes a repetition and is removed.

(2) Fr_f. We have $e \notin \mathrm{dom}((f, ?), \Xi')$ and since $\mathrm{rk}f > r$, the inference before and after multiplication by T looks as follows:

$$\frac{\Theta; (f, ?), \Xi'}{\Theta; \Xi'}\ \mathsf{Fr}_f \qquad \frac{\Theta, \Gamma; (f, ?), \Xi', (e, n)}{\Theta, \Gamma; \Xi', (e, n)}\ \mathsf{Fr}_f.$$

(3) $\mathsf{H}'_{f,m}$.

$$\frac{\Theta; (f, m), \Xi'_{\leq r'} - \Delta'}{\Theta; (f, ?), \Xi'}\ H'_{f,m} \qquad \frac{\Theta, \Gamma; (e, n), (f, m), \Xi'_{\leq r'} - \Delta'}{\Theta, \Gamma; (e, n), (f, ?), \Xi'}\ H'_{f,m}.$$

The H'-rule still applies to the new conclusion and is correct. ⊣

The subderivation ending with an uppermost cut of maximal rank has a form:

$$\begin{array}{ccc}
(e, ?), \Theta, \Gamma; \Xi_2 & & (e, n), \Theta, \Delta; \Xi_4 \\
\vdots & & \vdots \\
(e, ?), \Theta, \Sigma; \Xi_1 & & (e, n), \Theta, \Pi; \Xi_3 \\
\vdots & & \vdots \\
\multicolumn{3}{c}{\dfrac{d_0 : (e, ?), \Theta; \Xi \quad \ldots\ d_n : (e, n), \Theta; \Xi \ldots}{d : \Theta; \Xi}\ \mathsf{Cut}_e.}
\end{array} \tag{4.1}$$

We assume that $\mathrm{rk}(\Theta) \leq r = \mathrm{rk}(e)$ and d_i are derivations containing no cuts of rank $\geq r$. This implies $\mathrm{rk}(\Xi), \mathrm{rk}(\Xi_i) \geq r$. Now d is transformed similarly to Section 3.3, but Fr and H-inferences are added to account for completions.

$$
\begin{array}{llll}
\Theta, \Delta * \Gamma & ; & (e,n), \Xi_4 & \\
\vdots & & & \\
\Theta, \Pi * \Gamma & ; & (e,n), \Xi_3 & (d_n \to \{(e,n)\}) * \Gamma \\
\vdots & & & \\
\Theta, \Gamma & ; & (e,n), \Xi & \\
\vdots & & & \mathcal{FH}(\pi, \Gamma, e, n) \\
\dfrac{\Theta, \Gamma}{\Theta, \Gamma} & \dfrac{;}{;} & \dfrac{(e,n)}{(e,?), \Xi_2}\ \mathsf{H}'_{e,n} & \\
\vdots & & & \\
\Theta, \Sigma & ; & (e,?), \Xi_1 & d_0 \to \{(e,?)\} \\
\vdots & & & \\
\dfrac{\Theta}{\Theta} & \dfrac{;}{;} & \dfrac{(e,?), \Xi}{\Xi}\ \mathsf{Fr}_e . &
\end{array}
\tag{4.2}
$$

The lower part $d_0 \to \{(e, ?)\}$ is obtained by moving the component $(e, ?)$ in all sequents in d_0 from the fixed part into the completion. If the upper sequent of some branch in d_0 is not of the form $\mathsf{AxH}_{e,n}$ nothing further is done with that sequent.

If some upper sequent $(e, ?), \Theta, \Gamma; \Xi_2$ is $\mathsf{AxH}_{e,n}$, then the figure denoted by $(d_n \to \{(e, n)\}) * \Gamma$ is superimposed. This figure is obtained from d_n by moving the component (e, n) to the completion part and "multiplying" by Γ (Definition 4.1).

Definition 4.5. The figure (4.2) is denoted by $\mathcal{R}(d)$. $\mathcal{R}$ is called the *cut reduction transformation*.

Lemma 4.5. *The figure $\mathcal{R}(d)$ is a derivation.*

Proof. The proof is similar to Lemma 3.4. Since the Cut_e is the uppermost cut of rank $r = \mathrm{rk}(e)$, all cuts above it are of rank $< r$, hence $\mathrm{rk}(\Gamma) < r$ by Lemma 4.1. Now by the same induction as before, every line in (4.2) is a sequent. We note that the explicitly shown figure $\mathsf{H}'_{e,n}$ is indeed an application of the H'-rule and that other H'-inferences present in the derivation are preserved by our transformation.

The rank conditions for the new completion will be satisfied, since $\mathrm{rk}(e) = r$. Let us check the axioms.

(1) The axioms from d_0.
  (a) AxS, $\mathsf{AxH}_{f,v}$ for $f \neq e$. Use Lemma 4.2.
  (b) $\mathsf{AxF}((e, ?)\Theta, \Gamma)$. We have a computation

$$F[\![m]\!] \hookrightarrow \mathsf{FALSE}$$

for some $(f, m) \in \Theta$ with $f = \epsilon x F[x]$. Since the ranks of all formulas in Θ are $\leq r$, the deleted formula $(e, ?)$ of rank r cannot take part in the computation, hence $F[\![n]\!] \hookrightarrow_{\Theta} \mathsf{FALSE}$ as required for $\mathsf{AxF}(\Theta, \Gamma)$.

(2) The axioms from $(d_n \to \{(e, n)\}) * \Gamma$. Using $\mathrm{rk}(\Gamma) < r$ and Lemma 4.3, the proof is as before. ⊣

Theorem 4.6. *Every derivation of $\emptyset$ in $PA\epsilon'$ can be transformed into a cut-free derivation of $\emptyset$ by cut reduction transformations.*

Proof. Standard by induction on cut rank. ⊣

5. Original derivation and termination of the H'-process

Lemma 5.1. *For every finite system* Cr *of critical formulas one can construct (primitive recursively in* Cr*) a derivation of $\emptyset$ in $PA\epsilon'$ consisting of axioms and* Cut *inferences of rank $\leq r_0 = \mathrm{rk}(\mathrm{Cr})$ with the ordinal height $< \omega \cdot r_0 + \omega$*

Proof. The proof is given in detail in [11], Section 6.3. First (Lemma 6.11 in [11]) the empty sequent is expanded by bottom-up applications of the cut rule to compute all closed subterms of Cr . In particular every sequent has an empty completion, so the provisos in Definition 4.2 are obviously satisfied.

If for example Cr contains a subterm $\epsilon x A[x, \epsilon y B[x, y, \epsilon v D[v]]]$, the cuts are applied from the bottom up in the following order:

$$\cfrac{\cfrac{\cfrac{\ldots(\epsilon y B[u_a, y, u_d], u_b), (\epsilon x A[x, \epsilon y B[x, y, u_d]], u_a), (\epsilon v D, u_d)\ldots}{\ldots(\epsilon x A[x, \epsilon y B[x, y, u_d]], u_a), (\epsilon v D, u_d)\ldots}\ \text{Cut}}{\ldots(\epsilon v D, u_d)\ldots}\ \text{Cut}}{\emptyset}\ \text{Cut} \quad .$$

This stage leads to a tree of finite height bounded by the number of subterms of Cr.

After that each upper sequent $[(\epsilon x A[x, u_b], u_d), (\epsilon y B[x, u_d], u_b), (\epsilon v D, u_d)$ in our example] is similarly extended by the cuts upwards so that for every component $(\epsilon z C[z], n)$ present in the sequent all subterms of the formulas $C[\![n]\!]$ have values. This is done first for terms of rank r_0, then for rank $r_0 - 1$, etc. Reduction of rank by 1 increases the ordinal height by (at most) ω leading to a tree of height $< \omega \cdot r_0 + \omega$. Each of the uppermost sequents of this tree is either c.i., hence an AxF, or a solution, hence AxS, or AxH. $\dashv$

Theorem 5.2. *The* H′*-process terminates in a solution.*

Proof. Take the original derivation of $\emptyset$, transform it into a cut-free derivation d of $\emptyset$. It contains only rules Fr, H′, hence all sequents have empty fixed parts and there is only one branch. Since the derivation is well founded, this branch is finite. The top sequent of the branch is AxS, since AxF,AxH have non-empty fixed parts. Hence the result of erasing all Fr-inferences from d (as in [9,11]) is an indeterministic H′-process. Apply Lemma 2.4. $\dashv$

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