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Designing a neutral elliptic inhomogeneity in the case of a general non-uniform loading

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Abstract

We derive a general expression for an interface parameter which makes possible the design of a neutral elliptic inhomogeneity when the stress field in the surrounding matrix is a polynomial function of n th order and the composite is subjected to antiplane shear deformations.

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1. Introduction

A hole introduced into an elastic body will inevitably disturb the body's original stress field and often lead to a stress concentration. Mansfield [1] was among the first to recognize the feasibility of designing a reinforced 'neutral' hole which eliminates any stress concentrations introduced by the hole and hence does not disturb the original stress field in the uncut body.

The analogous problem of a neutral elastic inhomogeneity was studied by Ru in [2]. Here, it was shown that neutral elastic inhomogeneities cannot exist when a conventional perfectly bonded material interface is assumed to exist between the inhomogeneity and the surrounding elastic body. In addition, Ru introduced a method for the design of neutral inhomogeneities based on an established spring–layer model of an imperfect interface (see, for example, [3–10]). The neutral inhomogeneities designed in [2], however, assume the existence of a uniform stress field in the surrounding matrix. Of more practical

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interest is the case where the stress field in the matrix is *non-uniform*. In [11], Van Vliet et al. extended the techniques used in [2] to the case where the stress field in the matrix is *non-uniform* when the inhomogeneity is circular or elliptic and the composite is subjected to antiplane shear deformations. Here, however, the elliptic inhomogeneity is considered only for the simplest non-uniform (linear) stress field in the surrounding matrix, mainly because the complicated nature of the analysis involved precludes the extension of the method to cases of higher order. In [12], Schiavone employed a semi-analytical method to extend the results of [11] to the case of a quadratic stress field in the surrounding matrix.

In this work, we derive a general expression for an (imperfect) interface parameter which allows us to generalize the results in [12] to the case where the stress field in the surrounding matrix is characterized by a general polynomial of n th order.

2. Formulation

Consider a homogeneous and isotropic, linearly elastic body, finite or infinite in extent, simply or multiply connected, which is subjected to a given state of stress under a prescribed loading system. Assume that the same elastic body is then cut into a number of simply connected sub-domains each of which is filled with a different homogeneous and isotropic linearly elastic material (each sub-domain is now referred to as an inhomogeneity). Here, we are concerned with the design of the material interface between any single inhomogeneity and the elastic body such that the corresponding inhomogeneity is “neutral” in the sense that it does not disturb the original prescribed stress field in the uncut elastic body.

Considering antiplane shear deformations (see, for example, [13]), we represent the matrix by the domain S_1 and assume that the inhomogeneity occupies a region S_2 , with associated shear moduli $\mu_1 (>0)$ and $\mu_2 (>0)$, respectively. The inhomogeneity–matrix interface will be denoted by the curve Γ . Let (x, y) denote a generic point in \mathbb{R}^2 and $z = x + iy = re^{i\theta}$ the complex coordinate. In what follows, the subscripts 1 and 2 will refer to the regions S_1 and S_2 , respectively and $u_\alpha(x, y)$, $\alpha = 1, 2$, will denote the (harmonic) elastic (antiplane) displacement at the point (x, y) in S_α , respectively.

The ‘spring–layer type’ interface Γ can be defined by the condition [14]

$$\beta(x, y)[u_1 - u_2] = \mu_2 \frac{\partial u_2}{\partial n} = \mu_1 \frac{\partial u_1}{\partial n}, \quad \text{on } \Gamma, \tag{1}$$

where n is the outward unit normal to Γ and $\beta(x, y) : \Gamma(\subset \mathbb{R}^2) \rightarrow \mathbb{R}^+$ is the imperfect interface function. Practically, the interface Γ will represent the adhesive layer between the inhomogeneity and the body. Hence, β should be inversely proportional to the thickness or directly proportional to the density of the adhesive layer (see, for example, [3,4] and [7]). In accordance with [6] we note that if $\beta = 0$, the condition (1) reduces to the case of a traction free interface while if β is infinite (1) corresponds to a perfectly bonded interface. Thus, the function β can be selected by varying the properties of the adhesive layer. The only restriction is that β must be non-negative everywhere. The following boundary value problem describes the antiplane deformation of an inhomogeneity with an imperfect interface of the form (1):

$$\begin{aligned} \nabla^2 u_1 &= 0 && \text{in } S_1, \\ \text{and } \nabla^2 u_2 &= 0 && \text{in } S_2, \\ \beta(x, y)(u_1 - u_2) &= \mu_2 \frac{\partial u_2}{\partial n}, && \text{on } \Gamma \end{aligned} \tag{2}$$

$$\mu_1 \frac{\partial u_1}{\partial n} = \mu_2 \frac{\partial u_2}{\partial n}, \quad \text{on } \Gamma.$$

Denote by $v_i(x, y)$ the harmonic functions conjugate to $u_i(x, y)$. Since the external loading is self-equilibrated, $v_i(x, y)$ are single-valued and uniquely determined to within an integration constant and the corresponding complex potentials $\phi_1(z)$ and $\phi_2(z)$, with $z = x + iy$, are analytic within S_1 and S_2 , respectively. Thus,

$$\begin{aligned} 2u_i(z) &= \phi_i(z) + \overline{\phi_i(z)}, \\ \sigma_{13} - i\sigma_{23} &= \mu_i \phi'_i(z), \quad z \in S_i \quad (i = 1, 2). \end{aligned} \tag{3}$$

Noting that

$$2 \frac{\partial u_2}{\partial n} = \phi'_2(z)e^{in(z)} + \overline{\phi'_2(z)}e^{-in(z)}, \quad z \in \Gamma, \tag{4}$$

where $e^{in(z)}$ represents (in complex form) the outward unit normal to Γ at z , the boundary value problem (2) can be written in the following form:

$$\phi_1(z) = \delta\phi_2(z) + (1 - \delta)\overline{\phi_2(z)} + h(z)[\phi'_2(z)e^{in(z)} + \overline{\phi'_2(z)}e^{-in(z)}], \quad z \in \Gamma. \tag{5}$$

Here,

$$h(z) \equiv \frac{\mu_2}{2\beta(z)} \geq 0, \quad \delta \equiv \frac{\mu_1 + \mu_2}{2\mu_1} > \frac{1}{2}. \tag{6}$$

Taking the imaginary part of (5) yields

$$\text{Im } \phi_1(z) = (2\delta - 1)\text{Im } \phi_2(z). \tag{7}$$

Let the prescribed stress field be characterized by $\phi_1 = \sum_{n=0}^N A_n z^n$, where $A_n \in \mathbb{C}$ are given and $N = 1, 2, \dots$. According to the definition of a neutral inhomogeneity, the original stress field in the uncut elastic body remains undisturbed when the neutral inhomogeneity is inserted; then we have $\phi_1 = \sum_{n=0}^N A_n z^n$ in S_1 . Hence, for a neutral inhomogeneity, we require, from (5),

$$\sum_{n=0}^N A_n z^n = \delta\phi_2(z) + (1 - \delta)\overline{\phi_2(z)} + h(z)[\phi'_2(z)e^{in(z)} + \overline{\phi'_2(z)}e^{-in(z)}], \quad z \in \Gamma. \tag{8}$$

3. Elliptic inhomogeneity

Consider an elliptic inhomogeneity, centered at the origin, with axes of lengths a and b ($a \neq b$), coincident with the x and y axes, respectively. Suppose that the region S_1 (in the z -plane) is mapped onto the region $\sigma = \{|\xi| \geq 1\}$ (in the ξ -plane) by the function [15]

$$z = w(\xi) = R \left(\xi + \frac{k^2}{\xi} \right), \quad k \in (0, 1), \quad R > 0. \tag{9}$$

Then

$$z^m = R^m \left(\xi + \frac{k^2}{\xi} \right)^m = R^m \sum_{s=0}^m \binom{m}{s} \xi^{m-2s} k^{2s}, \quad m = 1, 2, 3, \dots$$

Suppose the stress field inside the inhomogeneity is characterized by

$$\phi_2(z) = \sum_{n=0}^N B_n z^n,$$

where B_n are complex coefficients to be determined. The interface condition in (8) now becomes

$$\sum_{n=0}^N A_n z^n - \delta \sum_{n=0}^N B_n z^n - (1 - \delta) \sum_{n=0}^N \bar{B}_n \bar{z}^n = h(z)[\phi_2'(z)e^{in(z)} + \overline{\phi_2'(z)}e^{-in(z)}], \quad z \in \Gamma.$$

In the ξ -plane, this is

$$\begin{aligned} & \sum_{n=0}^N A_n R^n \sum_{s=0}^n \binom{n}{s} \xi^{n-2s} k^{2s} - \delta \sum_{n=0}^N B_n R^n \sum_{s=0}^n \binom{n}{s} \xi^{n-2s} k^{2s} \\ & - (1 - \delta) \sum_{n=0}^N \bar{B}_n R^n \sum_{s=0}^n \binom{n}{s} \xi^{2s-n} k^{2s} \\ & = h(w(\xi))[\phi_2'(w(\xi))e^{in(w(\xi))} + \overline{\phi_2'(w(\xi))}e^{-in(w(\xi))}] \end{aligned} \tag{10}$$

where $\xi \in \partial\sigma = \{|\xi| = 1\}$. Next, we expand $h(w(\xi))[\phi_2'(w(\xi))e^{in(w(\xi))} + \overline{\phi_2'(w(\xi))}e^{-in(w(\xi))}]$ in Laurent's series to obtain

$$h(w(\xi)) = [\phi_2'(w(\xi))e^{in(w(\xi))} + \overline{\phi_2'(w(\xi))}e^{-in(w(\xi))}]^{-1} \sum_{n=-\infty}^{\infty} E_n \xi^n, \tag{11}$$

where

$$E_n = \frac{1}{2\pi i} \int_{\partial\sigma} h(w(\xi))[\phi_2'(w(\xi))e^{in(w(\xi))} + \overline{\phi_2'(w(\xi))}e^{-in(w(\xi))}] \frac{d\xi}{\xi^{n+1}}$$

are fixed. The interface condition (10) now becomes

$$\begin{aligned} & \sum_{n=0}^N A_n R^n \sum_{s=0}^n \binom{n}{s} \xi^{n-2s} k^{2s} - \delta \sum_{n=0}^N B_n R^n \sum_{s=0}^n \binom{n}{s} \xi^{n-2s} k^{2s} \\ & - (1 - \delta) \sum_{n=0}^N \bar{B}_n R^n \sum_{s=0}^n \binom{n}{s} \xi^{2s-n} k^{2s} \\ & = \sum_{n=-\infty}^{\infty} E_n \xi^n. \end{aligned} \tag{12}$$

Since the A_n are given and the E_n are fixed, we can equate coefficients of ξ^n in (12) and establish equations for the B_n in terms of the E_n and A_n .

Next [16], on $\partial\sigma$,

$$\begin{aligned} e^{in(w(\xi))} &= \xi \frac{w'(\xi)}{|w'(\xi)|} \\ &= \frac{\xi - k^2 \xi^{-1}}{|1 - k^2 \xi^{-2}|}. \end{aligned}$$

Thus, from (11), the interface function is given by

$$\begin{aligned}
 h(w(\xi)) &= [\phi_2'(w(\xi))e^{in(w(\xi))} + \overline{\phi_2'(w(\xi))}e^{-in(w(\xi))}]^{-1} \sum_{n=-\infty}^{\infty} E_n \xi^n \\
 &= \frac{|1 - k^2 \xi^{-2}| \sum_{n=-\infty}^{\infty} E_n \xi^n}{\phi_2'(w(\xi))(\xi - k^2 \xi^{-1}) + \overline{\phi_2'(w(\xi))}(\xi^{-1} - k^2 \xi)}.
 \end{aligned} \tag{13}$$

Finally, we must impose the conditions:

$$\text{Im } h(w(\xi)) = 0, \quad \text{Re } h(w(\xi)) > 0 \tag{14}$$

in order to maintain the physical meaning of the interface function β .

The conditions (14) allow us to write the B_n (found above) entirely in terms of the A_n and the (known) material constants. Finally, we can construct the interface function from (13).

4. General solution

Equating coefficients of ξ^n on both sides of the interface condition (12) we obtain the coefficients E_n as follows:

$$\begin{aligned}
 E_n^N &= \sum_{i=n, n+2, n+4, \dots}^N A_i R^i k^{i-n} \binom{i}{\frac{i-n}{2}} - \delta \times \sum_{i=n, n+2, n+4, \dots}^N B_i R^i k^{i-n} \binom{i}{\frac{i-n}{2}} \\
 &\quad - (1 - \delta) \times \sum_{i=n, n+2, n+4, \dots}^N \bar{B}_i R^i k^{i+n} \binom{i}{\frac{i-n}{2}}, \quad n = -N, \dots, N \tag{15} \\
 E_n &= 0, \quad n < -N, \quad n > N.
 \end{aligned}$$

(This result can be proved using mathematical induction—the details have been omitted for brevity.)

Let $h_0 = \frac{\mu_2}{2\beta} > 0$ corresponding to the case where $\beta > 0$ is uniform. The conditions (14) are satisfied if the coefficients in (15) take the form

$$\begin{aligned}
 E_n^N &= h_0 C_n^N, \\
 E_{-n}^N &= h_0 \bar{C}_n^N, \quad n = -N, \dots, N,
 \end{aligned} \tag{16}$$

where

$$\begin{aligned}
 C_n^N &= n B_n R^{n-1} + \sum_{i=n+2, n+4, \dots}^N B_i R^{i-1} k^{i-1} \binom{i}{\frac{i-n}{2}} - n \bar{B}_n R^{n-1} k^{2n} \\
 &\quad + \sum_{i=n+2, n+4, \dots}^N \bar{B}_i R^{i-1} k^{i+1} \binom{i}{\frac{i-n}{2}}.
 \end{aligned}$$

Since the constants A_0 and B_0 do not affect the stresses, we set them to zero. Eqs. (15) and (16) can now be solved to obtain the constants $B_i, i = 0, \dots, N$, in terms of the known constants A_j . In particular,

$$A_0 = B_0 = B_{N-1} = A_{N-1} = 0,$$

$$h_0 = \frac{R(1 - k^{2N})(\delta - 1)}{N(k^{2N} + 1)}, \quad N = 2, 3, \dots \tag{17}$$

Note that $h_0 > 0$, when $\mu_2 > \mu_1$. This requires that, for neutrality, the inhomogeneity has to be harder than the surrounding matrix material.

From (13), the interface function is characterized by

$$h(w(\xi)) = h_0 \left| 1 - \frac{k^2}{\xi^2} \right| > 0.$$

Writing $a = R(1 + k^2)$ and $b = R(1 - k^2)$, the interface function becomes

$$h(x, y) = \frac{h_0 b}{R} \left[1 + \frac{a^2}{b^2} \left(\frac{1}{b^2} - \frac{1}{a^2} \right) y^2 \right]^{\frac{1}{2}} = \frac{\mu_2}{2\beta(x, y)}$$

which leads to the interface parameter

$$\beta(x, y) = \frac{R\mu_2}{2h_0 b \left[1 + \frac{a^2}{b^2} \left(\frac{1}{b^2} - \frac{1}{a^2} \right) y^2 \right]^{\frac{1}{2}}}.$$

Example

Let $N = 3$. From (15) we obtain

$$\begin{aligned} E_{-3} &= A_3 R^3 k^6 - \delta B_3 R^3 k^6 - (1 - \delta) \bar{B}_3 R^3 \\ E_{-2} &= A_2 R^2 k^4 - \delta B_2 R^2 k^4 - (1 - \delta) \bar{B}_2 R^2, \\ E_{-1} &= A_1 R k^2 - \delta B_1 R k^2 - (1 - \delta) \bar{B}_1 R + 3k^4 A_3 R^3 - 3\delta B_3 R^3 k^4 - 3(1 - \delta) \bar{B}_3 R^3 k^2 \\ E_0 &= (A_0 + 2A_2 R^2 k^2) - \delta (B_0 + 2B_2 R^2 k^2) - (1 - \delta) (\bar{B}_0 + 2\bar{B}_2 R^2 k^2), \\ E_1 &= A_1 R - \delta B_1 R - (1 - \delta) \bar{B}_1 R k^2 + 3A_3 R^3 k^2 - 3\delta B_3 R^3 k^2 - 3(1 - \delta) \bar{B}_3 R^3 k^4, \\ E_2 &= A_2 R^2 - \delta B_2 R^2 - (1 - \delta) \bar{B}_2 R^2 k^4, \\ E_3 &= A_3 R^3 - \delta B_3 R^3 - (1 - \delta) \bar{B}_3 R^3 k^6, \\ E_n &= 0, \quad n < -3, \quad n > 3. \end{aligned} \tag{18}$$

From (16), we have

$$\begin{aligned} E_{-3} &= 3h_0 R^2 (\bar{B}_3 - k^6 B_3) \\ E_{-2} &= 2h_0 R (\bar{B}_2 - B_2 k^4) \\ E_{-1} &= h_0 \left(\bar{B}_1 - B_1 k^2 + 3\bar{B}_3 R^2 k^2 - 3B_3 R^2 k^4 \right) \\ E_0 &= 0, \\ E_1 &= h_0 \left(B_1 - \bar{B}_1 k^2 + 3B_3 R^2 k^2 - 3\bar{B}_3 R^2 k^4 \right), \\ E_2 &= 2h_0 R (B_2 - \bar{B}_2 k^4), \\ E_3 &= 3h_0 R^2 (B_3 - k^6 \bar{B}_3). \end{aligned} \tag{19}$$

Since the constants A_0 and B_0 do not affect the stresses, we set them to zero. Eqs. (18) and (19) can now be solved to obtain

$$\begin{aligned} A_0 &= B_0 = B_2 = A_2 = 0, \\ \operatorname{Re} A_1 &= \operatorname{Re} B_1 = \operatorname{Re} A_3 = \operatorname{Re} B_3 = 0, \\ B_1 &= \frac{i \operatorname{Im} A_1}{(2\delta - 1)}, \quad B_3 = \frac{i \operatorname{Im} A_3}{(2\delta - 1)} \\ h_0 &= \frac{R(1 - k^6)(\delta - 1)}{3(k^6 + 1)}. \end{aligned}$$

In other words, if the (quadratic) stress in the matrix is characterized by $\phi_1(z) = A_1 z + A_3 z^3$ ($\operatorname{Re}(A_1, A_3) = 0$) then the elliptic inhomogeneity is neutral with interior stress described by $\phi_2(z) = \frac{i \operatorname{Im} A_1}{(2\delta - 1)} z + \frac{i \operatorname{Im} A_3}{(2\delta - 1)} z^3$. The interface function is given by

$$\beta(x, y) = \frac{R\mu_2}{2h_0 b \left[1 + \frac{a^2}{b^2} \left(\frac{1}{b^2} - \frac{1}{a^2} \right) y^2 \right]^{\frac{1}{2}}}.$$

This agrees with the results obtained in [12].

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