Artillerymen and mathematicians: Forest Ray Moulton and changes in American exterior ballistics, 1885–1934

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Abstract

Mathematical ballistics in the United States until the First World War was largely dependent on the work of European authors such as Francesco Siacci of Italy. The war brought with it a call to the American mathematical community for participation in ballistics problems. The community responded by sending mathematicians to work at newly formed ballistics research facilities at Aberdeen Proving Grounds and Washington, D.C. This paper focuses on the efforts of Forest Ray Moulton and details how he dealt with various aspects of a single problem: differential variations in the ballistic trajectory due to known factors.

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1. Introduction

The enormous changes in technology that occurred in the United States in the last decades of the 19th and first decades of the 20th centuries were reflected in the developments in
its military branches. “The material of war has undergone greater changes in the past thirty years than in the previous hundreds of years since the introduction of gunpowder,” declared the author of a 1907 textbook on ordnance and gunnery. “The weapons of attack and defense have become more numerous, more complicated, and vastly more efficient... The science of gunnery constantly requires of the officer greater knowledge and higher attainments...” [Lissak, 1907, preface].

Among these changes was a revamping of the mathematical basis for exterior ballistics, the science of the trajectory of a projectile. This revision took place roughly in two waves. The first was initiated in 1885 by the appearance of the first book by an American army officer devoted exclusively to this subject, Exterior Ballistics in the Plane of Fire [Ingalls, 1886], by James Ingalls (1837–1927). It was continued by other writers such as Alston Hamilton (1871–1937) through the turn of the 20th century. These men were officers in the army, and their work was largely an adaptation of methods developed in Europe. The second wave was brought about by the World War and was overseen by Oswald Veblen and Forest Ray Moulton, the first a professor of mathematics at Princeton University, and the second a professor who taught astronomy and mathematics at the University of Chicago. The cumulative effect of these two waves was to lift the study of ballistics from chapter-length treatments in standard ordnance and gunnery textbooks to a more detailed, higher-level study using some of the most advanced mathematical analysis of the day.

The goal of this paper is to describe these two waves of reform, focusing on the work of the artillerymen Ingalls and Hamilton in their textbook treatments of the subject and the contributions of Moulton in his vast revision of this work. Various colleagues of Moulton and their efforts will be mentioned, as will figures whose lower-level expositions and critical analyses furthered the propagation of his new ideas. We concentrate on the army and coast artillery and their associated institutions, since the men in each wave had the closest association with these branches of the military. The role of the Coast Artillery School and its associated journal is central to our story. Developments similar to ours could also be related for the navy and field artillery, but our focus allows the discussion of particular mathematical contrasts between the waves of reform. The story can be seen as a study of the increasing mathematization, occurring in the “third period” of the evolution of mathematics in America as delineated by Parshall and Rowe [1994], of a technological problem of long standing.

We describe the fundamental ballistic problems treated by these men, singling out for closer analysis the differential variations in the ballistic trajectory. This subject treats the determination of the effects of relatively minor factors such as changing air densities, winds, and slight changes in ammunition, which can alter the path of the projectile. This problem is examined in detail for several reasons. It formed a far larger and more deeply analyzed part of the ballistic theories of the second wave than the first. The mathematics brought to bear on the issue by these men was much more sophisticated than that of the first wave; it also clearly dovetailed with ongoing research in pure analysis. Their study yielded many applications in range table construction. It also appears that this problem brought into

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1 The publication date is given as 1886, but the opening pages show that the book was “Approved and Authorized as a Textbook” in February 1885 by the United States Artillery School at Fort Monroe, Virginia. A volume with similar material was issued in 1885 by the Navy [Ingersoll and Meigs, 1885].
sharp focus the gaps in mathematical understanding between the users of the new theories and their developers.

We first give some background material on numerical integration in the context of applied mathematics in the United States prior to World War I, since one of the central contributions of the second wave of reformers was the introduction of this technique for computing trajectories. The reformers' role in raising the awareness of this topic in the larger mathematical community is one of the outcomes of the ballistics evolution we describe. There follow the history and development of ballistics in the United States, explanation of the central role of range firing tables, and a discussion of the differential variations of the ballistic trajectory, from all of which developed the work of the artillerymen and mathematicians we highlight.

2. Numerical integration as part of applied mathematics and computation in the United States before World War I

Since the calculation of the trajectory of a shell is a problem in applied mathematics that involves computation, some information about the status of these activities in the United States prior to and following World War I is appropriate. In this section we make some general comments about the role of applied mathematics in the American mathematical community during pre-War times and trace the development of the relevant subtopic of numerical integration.

Generally speaking, applied mathematics was a substantial part of mathematical activity in the United States in the late 19th century; Garrett Birkhoff has noted that “of the 200-odd NYMS [New York Mathematical Society] members listed in the Bulletin of November, 1891, forty per cent had at least partial professional responsibility for astronomy, physics, engineering, or actuarial work” [Birkhoff, 1977, 27]. Some of this work involved computation, be it of orbits of planets, changes in tides, routine evaluation of special functions, or solution of differential equations. A comprehensive account of the role of computation for astronomical and other purposes during this era can be found in Grier [2005]; here we find the history of such American institutions as the Nautical Almanac, the Harvard Observatory, the U.S. Naval Observatory, and the Coast and Geodetic Survey. Some of the computers in these institutions were drafted to work on trajectory computations during the World War.

In particular, numerical techniques were used by some astronomers in this era both in Europe and the United States. George Darwin, an English mathematician and geophysicist, used the methods in his investigations of periodic orbits, and George William Hill used them in his work on lunar theory [Moulton, 1930, 224]. But there is reason to believe that the techniques of numerical integration were not widely known among American astronomers and mathematicians at this time. Forest Ray Moulton, our central protagonist, claimed that “Probably not one out of twenty of them [astronomers] ever heard that such a method exists” [Moulton, 1919a, 18] and that these ideas were “almost wholly unknown to mathematicians of the present time” [Moulton, 1928b]. We shall attempt briefly to investigate circumstances relevant to Moulton's claims, restricting ourselves to publications in mathematical journals, books devoted to numerical analysis, and textbooks on differential equations.

In addition to several isolated publications by individuals in American mathematical journals, information about numerical integration of differential equations during this
period appears to have come from three main sources: the work of John Couch Adams and Francis Bashforth in England, the activities of the Mathematical Laboratory of Edmund Whittaker at the University of Edinburgh, and the publications of Carl Runge. The publication of Theories of Capillary Action by Adams and Bashforth [1883] made available from the astronomer Adams a numerical integration method that became a component of that introduced by Moulton in 1917, though its wider use during this era has been difficult to trace. An exposition of this method appeared in England and the United States in the text by Whittaker and Robinson [1924]. The pre-War influence of these men appears to have stemmed more from their ballistics work than from Capillary Action. Thus a 1905 paper notes Bashforth’s experiments with air resistance to fired shells and his “elaborate set of tables based on the method of quadrature for the integration of the final equations” [Gilman, 1905, 79]. This author introduced a numerical method for integration of the differential equations of motion of a projectile based on least squares. Bashforth’s experiments were also noted by American writers on ballistics; see [Ingalls, 1886, 31–40], for example. His tabulation efforts were also described, but no details of the method of derivation given. One of the participants in the American ballistics effort in the World War wrote of Adams’s own estimate of the suitability of his method for calculating trajectories [Hull, 1920, 223]. The effect of Theories of Capillary Action on ballistic calculation in America during this period, however, seems to have been nil.

The Edinburgh Mathematical Laboratory, established in 1913 under the leadership of Edmund Whittaker, was another source of numerical methods for dealing with the mathematics in a wide range of problems in science and engineering. Though the book regarded as a manual of its methods was not published until 1924 [Whittaker and Robinson, 1924], a 1915 publication from the Laboratory [Gibb, 1915] devoted to interpolation and numerical integration of functions was available for use in the United States. Differential equations are not explicitly mentioned in this text. Perhaps more important, the laboratory itself provided Joseph Lipka, a Ph.D. in mathematics at the Massachusetts Institute of Technology, with the experience of working with practical problems on a visit in the summer of 1913. This visit resulted in the creation of a mathematical laboratory at his home institution in 1914 and his 1918 volume on numerical and graphical computation [Lipka, 1918]. His book contains a section on approximate integration (Chapter IX), but does not mention Adams’s or any other extant method explicitly. The book appears to be one of the, if not the, earliest by an American mathematician to treat a broad spectrum of computational topics.3 The authors of a recent book on American mathematics instruction [Kidwell et al., 2008, 121] note that, “perhaps tellingly,” Lipka’s course was not required by engineering departments; apparently the time was not available for such instruction, or perhaps the need was not strongly felt.4

2 Adams is perhaps best remembered as the astronomer who lost out to Le Verrier in the discovery of Neptune.

3 Mentioned in this volume is an original method, due to William F. Durand, for numerical integration of differential equations of a certain type. Durand, then a marine engineer at Cornell University, made reference in his paper [Durand, 1898–1899] to an earlier publication of 1897 in an engineering journal where the method is applied. This latter paper is the earliest America-journal-reported use of a numerical method for solving differential equations in an applied setting that the author has discovered.

4 Pages 120–122 of this volume contain surveys of the prevalence of such instruction in computation during these years, citing some ambivalence on the part of those polled toward the subject.
The final major source for methods of numerical integration of differential equations was the writings and lectures of Carl Runge, a German mathematician who published his method in Runge [1895]. Runge’s work seems to have made the biggest impression on the American mathematical community, perhaps because of the journal chosen for publication and his status as a mathematician, but also because of his lecturing activities in Germany and the United States. While in America in 1910 he gave a series of lectures at Columbia University on graphical methods for solution of mathematical problems of various kinds, and the notes for these lectures were published in book form [Runge, 1912]. Some pages are devoted to his numerical method (pp. 120–124). His lectures at Göttingen on similar topics were the basis for a volume that includes his method, whose English translation appeared as [von Sanden, 1913]. A small measure of the early awareness of Runge’s approach can be found in Epsteen [1904].

The popularity of Runge’s method can also be gauged from its inclusion as a topic in textbooks on differential equations. As late as 1933 the author of such a text in America could say that the inclusion of “A chapter on interpolation and numerical integration is unusual for a text on differential equations” [Ford, 1933, v], and a perusal of typical volumes from 1895 through 1918 bears this out by the absence of such sections. An exception can be found in the third edition of the English mathematician Andrew R. Forsyth’s classic Treatise on Differential Equations [1903, 53–56]; this edition was the first to include a numerical technique. Another text including Runge’s approach is the work of the eminent applied mathematician Harry Bateman [1918, 227–230].

The calculation of a trajectory involves moving from acceleration information to position information via integration, a process that, in the absence of integrable functions in the differential equations, requires the use of a numerical technique. The use of numerical integration for this purpose was not completely unknown in America at this time; in addition to the 1905 effort previously mentioned, there were at least two other attempts to apply numerical integration techniques to the calculation of the ballistic trajectory. One such method was introduced by Alston Hamilton in a document issued from the Sandy Hook Proving Ground [Hamilton, 1917]. This technical report, in which methods independent of existing ones are used, apparently had no influence on subsequent events and appears to have been Hamilton’s sole new mathematical contribution to wartime ballistics. Similarly, Arthur Gordon Webster, a physicist at Clark University and founder of the American Physical Society, made an original calculation of the trajectory of a German long-range gun [Webster, 1919]. Webster claimed that he was the first to publish trajectories of this gun and that he was the first professor to deliver lectures on ballistics at an American university [Webster, 1920, 368]. He maintained a small “Ballistics Institute” at Clark University throughout the war; this institute apparently produced additional trajectory calculations. Webster showed his results to Forest Ray Moulton just prior to the latter’s beginning his service in the army, but again, this effort did not affect subsequent methods established by the second wave reformers.

This limited survey suggests a fragmented state of affairs for the topic of numerical integration of differential equations in the United States prior to 1918. We see some awareness of methods originated abroad and sporadic publications of novel methods and related

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5 One has for example [Craig, 1889; Murray, 1898; Johnson, 1906; Cohen, 1906; Campbell, 1913; Maurus, 1917].

6 Note that though the publication date is 1919, the author states that the calculation was first presented in 1918.
formulae by isolated mathematicians and engineers. There are sparse appearances of the subject in textbooks and treatises on differential equations and inclusion in works devoted to numerical and graphical methods. No textbook devoted exclusively or even largely to the exposition of the various techniques of Adams, Runge, and others existed in English. The only work the author was able to find that compared the various methods for effectiveness was Bateman’s volume (pp. 228–230), in which the author quotes a comparison from an existing article. The ballistic calculations of Gilman, Hamilton, and Webster are independent of existing methods and of each other. It is likely that there were more scientists, mathematicians, and engineers than listed here who were versed in the subject, but overall the traces of their knowledge are not readily visible.

3. A short history of ballistics and its instruction in the United States

In a paper read before the International Congress of Mathematicians in Toronto in 1924, P. Charbonnier, Engineer General of Naval Artillery in France, made the following remarks:

Exterior Ballistic Theory . . . is often called the terrestrial sister of celestial mechanics; it is an older sister, since Galileo created it in its modern form well before Newton, who generalized its fundamental laws to the world of the planets. The relationship between the two sciences, at their source, is so close that Newton’s second book of *The Principles of Natural Philosophy* is actually a treatise of pure ballistics. The two sciences later split: the one, which occupied itself with unchanging objects, eternal and perfect, developed itself in the pure domain of mathematics and absolute laws; the other, which participates more in terrestrial imperfection, and is used for very few speculative ends, goes very slowly, step by step, hesitates, turns back . . . ; as a science of application, it must follow the requirements of the fashionable ideas of the moment. [Charbonnier, 1924, 317]7

This passage gives a concise summary of the relationship between mathematics, ballistics, and celestial mechanics from the point of view of a career ballistician of the day. Most important for our account is the close association of numerical approximation techniques with astronomical and ballistic problems that was exemplified by the work of Leonard Euler. His translation of Benjamin Robins’s 1742 work *New Principles of Gunnery*, which was accompanied by an extensive critical analysis and consideration of topics not treated by Robins, constituted a “ballistics revolution” according to Brett Steele [1994, 366–369]. Steele makes the argument that “The ballistics revolution [credited to Euler and Robins’s work] generated new theories that offered a rational understanding of gunnery, the technology of controlling gunfire. This made the teaching of calculus and mechanics to artillery and engineering officers profitable for Western governments during the second half of

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7 La Balistique Extérieure. . . est souvent dite la soeur terrestre de la Mécanique céleste; c’est une soeur ainée, puisque Galilée la créa sous la forme moderne, bien avant que Newton n’en généralisât la loi fondamentale en l’étendant au monde des planètes. La parenté des deux sciences à leur origine est si étroite que le second volume des “Principes de la Philosophie naturelle” de Newton n’est en réalité qu’un pur traité de Balistique. Les deux sciences se séparèrent dans la suite: l’une qui s’occupe d’objets immuables, éternels et parfaits, se développe dans le pur domaine des mathématiques et des lois absolues; l’autre, qui participe davantage des imperfections terrestres et est utilisé à des fins très peu spéculatives, marche terre à terre et pas à pas, hésite, revient en arrière . . . comme science d’application, elle est obligé de se plier aux exigences, aux idées faits, aux modes du moment.
the 18th century” (p. 350). He goes on to detail how these ideas came to the newly emerging United States, and the role they played in the establishment of the military academy for engineering and artillery officers at West Point, New York, in 1802.

From West Point came textbook treatments of exterior ballistics, contained in a series of volumes for courses of instruction in ordnance and gunnery. These volumes continued to be issued well into the 20th century. Typical of these were the books of James G. Benton, first appearing in 1859 and revised in editions as late as 1889, which were held in high regard. In Benton’s books ballistics is but a single chapter among, for example, 13 in the 500+ pages of the third edition [Benton, 1867]. But in the following decades a single textbook could no longer encompass all that needed to be known. In an 1892 review of a new West Point textbook appearing the previous year, the reviewer writes,

When Benton’s ordnance and gunnery was first published, its excellence as a text-book was at once established. For many years the growth of artillery science had been relatively slow. Under these conditions it was possible for a single text-book to present both fully and accurately, that state of military science.... To-day, however, it is for obvious reasons, a task of no ordinary difficulty to write a text-book of ordnance. We cannot reasonably expect to-day any textbook to fill the place once occupied by Benton. [Anonymous, 1892]

The increasing complexity of military science in these years was mirrored in the founding of a series of schools relevant to our story, namely those culminating in the Coast Artillery School of 1907. All were located at Fort Monroe, Virginia. It was in the journal associated with this institution that articles about the second wave of ballistic reform appeared; this movement was referred to therein as the New Ballistics. An article from the journal, appearing in 1927, gave the history of the school [The Coast Artillery School, 1927]. We shall briefly mention some highlights, since this institution has not received the attention it warrants in connection with military education in the United States. The author refers to the military academy at West Point as having become a “preparatory school” (p. 1) in the years following its founding, a school in need of supplementary advanced institutions. These came in the form of a series of schools, in each of which instruction in mathematics was considered key. For some subject matter it was also realized that existing literature was inadequate for the school’s advanced goals, and in the years 1870–1890 some instructors wrote monographs of their own. It would appear that the school considered itself in the vanguard of progress in the military science in the United States because of both its focus on artillery and its attention to the defense of the coast. The school’s journal began publication in 1892 as The Journal of the U.S. Artillery; its title changed to the Coast Artillery Journal in 1922. We will from now on refer to this series as JUSA/CAJ. Illustrative of the new level of sophistication sought by the school were articles appearing in its journal that tried to advance the discussion of ballistics. Issues contained book reviews, including those of mathematical works of the day such as Greenhill’s The Applications of Elliptic Functions [Craig, 1893] and various differential equations texts. New artillery instruments were highlighted, and field ballistics problems modeling battlefield situations were posed and solutions published. The nature and content of instruction for gunners was hotly debated.

Two advocates of the first wave of ballistic reform, James Ingalls and Alston Hamilton, had connections to this institution. Ingalls, an American Civil War veteran, was an 1872 graduate of the Artillery School who in turn taught many subjects there. He was put in charge of the Department of Ballistics in 1883, and his text was issued by the school in...
1885. Hamilton, whose book was also issued by the school, was a career Army man, a graduate of West Point, and served in the Spanish–American War.\textsuperscript{8} 

The establishment and development of the Coast Artillery School can be seen as part of a general trend in the U.S. military, that of the increased professionalization of American army officers, itself a reflection of forces at work in American society from 1880 through 1917.\textsuperscript{9} In the late 19th century American society had come to regard professionalization as involving more than simply gaining experience through spending a career in a certain line of work; education by knowledgeable instructors who upheld standards for both entrance and advancement became a requirement. This trend in the Army was consistent with the expectations of the Progressive Era in the early 20th century, with its emphasis on the supposed ability of rational, educated managers to solve the problems of society. America’s emergence on the world stage after the Spanish–American War also impelled the American military to operate like, act like, and even look like the professional European armies. The emphasis on continuing education became increasingly important in the post Civil War period with the establishment of Reserve Officer Training Corps [ROTC] military education in land-grant colleges to supplement the pool of West Point officers, and the creation of the General Staff College in 1881 and the Army War College in 1901 are considered major milestones in the professionalization of the U.S. Army. Instruction in these institutions would presumably provide opportunity for exposure to mathematics.\textsuperscript{10} 

4. Range firing tables, the mathematics of exterior ballistics, and the first wave of reform 

As we come closer to the mathematics of exterior ballistics as used at this time, it will be helpful to distinguish three levels of discussion. The first is the theoretical level, at which mathematical models of the trajectory are formulated, using whatever scientific laws are deemed necessary, and solved by whatever methods seem appropriate, without regard to use. The model is usually a system of differential equations incorporating analytic expressions for such variables as air resistance, shape of shell, and density of atmosphere, and solved either by exact techniques or by numerical integration. A contemporary example was the application by Sir George Greenhill of elliptic functions to solve the equations of motion for a certain form of the drag law [Greenhill, 1892, 244–253]. This achievement, in Charbonnier’s view (expressed in an article translated for JUSA/CAJ in 1908), elevated the status of these functions so much that they began to be taught in courses of analysis in France [Charbonnier, 1908, 62]. The second level we will call the firing range level: here ballistics is studied under scientifically controlled conditions, where firing practices are held, data are collected, theories are tested, and firings for range tables conducted. The Coast


\textsuperscript{9} For the information in this paragraph I am indebted to Richard J. Sommers, Ph.D., Senior Historian, U.S. Army Heritage and Education Center at the U.S. Army War College, Carlisle, Pennsylvania, whose remarks to the author are closely paraphrased here. 

\textsuperscript{10} Standard references for these schools are [Nenninger, 1978] and [Ball, 1983, 1994], respectively.
Artillery School had its firing ranges, as did many arsenals; Aberdeen Proving Grounds was the prominent firing range in America. We include textbook presentations of ballistics and journal articles at this level, since results of firing range tests are reported and explained in these media. The third level is on the battlefield, where tactical and strategic factors enter into the use of ballistics, and decisions are made on imperfect information. These levels clearly overlap: any theory can be tested on the firing range, and the information gathered about the performance of a gun is needed for battle.

This information was compiled in a range firing table, the material product of exterior ballistics work. The tables included such basic “elements” as range and time of flight for a specific type of projectile and gun fired at a given angle of elevation, though many more elements came to be included over time. The first such tables in modern times were those presented to the Royal Society by Benjamin Robins in 1746. More information on firing tables in this era can be found in Steele [1994, 370], and details of the subsequent historical development of range firing tables can be found in the Historical Appendix of McShane et al. [1953].

The relationship between mathematical ballistic theories, actual firings, and construction of range firing tables is complex. One can imagine two extremes: a table based only on computations following from a ballistic theory and involving no firing, and a table based only on the results of firing and collecting data. Thus Galileo’s parabolic arcs could be considered an early example of the former, with Robins’s tables as a later instance. Clearly, though, the accuracy of any theory needs to be tested by actual firing; this was a routine activity at firing ranges. The matter is complicated further by the fact that, as more elaborate theories developed, firing data were needed to provide accurate numerical values for constants occurring in the theory itself. An example of this is the introduction of the “coefficient of form,” denoted by $i$ (less politely referred to in some American texts as the “coefficient of ignorance”). This constant, appearing in the differential equations of motion, was an attempt to take into account the shape of the projectile. (See Fig. 1.) A preliminary value of $i$ was determined from the ogival, or nose angle of the projectile, and two successive firings were then necessary, with data collected from each one, to arrive at a satisfactory value [Charbonnier, 1907, 255]. Other similar constants were used in the theories. The most widespread was the “ballistic coefficient,” a constant that included $i$ among other factors, and
was roughly described as measuring the ability of a projectile to penetrate the atmosphere. Ignoring all these difficulties, one could compile a table on the basis of a few firings at different elevations followed by linear interpolation for the remaining trajectories.

In the United States the tables show evidence of the two extremes previously mentioned. In the *Confederate Field Manual* of 1862, used by the Southern states in the American Civil War, ranges for different kinds of guns are given as “the mean results of such trials of the ranges of our ordnance as have been made from time to time by the ordnance department” [The Confederate Field Manual, 1862, 110]. The information for the 6-lb field gun is composed of amount of powder used, type of ball, and six elevations with their corresponding ranges. Benton’s book [1867] contains both tables resulting from theory-based calculation (including the same 6-lb field gun) and those determined “by practice.” The table for the 6-lb gun takes about a half a page.

As the century progressed the two extremes were explicitly discussed: an 1877 Navy pamphlet [Very, 1877, 3–4] describes “two ways of constructing such a table”:

One depending upon the application of formulas which have been deduced partly from practice and partly from abstract laws. . . . [With regard to this method] As yet, however, it has been impossible to form any law of resistance [of air] that will apply to all the circumstances attending the flight of a projectile; and, even were such laws known, their application to the known formulas would be extremely difficult. . . . The second means of arriving at the required results is by making observations of the effects of the firing of any gun as actually carried on. . . . This second method may be called the practical one in contradistinction to the first, which is purely theoretical. It is simpler than the first, and, in the hands of others than the most skillful mathematicians, is less liable to error.

The author described in detail the second method, which was used at a naval experimental battery for a “3-inch B.L. Rifle.”

The artillerymen of the first wave of reform, beginning in the 1880s, presented new theories as specifically tied to range table construction: Ingalls, in the introduction to his book, stated, “The aim has been to present in one volume the various methods for calculating range tables and solving important problems in trajectories, which are in vogue at the present day…” [Ingalls, 1886, Preface]. The tables also became more complex, introducing additional elements such as angle of fall, terminal velocity, ordinate, and height of the summit. The increase in types of ordnance and the rapidity of change in their performance seems to have initiated a hope that table-based calculations could provide information that would otherwise have to come from costly firings. *Artillery Circular M* (1900) [Ingalls, 1900], a widely used set of tables, contains a suggestive example using the United States magazine rifle, caliber 0.3 inches. “We will . . . make similar computations for other muzzle velocities, forming the following table, which gives important information, more accurate than could be obtained by experiment, and which costs nothing save a little labor” (p. iv). The example demonstrates an economic motive for the preference of mathematics over firing to determine trajectory information.

This economic motive became an explicit concern in the second wave of reformers. In 1919, Forest Ray Moulton wrote, in a comprehensive unpublished internal history of the contributions of his ballistics team: “At present the performance of a gun at all ranges can only be determined by trial at the proving grounds. The range firing of a large gun costs tens of thousands of dollars besides wearing out the gun itself. This expense in time and money is only an expression of our ignorance of the fundamentals of the problem. The flight of a projectile is as much subject to the laws of physics as the performance of an
engine. There is no doubt that at some time a few shots will be sufficient for the accurate prediction of what would happen under any conditions” [Moulton, 1919a, 90]. The conditions referred to by this time had been extended to include larger angles of elevation, several types of shells for the same gun, and a host of weather-related factors such as wind, density of air as a function of altitude, and temperature. Each of these factors had to be considered in the firing of each type of weapon. Moulton spoke here with characteristic imperious optimism about the power of the New Ballistics to handle all these elements with a minimum of actual firing. He also directed his ire at the work of his predecessors: “the range tables were based on large numbers of experimental firings at various angles of elevation and were largely independent of any mathematical theory. The theory was adjusted to the firings by choosing such a ballistic coefficient at each elevation that the ballistic tables and practice would be in harmony” (p. 95). This claim is one contributor to his criticism of the ballistic coefficient: “For a quantity that was treated as a constant in the mathematical theory the ballistic coefficient was made to carry heavy burdens” [Moulton, 1919a, 106]. Moulton characterized the tables as “not much more than interpolation formulas for supplying results in the intervals” (p. 10).

The tables had several uses. Some of these occurred at what we have called the firing range level of discussion. Ingalls’s work, as well as that of his successor, Alston Hamilton, involved designing classroom exercises using their tables in problem books [Ingalls, 1890]. Ingalls’s textbook promoted their use “for determining in advance the ballistic efficiency of those [guns] which may be proposed in the future” [Ingalls, 1886, Preface], that is, as a planning tool, as pointed out by Grier [2001, 924].

But it is also clear that the tables were at least envisioned as being used in battle. Moulton, describing some of the new ballistic problems brought about by the World War, cited “the barrage, and especially the moving barrage, [which] made it more necessary than before to direct artillery fire with great accuracy by theory alone. Under such conditions with hundreds, and even thousands, of guns in action, it would be essentially impossible to correct imperfect firing, even if there were time” [Moulton, 1919a, 11]. Thus he is speaking of the need to supplant adjustment by sight with accurate tables providing information for one-time firing. This sentiment is supported by a quote from MacFarland’s Ordnance and Gunnery of 1929, the first of the West Point ordnance and gunnery volumes to include the New Ballistics: “The Firing Tables are the means by which the commander of an artillery unit determines the range and deflection settings that will enable him to place his shells on the target with a minimum of adjustment after firing is open.... Familiarity with these tables and facility in their use are required of all officers” [MacFarland, 1929, 428]. Such familiarity by this time would have been no small task. The volume of information to be digested had risen to sixty pages for the Provisional Range Tables used with the British 75-mm. Gun, Model 1917, issued under Moulton’s direction. The 1921 edition of this same volume added detailed instructions for the use of the “meteorological message” to account for weather factors among many others in adjusting the elevation of the gun [Range Tables 1921, v–vii]. Moulton also spoke with pride of his deference to “those who use range tables in battle”; the Field and Coast Artillery officers using them “should be the final authority on questions of their contents and arrangements” [Moulton, 1919a, 80].

11 This criticism was noted by Grier [2001, 928].
12 The latter volume was issued on May 24, 1918 as a handy 4 by 7” volume [Provisional Range Tables, 1918].
And what of the mathematics used to construct these tables? The United States had no native tradition of theoretical exterior ballistics comparable to those developing in European countries in the 18th and 19th centuries. This state of affairs continued into the period prior to the World War, a fact frankly acknowledged by Alston Hamilton in the introduction to his 1908 text: “On account of the finished classical methods current at the present time little remained for the writer to do beyond selecting the best for use in this school [the Coast Artillery School]” [Hamilton, 1908b, introductory page].

The first wave of innovation came with the publication in Revue d’Artillerie in October 1880 of an article by Major Francesco Siacci of the Italian artillery [Siacci, 1880]. This article described a method of approximate solution of the equations of motion giving the ballistic trajectory. The article was translated in the United States in 1881 and included in that year in a report to the Chief of Ordnance of the Army. Siacci’s method was so successful that it eventually became used by all the world’s major military forces. This was the theory first explicated for the United States Army by James M. Ingalls [1886]. The method was officially adopted at West Point in 1891 and replaced all other textbook approaches by order of the Academic Board. We will give an idea of the theory, combining the treatments given by Ingalls and Hamilton.

Suppose that a projectile is fired with initial velocity $V$ at an angle of inclination $\theta$. Letting $x$ and $y$ be the Cartesian co-ordinates of the position of the projectile at time $t$, taking the origin to be the location of the gun and $x$ and $y$ positive, let $h$ be the angle tangential to the curve of travel at time $t$, and $v$ the velocity at time $t$. (See Fig. 2.)

The projectile is considered as being acted on by two forces: gravity, denoted by $g$, and the force of retardation due to the atmosphere, which acts opposite to the tangential direction of the projectile with magnitude $F(v)$. Resolution of the forces along horizontal and vertical lines through the projectile easily gives the so-called Principal Equation,

$$\frac{d(v \cos(\theta))}{d\theta} = \frac{vF(v)}{gC},$$

(4.1)

where $C$ is the notorious ballistic coefficient. The function $F(v)$ was to be experimentally determined for a standard projectile under standard conditions of fire, and $C$ was an attempt to account for any variation from those conditions, so that $\frac{F(v)}{C}$ was effectively an adjusted atmospheric force function. At its simplest, $C$ is defined as $C = \frac{w}{2\gamma d^2}$, where $w$ is the weight of the standard projectile, $d$ its diameter, and $\gamma$ the coefficient of form. This definition had a theoretical justification involving the manner in which diameter and weight could affect the force of retardation. However, the formula became swollen to

$$C = \frac{\delta}{\delta f} f w \frac{w}{i\gamma d^2},$$

(4.2)
where the other constants are \( f_w \) for a wind, \( f_a \), an altitude factor, \((\delta_1/\delta)\), a ratio of empirically determined density factors to account for the atmospheric density, and \( \gamma \), a “constant of curvature,” to account for the curvature of the trajectory. This complication could also have been adduced by Moulton as an instance of the “heavy burdens” borne by the coefficient. One of Moulton’s contributions, to be explained in this paper, was his attempt to remove the variable sources from \( C \) and account for them in a functional way in the differential equations of motion.

The function \( F(v) \) for a standard projectile was first taken to be proportional to a monomial in \( v \); Newton argued on theoretical grounds for \( v^2 \). A vast literature devoted to the form of this law eventually developed. Test firings at various European firing ranges were undertaken to determine this law more accurately; among these were the work of Francis Bashforth. At the time Siacci formulated his approach to the solution of the Principal Equation, several such results were available. Ingalls and Hamilton used those provided by the Krupp firings at Meppen in 1881. The mathematical form of \( F(v) \) that resulted from these firings is a continuous function of piecewise monomials in \( v \), including some fractional powers; see [Grier, 2001, 926] for the specific function.\(^{13}\)

The difficult part of the analysis comes in the attempt to integrate the Principal Equation. (“From the non-integrability, in general, of this equation follows all the miseries and all the work of the ballisticians,”\(^{14}\) as Charbonnier put it in his 1924 Toronto address [Charbonnier, 1924, 317].) Many approaches to the integration problem were given by mathematicians throughout the years for \( F(v) \) of various forms. Siacci’s major contribution was the introduction of an approximation that made this integration possible for all laws simultaneously. He defined a quantity \( u \), called the pseudo-velocity, by

\[
\cos(\phi) = \cos(\theta) u \cos(\phi) = \cos(\theta).
\]

This quantity has a physical interpretation as a velocity parallel to the muzzle velocity and having the same horizontal component as the actual velocity \( v \); see Fig. 2. He then introduced a function \( \beta \) defined by the equation

\[
F(v) = \beta F(u) \cos^2(\phi) \sec(\theta) \tag{4.3}
\]

“with the practical certainty that, for direct fire, the value of \( \beta \) will never differ greatly from unity” [Hamilton, 1908b, 18].\(^{15}\) The constant \( \beta \) had to be determined for any particular law and, since it was generally a function of \( \theta \) and \( \phi \), replaced by an appropriate constant approximation, thus making the Principal Equation integrable. These definitions and approximations, when substituted into the Principal Equation, eventually yield

\[
\frac{du}{uF(u)} = \cos^2(\phi) \frac{d(\tan \theta)}{gC}, \tag{4.4}
\]

where the \( C \) now is the original \( C \) divided by \( \beta \). Integrating between \( \theta \) and \( \phi \), equivalently from \( V \) to \( u \), gives

\[
\tan(\theta) - \tan(\phi) = \frac{-C}{2 \cos^2(\phi)} \int_V^u \frac{-2g}{uF(u)} du = \frac{-C}{2 \cos^2(\phi)} (I(u) - I(V)), \tag{4.5}
\]

\(^{13}\) A trace of an older mathematical tradition is evident when Ingalls refers to this law as “these discontinuous functions” [Ingalls, 1900, iii].

\(^{14}\) “De la non-integrabilité, en général, de cette équation, découlent toutes les misères et tous les travaux des balisticiens.”

\(^{15}\) Direct fire was defined as occurring when \( \phi < 15^\circ \).
where

\[ I(u) = \int_0^u \frac{-2g}{uF(u)} \, du. \tag{4.6} \]

Thus the tangent of the angle of inclination, and consequently the angle itself, is given as an integral function of the pseudovelocity \( u \). The differentials \( dt \), \( dx \), and \( dy \) can also be written in terms of \( du \), yielding, upon integration, formulae for \( t \), \( x \), and \( y \). (In this treatment time is not considered an independent variable.) All the elements of the trajectory are simply expressible in terms of these four functions. The law \( F \) itself does not need to be given explicitly at this point. The formulae for the elements can be found in Grier [2001, 925]. For a specific \( F \) such as that given by the Krupp firings, these functions were computed and tabulated. A determination of a constant to approximate the function \( \beta \) was also necessary to incorporate into the ballistic coefficient.

Ingalls’ book contains many sample problems using these four functions and various initial or terminal data; the ballistic coefficient was given as part of the data. The solutions are given and involve only table lookup and knowledge of logarithms and trigonometry.\(^{16}\)

A miniature range table for the 8" rifle was developed giving angle of elevation, angle of fall, range, striking velocity, and time of flight [Ingalls, 1886, 123]. Adaptations of the method by Hamilton address higher-angle fire as well, invoking a host of new variables. A striking feature of the work is the number of approximations necessary to carry out the solutions. This was not a new feature peculiar to Siacci’s method or the Americans’ application of it. Approximations presented by Ingalls and Hamilton include, among others, the assumption that the time from the point of projection to the summit is one-half the time of flight and the use of a mean constant resistance at low velocities or a square law throughout for mortar fire. For the calculation of a mean value for \( \beta \) as used in Siacci’s theory, several techniques are given in Hamilton’s book. For example, assuming a square law of resistance throughout, \( \beta \) is in fact just \( \sec(\theta) \); a mean value for this is calculated by the formula

\[
\frac{1}{\beta} = \int_{-\theta}^{\phi} \frac{1}{\sec(\theta)} \sec^3(\theta) \, d\theta = \frac{\tan(\phi + \tan \omega)}{(\phi + (\omega))}
\]

\[
(\phi) = \int_{0}^{\phi} \sec^3(t) \, dt, \tag{4.8}
\]

where \( \phi \) is the angle of fire, and \( \omega \) is the terminal angle.

These techniques of approximation, though presumably anathema to a pure mathematician, were part of ballistics at our first and second levels of discussion as practiced at this time. This style was characterized as the “rough, old, formal ballistics” by Norbert Wiener, a participant in the activities at the Aberdeen Proving Grounds during the World War [Wiener, 1953, 256]. It required familiarity with the many types of ordnance and the judgment to choose the right technique for a given problem, leading to the large number and variety of examples in the books. This experience would presumably be a prerequisite for intelligent study of these problems by artillerymen. The style included rules of thumb such as the Principle of the Rigidity of the Trajectory, routinely assumed in direct fire:

This principle assumes that, if the angle of departure necessary to reach a certain point at a horizontal range, \( x \), from the gun and on the same level, is known, it will only be

\(^{16}\) Numerical work is of course present; among other places, it appears in the form of sophisticated methods of interpolation from tables.
necessary, in order to reach another point \( h \) feet below the former and at the same horizontal range, \( x \), to subtract from the first angle of departure, the angle \( e \), called the position angle and given by the equation \( \tan(e) = h/x \ldots \). The term ‘rigidity’ in this connection refers to the supposed rigid shape of the trajectory and its chord (drawn from the muzzle to the target or point of impact); and the assumption practically involves the hypothesis, that the figure whose outline is composed of the trajectory and its chord, behaves as if it were cut out of cardboard and rotated up or down with the muzzle of the gun as a center. [Hamilton, 1908a, 4–5]

Wiener stated that this manner of considering ballistic problems was giving way in his time to the “point by point solution of differential equations” [Wiener, 1953, 256], i.e., to the numerical solution of the differential equations of motion ushered in by Moulton. This method made some of the approximations unnecessary. These new techniques presumably would be accompanied by a less ad hoc treatment of the problems of exterior ballistics. It would not be clear that newer methods were more effective, however.

The result of the first wave of reform in exterior ballistics in the United States was the proliferation of techniques based on the Siacci theory illustrated by problem books and range firing tables constructed by Ingalls and Hamilton. The latter’s work introduces algorithms and variables to stretch the application of the Siacci theory to all types of ordnance, resulting in a book that has the appearance of complication but whose level of pure mathematics goes no deeper than calculus and separable differential equations. There was no approaching, for example, the mathematical level of Siacci himself in his 1892 French edition of Ballistique Extérieure [Siacci, 1892]. But the results of Ingalls’ and Hamilton’s work appeared sufficiently complex in the eyes of some military men to be off-putting. In a document prepared after a post-Armistice tour of Europe, Oswald Veblen opined that “The theoretical methods which cluster about these tables are objectionable because the mathematics employed is at once too recondite and clumsy. It is largely because of this mathematical defect that exterior ballistics has established for itself a reputation in the American Army of being an extremely difficult subject” [Veblen, 1919, 1]. It is hard to gauge the truth of this view, and the degree to which the antipathy described could be attributed to run-of-the-mill distaste for mathematics itself.

### 5. Differential variations in the ballistic trajectory

All of the preceding considerations assume that the weapons used and discussed were sufficiently accurate so that roughly the same trajectory could be expected on each firing if all conditions could be kept the same. This state of affairs was in fact only achieved during the 19th century as a result of advances in engineering. A consequence of this was that one could with confidence begin to speak of the “differential effects,” as they came to be known, of phenomena that could slightly vary the “normal” trajectory. These included some factors that the first wave ballisticians attempted to include in the ballistic coefficient, such as winds in the plane of fire, density of the atmosphere, temperature, and certain features of the shell, as well as those that could not be so incorporated: a slight change in the angle of fire, change in muzzle velocity, and cross winds. These forces were sometimes called the “secondary effects,” as opposed to appreciable changes in, for example, angle of fire. The astronomical analogy of the “secondary forces” of distant planets in determining the orbit of a given planet around the sun was also cited. Some of these factors could be anticipated; others had to be considered accidental. Occasionally a stricter definition of the effects was given, for example, as a factor that caused, say, no more than 3% change...
in the range [Bliss, 1919, 296]. It was of course understood that, regardless of how similar conditions were from one firing to the next, different trajectories were inevitable, due partly to undetected variations and the realities of gun performance. How can one provide a mathematical account of all the factors that vary a “normal” trajectory and a description of their effects?

This question is relevant at all levels of ballistic discussion. From a modern point of view, mathematical tools that provide measures of variation run from simple subtraction and differentials of elementary calculus through differentials of multivariable calculus, the calculus of variations, and perturbations of differential equations, when deterministic theoretical models are used. At the firing range level the problem manifests itself in the ever-changing conditions of fire. Once it is admitted that weather factors, for example, have a measurable if small effect on the range of a fired shell, some set of weather conditions needs to be accepted as standard for range table data. The results of firing under nonstandard conditions have to be adjusted to standard by taking into account the effect that the differences in the factors would have. This required having acceptable results describing these effects available in some combination of empirical laws and theoretical constructs. On the battlefield, the battery commander needs to have constant updates on meteorological data for accurate fire control.

The changing attitude toward factors altering an accepted trajectory can be traced in the West Point textbook sources. [Benton, 1867, 424–434] discusses these causes, noting variations in the weights of powder and projectile, temperature of the gun, manner of loading, wind, and atmosphere among others. He gave the following advice on wind: “It is difficult to calculate the effect of the wind in any particular case; in making allowances for it, therefore, the gunner should be guided by experience and judgment. For the same projectile, velocity, and wind, the deviation [from the vertical plane of fire] varies nearly as the square of the range” (p. 433). No justification is given for this rule. By the first wave, the situation had evolved, as we have seen in the changing form of the ballistic coefficient.

Hamilton offered two approaches to the problem. In Part I of his ballistics book he devoted two chapters to methods of calculating range and deviation corrections. These largely involve table lookup, using undervived empirical formulae from interior and exterior ballistics. In Part II, however, he uses the relations among \( X, C, \) and \( V \) developed in the Siacci theory and derives changes in these quantities by calculating their differentials. One such relation (p. 30), for example, is given by

\[
\frac{dX}{X} = \left( 2 - \frac{nN}{2N+1} \right) \frac{dV}{V} + \frac{N+1}{2N+1} \frac{d(\sin(2\phi))}{\sin(2\phi)} + \frac{N}{2N+1} \frac{dC}{C},
\]

where \( N = V^2 \frac{\sin(2\phi)}{gX} - 1 \), and \( n \) is the power of \( v \) used in \( F(v) \). A value of \( n \) is chosen as either 2 for curved fire (higher elevation than 15°) or 3.5 for field artillery. These methods were still in use after those of the second wave of reformers were introduced [Alger, 1919].

6. Forest Ray Moulton and his associates

The World War brought about new phenomena, which required extensions and revisions of ballistic theory in all countries. In battle, guns had to be fired for greater ranges, which

\[\text{Technological advances in instrumentation allowed more accurate measurement of these forces: new chronometers to measure muzzle velocity and improved weather-related devices were profiled in JUSA/CAJ.}\]
required information on firing at higher elevations, not just at direct fire levels. The older methods, when applied to these situations, did not provide adequate results. We have already cited the moving barrage, which required many guns to be fired with accuracy “by theory alone,” without possibility of correction by sight. The airplane as an offensive weapon made it necessary to be able to fire at all elevations and at a moving target. A knowledge of the entire trajectory, not just the elements, of a projectile was needed to deal with the airplane, since fuses on the projectiles had to be timed to explode in the air. Another new weapon with which to contend was the long-range gun used in the German offensive on Paris in the Spring of 1918. This gun (commonly referred to as the “Paris cannon” or “75 mile gun” in the American artillery) presented a frightening new prospect—with a range of 75 m, several such guns laid siege to Paris for weeks, well out of sight. Clearly these weapons required more of ballistic theory; a knowledge at least of the effect of the upper atmosphere was needed. The time of flight of the projectile was so long that the effect of the rotation of the earth needed to be considered—this became a new differential variation in the second wave treatment.

Forest Ray Moulton (1872–1952) was the man responsible for organizing and overseeing the second wave of reform in exterior ballistic theory during and after the World War. He was raised on a farm in southern Michigan and did his first teaching at age 16 in a rural school near his home. He received a B.A. at Albion College in 1894, having taught astronomy there. After working his way through the instructor ranks at University of Chicago, Moulton received his Ph.D. in astronomy in 1899, after which he rose in professorial ranks to full professor in 1912. He headed the Department of Astronomy at Chicago until 1926. From 1898 until 1904 he worked with Thomas Crowder Chamberlain, the chair of the geology department, on a theory of solar system formation. He published the first edition of his *Introduction To Celestial Mechanics* in 1902 [Moulton, 1914], and authored several other books on astronomy before the war. Mathematics formed an important part of his approach to astronomy. In an expository paper [Moulton, 1911], he described the influence of astronomy on mathematics, stating among other claims that “Astronomy not only turned the attention of mathematicians to analysis, but it often determined the precise form their theories should take” (pp. 362–363), and continuing by naming five methods for solving differential equations, “all of which were devised under the pressure of astronomical problems...” (p. 363). A recent biography of a contemporary astronomer, Princeton University’s Henry Norris Russell, refers to Moulton as a “mathematical theorist” [Devorkin, 2000, 107] and a “celestial mechanician” (p. 4).

At the beginning of the American involvement in the War Moulton was posted at Fort Sill, Oklahoma, as a major in the army ordnance. He was appointed head of the newly organized Ballistics Branch of the Artillery Ammunition Section, Engineering Division of the Ordnance Department, to give it its full title, in April 1918. According to Moulton, “During 1917 and the first three months of 1918 the demand for ballistic results and researches had increased to such an extent that the Ballistics Branch [as a separate unit] was organized” [Moulton, 1919a, 2]. The magnitude of the efforts coordinated by Moulton can be grasped by reading the 98-page document [Moulton, 1919a], the most comprehensive account of his contributions. One can readily agree with the assessment in Fenster et al. (2009) that “Moulton appears to have taken it on himself to wholly recast ballistics.” Not including

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18 For further information concerning Moulton’s life and work, we refer the reader to [Gasteyer, 1970; Tropp, 1973], and Moulton, F. R., entry in Notable Scientists: From 1900 to the Present, Gale Group, 2001, Anonymous.
his own considerable mathematical papers, he devised instrumental aids and nomograms for computing trajectories, authorized tests to determine maximum muzzle velocities for existing guns, published range firing tables based on the new theories, conducted extensive experiments on the effect of center of gravity and rotating bands on projectile performance, involved himself in wind tunnel experiments to observe the forces acting on a projectile, and coordinated efforts to train students in all branches and ranks of the armed forces in the results of the new work. He insisted on strong coordination between theory and experiment, criticizing existing theory and reporting the results of newly authorized investigations. He also wanted to make these results usable on the field: the Princeton topologist J. W. Alexander II was commissioned to design a pocket-sized range and deflection corrector for abnormalities in muzzle velocity, wind, air density, or weight of projectile [Moulton, 1919a, 69–70]. It is not an exaggeration to say that Moulton had a vision of the entire subject of exterior ballistics, its deficiencies, its possibilities, and its methods; these took place at all three levels we have described.

It is best to quote Moulton himself on his criticisms of his predecessors:

From the mathematical point of view the methods of Siacci and his followers have several imperfections. In the first place, the method of representing the resistance by a series of functions of the form $c_n v^n$ has the defect of giving solutions whose higher derivatives are discontinuous, though doubtless this defect could be remedied by adopting another form for the resistance function $[F(v)]$. In the second place, the method of taking into account the effects of the decrease of density of the atmosphere with increase in altitude has been on no solid logical basis. In introducing the so-called “altitude factor” in the ballistic coefficient the mean value theorem of integral calculus has often been incorrectly applied. And even if no error were committed in this connection . . . no solid foundation would be laid for taking into account the effects of the wind and other disturbing factors . . . the methods of determining the effects of such disturbing factors as winds, abnormal air densities, and the rotation of the earth were imperfect in the extreme and in some cases even gave results of the wrong sign. [Moulton, 1919a, 9–10]

This paragraph not only summarizes some of Moulton’s objections to specific aspects of Siacci’s theory and practice but introduces often-repeated themes and mathematical stylistic issues. There is a raising of the mathematical level of discourse: questions of continuity and differentiability of functions and correct applications of the integral mean value theorem are posed. Clearly these were not of concern to Ingalls or Hamilton. They were, however, characteristic of the second wave expositors. For example, Dunham Jackson, a Ph.D. in mathematics and a representative of the second wave of reformers, wrote in his 1919 ordnance textbook “The resistance function which is tabulated for use in applying the method described in the following pages . . . has a continuous derivative, in addition to being continuous itself.” [Jackson, 1921, 9]. Thus the new discussions were taking place at the level of advanced calculus or analysis, as opposed to the elementary calculus and differential equations of the first group of reformers. This difference is also reflected in the mathematical notation chosen by each group. Ingalls and Hamilton largely used differential notation in their expositions, perhaps the better to emphasize to their readers the small changes in the quantities considered. Moulton and his associates used function notation freely along

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19 An officer from the Coast Artillery School was detailed to Moulton’s office to learn the new methods, and one senses from the account [Moulton, 1919a, 81–82] the inevitable conflict between the career army men and new mathematical practitioners, though “in the end satisfactory agreements were almost invariably reached, and the discussions were always found valuable.”
with differentials. This choice fit in with their concerns about properties of functions and
the literature on Picard iteration on which much of Moulton’s numerical processes were
based.

The need for a logical foundation for any ballistic method is stressed; this type of need
was repeatedly stated by Moulton on different occasions and in different contexts: “They
[the earlier methods] contained defects of reasoning, some quite erroneous conclusions, and
the results arrived at by singularly awkward methods” [Moulton, 1926, 1]. “Reliance will
not be placed on intuitions, or insecure methods of reasoning, but a direct treatment of
the problem [of the variations of the trajectory] will be taken up” [Moulton, 1926, 81]. From
his 1930 textbook on differential equations, “It would be inexcusable to discuss general the-
ories of differential equations and not to treat them with rigor and completeness that are so
characteristic of the present day. ... This aim has been inspired, in part ... by the firm con-
viction that the time is at hand for insisting on equally high standards in the application of
differential equations to the physical world” [Moulton, 1930, vi]. Moulton the astronomer
was attempting to deal with the “terrestrial imperfections” of ballistic science with
advanced mathematical tools.

Moulton’s introduction of numerical methods into the solution of the ballistic differen-
tial equations is noteworthy; it highlights his concerns and style, and was used to compute
the trajectory as well as the differential variations in the trajectory. We first describe the
form in which Moulton treated these equations, because this became the standard in Amer-
ican practice. Locating the muzzle of the gun at the origin of the Cartesian coordinate sys-
tem, the range direction along the positive \( x \)-axis, \( y \)-axis vertical with positive values
upward, and positive \( z \)-axis to the right of the line of fire, Moulton set

\[
\begin{align*}
\frac{d^2x}{dt^2} &= -Fx' \\
\frac{d^2y}{dt^2} &= -Fy' - g \\
\frac{d^2z}{dt^2} &= -Fz',
\end{align*}
\]  

(6.1)

where \( g \) is the acceleration of gravity, \( F = \frac{G(v)H(y)}{C} \); the derivatives are with respect to time.\(^{20}\)

In the latter, \( v = v(t) \) is the velocity, \( G(v) \) a new resistance function established by the
French ballisticians at the naval proving grounds at Gâvre in 1883. Moulton forthrightly
stated that “It is not certain that \( G(v) \) is the same for projectiles of all shapes and sizes;
in fact, it is very probable that it depends upon the shape of the projectile ... but the depen-
dence at present is quite unknown” [Moulton, 1926, 84]. Then \( H(y) = e^{-0.0001036y} \) is a factor
for the atmospheric density, which varies with altitude, and \( C \) is the ballistic coefficient.
Moulton went into no detail about \( C \), only occasionally stating that it depends on the
shape, size, and weight of the projectile. The initial conditions are given by
\( x(0) = y(0) = z(0) = 0, \ x'(0) = x'_0, \ y'(0) = y'_0 \), and \( z'(0) = 0 \). These equations replace the
“Principal Equation” of the Siacci theory.\(^{21}\)

Moulton introduced a numerical integration technique based on the Picard iteration pro-
cedure to give an approximate solution to these equations; this kind of technique was

\(^{20}\) The \( z \)-axis is introduced to track the effects of cross winds and other differential corrections such
as rotation of the earth.

\(^{21}\) The Siacci equations can in fact be easily derived from Moulton’s, as shown in Dederick [1940].
known at the time as a “small arc” method. The scheme had a connection with previous techniques for numerical integration. According to Hermann H. Goldstine’s account in *The Computer from Pascal to von Neumann*, Moulton’s method is a modification of the established technique of Adams; see [Goldstine, 1972, 76–77]. It has occasionally been summarized as a variation on Adams’ method using the calculus of finite differences [McShane et al., 1953, 786–787; Charbonnier, 1927, 733], though a reading of these sources shows matters to be more complicated. The resulting procedure has become known as the Adams–Moulton method; it is now a staple of numerical analysis. In 1922 Moulton published a treatment of his method in a volume issued under the auspices of the Smithsonian Institution [Moulton, 1922]. A more general approach can be found as Chapter 3 of [Moulton, 1926]. The reader is referred there for technical details of the method, as given by Moulton himself.23

Although this is the mathematical war contribution for which Moulton is most frequently cited, several comments are in order. The use of these methods in contemporary times was relatively common in other countries. France, for example, had been using numerical methods at Gâvre since 1887. Moulton was aware that others in the United States, such as Arthur Gordon Webster, could produce such computations, and he downplayed this aspect of his work, stating that “The introduction of the method of solving numerically differential equations is so simple and obvious that any one familiar with the general field of differential equations would hardly fail to do substantially what I did” [Moulton, 1928a, 246]. Moulton himself had written on the use of numerical methods for astronomical purposes in his book on celestial mechanics [Moulton, 1914, 425]. However, of greater importance to Moulton was his proof that the solution of the differential equations exists and that his solution method converges to the sought-for solution to within any desired degree of accuracy. The existence proof involves a two-parameter process that includes those of Picard and the Cauchy–Lipschitz method. It can be found in the 30 pages of Chapter 5 of Moulton [1926] and could easily have appeared in a journal of pure mathematics at that time. Moulton stated that the proof was completed in 1914 [Moulton, 1930, 224]. In a handwritten set of notes taken during Moulton’s lectures on *The Modern Theory of Differential Equations* for the academic year 1916–1917 at the University of Chicago one finds extensive coverage of these processes; the material was clearly at hand upon the beginning of his war work [Moulton, 1916–1917, 65 ff]. Of this work Moulton said,

I laid down for the first time explicit conditions under which the process is valid in a strict mathematical sense. One having any considerable degree of mathematical sophistication would not feel at liberty to ignore the question of the validity of the process on which he bases all his conclusions. . . . Most of those who have commented on my work either have been blind to the necessity of proving that the process is mathematically sound or have assumed that its validity was established in some indefinite past. [Moulton, 1928a, 247]

22 More detailed accounts of the relation between Adams’s and Moulton’s work can be found in Bennett et al. [1921, 74–75], Gear and Skeel [1990, 89–91], and Goldstine [1977, 297–298].

23 The method underwent modifications by others shortly after its introduction: J. J. Arnaud, a recruit from the Naval Observatory, produced “integrating forward” [Jackson, 1921, 13]. Albert Arnold Bennett, a professor of mathematics at the University of Texas drafted to Moulton’s group from service in the Coast Artillery, found a change of variables that led to another approach, the “tangent reciprocal method” [Jackson, 1921, 19].
Thus Moulton’s concern here as elsewhere was with logical foundation and mathematical rigor, in this case in the numerical techniques that provided the entries in the range firing tables. He was in effect distancing himself from the “rough, old, formal” style of ballistics by bringing in the spirit of Weierstrassian rigor as practiced at the University of Chicago, personified for example by the analyst Oscar Bolza.

This view finds confirmation in a one-page “Historical Sketch” in Moulton’s text on differential equations [Moulton, 1930, 224], which follows his chapter on numerical integration of differential equations. Despite his later protestations that his career as an astronomer did not equip him with any special qualifications for introducing numerical methods into ballistics, the sketch mentions the work of Newton and his successors. This is followed by the statement that the methods developed in the chapter were used to calculate trajectories of projectiles in the World War. The final paragraph states that a proof of the validity of the process could not have been given until a “logical foundation” had been laid for it on a restricted interval, as Picard had done, and that Moulton himself was responsible for the proof valid for the entire region for which the process held. Thus in one page we have mention of the astronomical predecessors whose work was familiar to Moulton, his statement of his related process and its use in the war, and a final announcement of his analytic convergence proof.

Moulton’s level of exposition was not always suitable to a military audience. He contributed an article to the JUSA/CAJ, one in a series of ten pieces on the New Ballistics by various authors, describing his method [Moulton, 1919b]. He noted that, in typical developments of differential equations, the student assumes that if an equation cannot be solved by separating variables (as the first wave of reformers’ writings may have implied by their restriction to this type), that it cannot be solved at all. In contrast to this, his method is applicable to any differential equation, separable or not, it “rests on a solid logical basis,” and is very convenient in practice. The results of the method and the results of actual firing are claimed to be “astonishingly harmonious.” He repeated his concern with rigor and acknowledged its possible hindering of the understanding of technique: “Unfortunately for practice, the requirements of modern analysis make it necessary to pay much attention to delicate points of logic in which the practical man has little interest, with the result that the value of a method in applications is obscured by the theory of its correctness” ((p. 41) for the preceding three quotation). He included a computation illustrating the method for the calculation of the orbit of a planet. Although this is a good account, readable by a serious student of mathematics, it is interesting that the article was followed quickly by another, authored by First Lt. J. J. Johnson [1921], which contains only the results of a computation laid out step by step. This article does “not presuppose a knowledge of mathematics beyond plane trigonometry” (p. 50) for those who were interested only in the mechanics of the method.

24 The first wave writers’ and students’ knowledge of differential equations did not necessarily stop with separation of variables. The Ordnance School of Application, another professional educational institution, established in 1902, had a course on differential equations with an accompanying booklet of problems [Dickson, 1910]. The work contains all standard elementary types taught in a typical undergraduate course, including linear systems. But the methods are straightforward and algorithmic: systems are solved by elimination, for example. The higher-level topics used by Moulton and his associates, such as variation of parameters, Wronskians, and basis of a solution set for a linear system—all topics mentioned later—are absent.
In order to give an example of Moulton’s style and concerns, we now examine in detail his treatment of the differential variations problem. He framed the question in terms of range firing tables: “It is obviously impossible to use range tables having as arguments all the factors upon which the motions of projectiles depend. Hence the effects of the minor factors are given in supplementary tables [in his newly proposed scheme] and are applied as corrections to the results for normal conditions” [Moulton, 1926, 80]. His approach is grounded in the theory of linear differential equations and, as presented in Chapter 4 of Moulton [1926], is straightforward in execution in spite of the 45-page length. It should be emphasized that in this work Moulton treats the projectile as basically a particle modified by the ballistic coefficient, as earlier authors had done. Thus his work on the variations ignores many aspects that were in fact receiving increased attention at this time: tumbling, precession, and air compression, for example. Moulton was, however, well aware of these forces, as a glance at his volume shows.

His main tool is variation of parameters. Beginning with (6.1) as the “undisturbed” trajectory, i.e., the trajectory given as normal with normal initial conditions, he assumes that the solutions to these equations are known (having been computed by the short arc method), given as

\[
\begin{align*}
  x &= \phi(t), \quad y = \psi(t), \quad z = 0 \\
  x' &= \phi'(t), \quad y' = \psi'(t), \quad z' = 0.
\end{align*}
\]

(6.2)

He then assumes that any disturbing forces such as wind or abnormal densities of atmosphere acting on the projectile have components represented by \(X\), \(Y\), and \(Z\), and thus sets up the differential equations of the disturbed trajectory as

\[
\begin{align*}
  \frac{d^2x}{dt^2} &= -Fx' + X \\
  \frac{d^2y}{dt^2} &= -Fy' - g + Y \\
  \frac{d^2z}{dt^2} &= -Fz' + Z,
\end{align*}
\]

(6.3)

using the same initial conditions as (6.1). The solution of this disturbed trajectory is expressed using the solutions to the original system as

\[
\begin{align*}
  x &= \phi + \xi, \quad y = \psi + \eta, \quad z = 0 + \zeta \\
  x' &= \phi' + \xi', \quad y' = \psi' + \eta', \quad z' = 0 + \zeta',
\end{align*}
\]

(6.4)

where it is assumed that \(\xi, \eta, \) and \(\zeta\) are small quantities relative to \(\phi\) and \(\psi\). The function \(F\) is expanded in a power series and all terms of order higher than the first are dropped, since products of \(\xi, \eta, \) and \(\zeta\) can be assumed negligible. (On the expansion of \(F\), Moulton said that, even if its smoothness cannot be justified experimentally, it can be approximated by polynomials, which clearly have such an expansion [Moulton, 1926, 82].) When (6.4) are
substituted into (6.3) and terms cancelled, he obtained a system of differential equations for \( \xi, \eta, \) and \( \zeta \), given by

\[
\begin{align*}
\frac{d^2 \xi}{dt^2} &= P_1 \xi' + P_2 \eta' + P_3 \eta + X \\
\frac{d^2 \eta}{dt^2} &= Q_1 \xi' + Q_2 \eta' + Q_3 \eta + Y \\
\frac{d^2 \zeta}{dt^2} &= -F'_{\zeta} + Z,
\end{align*}
\]

where \( P_1, P_2, P_3, Q_1, Q_2, \) and \( Q_3 \) are functions of \( F, \psi, v, \phi, G, \) and various derivatives of these. Let us repeat that these expressions are assumed known from the original trajectory, and have their values at each \( t \) already determined from the original trajectory, computed by the short arc method. The central goal of Moulton’s approach is to solve this system in general for \( \xi, \eta, \) and \( \zeta \) by variation of parameters. Then, by specifying \( X, Y, \) and \( Z \) for the particular force in question, he derives formulae for \( \xi, \eta, \) and \( \zeta \) as functions of \( t \) for the entire trajectory corresponding to that force, not just at the terminus. Variants of this technique give \( \xi, \eta, \) and \( \zeta \) for abnormal initial conditions. These three variables are known as the differential variations due to the force considered.

We will only sketch the remaining steps in the procedure, since it is standard variation of parameters. The third equation is independent of the first two and can be solved on its own. The first two equations are expanded into a system of four equations in the usual way as

\[
\begin{align*}
\frac{d \xi}{dt} &= \xi' \\
\frac{d \eta}{dt} &= P_1 \xi' + P_2 \eta' + P_3 \eta + X \\
\frac{d \eta'}{dt} &= \eta' \\
\frac{d \eta}{dt} &= Q_1 \xi' + Q_2 \eta' + Q_3 \eta + Y.
\end{align*}
\]

To effect variation of parameters, four linearly independent solutions to the corresponding homogeneous system are sought, forming a basis, or “fundamental set” in Moulton’s terminology, for the general solution. Two of these can be easily found and expressed in terms of \( F, \phi, \psi, \) and their derivatives with respect to \( t \). The remaining two are those computed trajectories corresponding respectively to a change \( \Delta \theta_0 \) in initial angle of elevation and \( \Delta v_0 \) in initial muzzle velocity. Linear independence of the four is verified by computing the Wronskian. These four solutions are then linearly combined with coefficients \( C_1, C_2, C_3, \) and \( C_4 \) to given the general solution to the homogeneous system. The solutions with the \( C_i \) regarded as functions are then substituted into (6.6) to get the equations for \( \frac{dC_i}{dt} \), as required by variation of parameters. These derivatives are found using Cramer’s rule and then integrated to get the \( C_i \). When substituted back into the expressions for \( \xi, \xi', \psi, \) and \( \psi' \), they give the required differential variations in terms of \( X \) and \( Y \) and the functions from the original, undisturbed trajectory. The individual stages required in the computation of the solution were organized in an economical eight-step procedure that minimized the number of integrations necessary and introduced other short cuts.

To give an example of the use of the procedure, Moulton stated that the normal trajectory assumes that the air is stationary with respect to the earth’s surface, and that winds can
be regarded as an abnormal force. “Let $u$ be the wind component in the plane of fire, taken positive in the direction of fire; $s$, the vertical component taken positive upward; and $w$, the cross-component taken positive toward the right of the direction of fire” [Moulton, 1926, 108–109]. Then the differential equations for the disturbed trajectory become

$$
\frac{d^2 x}{dt^2} = -F(x' - u, y' - s, z' - w, y)(x' - u) \\
\frac{d^2 y}{dt^2} = -F(x' - u, y' - s, z' - w, y)(y' - s) - g \\
\frac{d^2 z}{dt^2} = -F(x' - u, y' - s, z' - w, y)(z' - w).
$$

Now the right-hand members are expanded in a power series in $u$, $s$, and $w$ and, since these are small relative to $v$, all terms of the second order and higher are again dropped, giving

$$
\frac{d^2 x}{dt^2} = -F(x', y', y) x' + \left[ F + x' \frac{\partial F}{\partial x'} \right] u + x' \frac{\partial F}{\partial y'} s \\
\frac{d^2 y}{dt^2} = -F(x', y', y) y' - g + y' \frac{\partial F}{\partial x'} u + \left[ F + y' \frac{\partial F}{\partial y'} \right] s \\
\frac{d^2 z}{dt^2} = -F(x', y', y) z' + Fw.
$$

In this case the functions $X$, $Y$, and $Z$ are given by

$$
X = \left[ F + \phi' \frac{\partial F}{\partial x'} \right] u + \phi' \frac{\partial F}{\partial y'} s \\
Y = \psi \frac{\partial F}{\partial x'} u + \left[ F + \psi \frac{\partial F}{\partial y'} \right] s \\
Z = Fw.
$$

These expressions are then substituted along with the other necessary ones into the integrals to be evaluated in determining the differential variations due to wind.

Of course, this assumes that the winds are constant and known ahead of time. To account for the more realistic situation of winds of varying direction and velocity, that could be known only at the time of battle, it was noted that the integrals that are necessary to express the variations are of the form

$$
\int_0^T f_1(t) u(t) \, dt; \quad (6.10)
$$

for example, where $T$ is the time of flight, $f_1(t)$ is a combination of the known functions from the undisturbed trajectory and $u(t)$ is the (usually varying) wind component in the range direction. Moulton proposed that the meteorological service would transmit data to the battery commander at various prescribed intervals of time. These data would allow approximation of $u(t)$ by a step function having as values those extracted from the message. Thus (6.10) would be simplified to a sum of integrals, over time intervals, of $f_1(t)$ with the constants, extracted from the message, multiplying them. Since these integrals only involve $f_1(t)$ with the wind components as constants, they could be evaluated before hand, reducing
the work of the battery commander to arithmetic. The integrals without the \( u \) present were called “weighting factors”; each disturbing force had its own set. In this manner Moulton attempted to account for the third level of ballistic practice, actual engagement. From this point of view one concurs with the opinion expressed in Fenster et al. (2009) that “this chapter is probably the most important part of the book practically,” although actual implementation of this scheme would appear difficult. These ideas began to be realized upon the inclusion of weighting factors for wind in “some range tables issued by the Technical Staff of the Ordnance Department,” according to a JUSA/CAJ article by Joseph Ritt [1920, 404].

As with the ballistics of the earlier practitioners, the mathematics used and the surrounding milieu both need to be appreciated. For Moulton it was important to justify every approximation by reference to an appropriate mathematical result, as we have seen in the case of the series expansion of \( F \). While it is true that the level of mathematics used here is higher than that of Hamilton and Ingalls, so is the level of justification. The artillerymen often accepted their approximations by seeing if the agreement between theory and firing warranted them, but Moulton sought mathematical and computational rationales. For example, in the discussion of the weighting factor, Moulton expresses concern about replacing \( u(t) \) by a constant approximation, and references the integral mean value theorem in a general discussion:

The integrals arising in the determination of [the weighting factors] are strictly speaking of the form \([6.10]\), where actually \( u \) is variable. It follows from the mean value theorem that if \( f(t) \) does not change sign in the interval of integration, then there is a number \( u_1 \) between the smallest and largest values of \( u \) such that

\[
\int_0^{\ell_1} f_1(t)u \, dt = u_1 \int_0^{\ell_1} f_1(t) \, dt.
\]

This is not in general the mean value of \( u \) on the interval [as the earlier men may have assumed], but if \( u \) does not vary too widely and if the zone is narrow enough so that \( f_1(t) \) does not vary greatly, then the mean value of \( u \) may be taken for \( u_1 \) without making appreciable error. There would be no difficulty in examining numerically the question of the possible extent of the error in typical cases. [Moulton, 1926, 121]

This expression of concern with the mathematical validity of his techniques and computations to back up his assertions is typical of Moulton’s work. Five pages of the chapter on differential variations (pp. 102–106) are devoted to considering what would happen if terms of higher order were retained in the expansion of \( F \) used in this approach.

Moulton’s background in astronomy can be seen in his choice of variation of parameters as a method of solution for the problem of differential variations. He was of course aware of the history and application of the technique to problems in astronomy; he made historical comments to this effect in his books on differential equations and on celestial mechanics. In the former he introduced variation of parameters in the fourth chapter in full generality; in the latter, Chapter 10 is devoted to perturbation theory and its use of the technique. We must emphasize that Moulton’s approach to the variability problem in ballistics involves applying variation of parameters to a linearized version of a system for the variations themselves, however. The author has been unable to determine what degree of originality to attribute to Moulton for this idea.

Moulton found another use for the differential variations, in the construction of general ballistic tables. “... in view of the great variety of guns and projectiles in modern artillery practice, it is desirable to have general ballistic tables from which all special range tables
can be interpolated, or otherwise determined, by a relatively small amount of work," he declared [Moulton, 1926, 124]. Rather than compute new trajectories for each type of gun together with a set of differential variations for it, why not allow the parameters $C, x_0, y_0, \theta_0$ in the equations (6.1) to assume a large set of reasonable values separated by small intervals and compute a comprehensive collection of trajectories once and for all? This being done, any particular gun would yield its values for the parameters, furnishing specifications for a trajectory that would then be selected or approximated from the existing collection. The idea was not new with Moulton. Charbonnier [1907, 256–258] described this idea and its attempted execution in France, which began in 1894. Charbonnier pointed out, however, that such tables still depend on some resistance law, and existing tables become useless if that law is revised. Moulton proposed to deal with this problem by claiming that when the retardation law is improved, trajectories from the older tables would be corrected by using the technique of differential variations on the law itself [Moulton, 1919a, 78].

In fact the differential variations were to play a role in the construction of the general tables themselves. Moulton observed that each trajectory calculated by short arcs from (6.1) simultaneously furnished a whole set of “secondary trajectories” [Moulton, 1926, 128]. Each such trajectory is found by taking as its initial point a fixed point of the already computed trajectory and following the original trajectory from there. The secondary trajectory corresponds to the flight of a projectile with different $C, \theta_0,$ and $v_0$ than the original, but these values were easily determined (p. 125 ff). Thus computing a single trajectory could supply an entire family of others, which could then be used for the general tables. This idea clearly cuts down the computing labor. One difficulty in this plan is that the secondary trajectories do not differ from each other by unit multiples of $C, v_0,$ and $\theta_0,$ and so were not necessarily suitable for table use as they stood. Moulton found a remedy for this problem by using his differential variations (pp. 127–129); it is at least clear that the variations could play a role in such adjustments.

The construction of the general ballistic tables proposed by Moulton was begun under his direction by A. A. Bennett, and three volumes were eventually released. When framed in terms of the spectrum of possibilities for range table construction given earlier, one can see the evolution from the gun-specific option of “firing only” to the mathematically created general firing tables, which acted as a sort of universal trajectory bank for which a specific gun provided some parameters. The contrast with Ingalls and Hamilton, whose textbook problems are often tied to a particular gun, is evident.

We conclude this section with a brief account of the contributions of Gilbert Ames Bliss to the problem of differential variations. Bliss, a colleague and one-time student of Moulton’s at the University of Chicago, was recruited by Oswald Veblen at Aberdeen to help with the mathematics of the war effort. Bliss was immediately attracted to the variability problem, and he devised a method, based on the use of an adjoint system of differential equations, to deal with it. This adjoint is described explicitly in Gluchoff [2005, 332–333]. His contribution is at the first and second levels of ballistic discourse. In his approach, all sources of variability can be treated simultaneously, as they are all contained in a single equation. Consequently Bliss’s solution involves only a single numerical integration, namely that of the adjoint system, to get all range variations. This contrasts with Moulton’s technique, in which a separate integration is required for each one. Bliss stressed this time-saving aspect: “the results explained here enable one to compute a range table . . . with

---

26 A detailed account of Bliss’s life and work may be found in Bliss [1952].
amount of labor not excessively greater than that required by the Siacci–Ingalls theory” [Bliss, 1919, 296]. Moulton included a summary of this approach in his 1926 text, and stated, “The method is characterized by a singular elegance and great simplicity in application” [Moulton, 1926, 135]. In practice Bliss’s algorithms appear to have been the more widely used. In a subsequent paper Bliss showed how, using his methods, corrections in the ranges due to a change in the retardation law could be effected [Bliss, 1920b, 101], substantiating Moulton’s claim that the variations could be used to update trajectory calculations for any newly determined law.

7. Reception of the second wave ideas

Given that Moulton frequently expressed the novelty of his approach to ballistic problems, the “complete independence of the present developments from those that have gone before” [Moulton, 1926, 1], it is natural to ask how successful his efforts were. There are many surface indications that his work was well received and made a lasting impact, though this statement has qualifications. At the outset, it is obvious that the mathematical revision of a ballistic theory would not make as large an overall impression as, say, the introduction of a new weapon or new method of communication. One could expect such a mathematical innovation to be of little general interest. But several measures of reception are noted below. Most helpful for the propagation of Moulton’s ideas on differential variations was a simplification of their derivation due to William E. Milne that appeared in JUSA/CAJ [Milne, 1919] and that requires no more than a knowledge of calculus and elementary differential equations. Milne had earned a Ph.D in mathematics from Harvard in 1915, and spent his pre- and postmilitary career at Oregon State University.

Tables were quickly constructed using Moulton’s numerical and mathematical methods. The 60-page Provisional Range Tables for the British 75 mm. Gun, Model 1917 [Moulton, 1919a, 39] has sections for French shrapnel, American shrapnel, and short and long “fuzes,” separated by different colors for easy use; again these divisions were due to the differential variations for type of ammunition. From time to time results comparing the actual firing of a gun with the ranges predicted by Moulton’s theories were reported in professional journals; these reports ranged from cheerful confirmations of success to cautious withholding of support in view of further tests [Moulton, 1919b, 41; Schwartz, 1924]. MacFarland’s Textbook of Ordnance and Gunnery gives an example of numerical integration (pp. 422–426) as well as a long section on fire preparation (pp. 430–443) that has much in common with the procedure specified in the 1921 75 mm. Gun volume. The reviewer of this text for the JUSA/CAJ praised the book, expressing a grudging respect for numerical integration and the progress which it engendered: “To the average person, mere mention of this method of integration is likely to result in the feeling that it is something entirely too tedious for consideration. Truly, it does require concentration and accuracy for its application, but the method has opened up new fields of knowledge concerning the behavior of a projectile. The recent advances in exterior ballistics are due in part to this method of calculation, and its importance must not be minimized” [Anonymous, 1929].

The new methods, including differential variations, were adopted as standard for computing range firing tables, though the Siacci method was retained [Dederick, 1940]. A volume devoted to table construction [Hitchcock, 1934] has Moulton’s equations of motion on p. 40, and explains the numerical integration technique in copious detail (p. 50 ff). There is a lengthy explanation of differential variations, credited to Moulton, but given in the form developed by Bliss (224 ff.). Jackson’s ordnance text, issued in October 1919, is divided
equally between coverage of numerical integration and differential variations, using Milne’s simplification of Moulton’s work as well as Bliss’s new approach. In 1927 a “Range, Deflection and Wind Correction Computer,” an analog circular device, eight inches in diameter and containing scales for variations in the usual quantities, was constructed at the Frankford Arsenal in Philadelphia, Pennsylvania [Frankford Arsenal, 1927]. It resembles the model described nearly a decade early by Alexander and incorporated the ideas of Moulton and Bliss. These activities took place after the Armistice, and it is unlikely that even the provisional tables, for example, actually saw use in the World War.

Moulton’s own 1926 volume was released upon his retirement from the University of Chicago, and was the result of his earlier technical reports and the courses in ballistics he taught there. A course in exterior ballistics was offered in 1919 in the Department of Astronomy and Astrophysics, and for the academic years 1920–1923 “Exterior Ballistics” as well as “Advanced Ballistics” was offered [University of Chicago]. The courses were attended by representative officers from the different branches of the military, though this practice seems to have died out after several years. This attendance followed from one of Moulton’s recommendations [Moulton, 1919a, 87]. The Coast Artillery School considered itself as a source of postgraduate education for officers, but now Moulton suggested university-level graduate instruction.

The reception of Moulton’s book ran the gamut from extravagant praise to respect tempered with reservations. An example of the former was a review by Roger Sherman Hoar, a lawyer by training who enlisted in the Army upon the entrance of the United States into the World War. Hoar’s interest in mathematics led eventually to a posting at the Coast Artillery School, where he taught advanced orienteering post-Armistice. He was also the first to prepare a course in exterior ballistics based on the second wave methods for the new Ordnance School of Application at Aberdeen; the results of these notes were published in book form by the army [Hoar, 1921]. This work attempts an exposition of Moulton’s ideas at an elementary level. It describes in detail a method for range table construction, complete with forms for computing trajectories by numerical integration. Among Hoar’s other writings was the introductory article to the 10-article series on the New Ballistics appearing in JUSA/CAJ [Hoar, 1919]. Familiarity with these practical aspects of ballistics put Hoar in a position to attempt an evaluation of Moulton’s work. In his review of Moulton’s book he claimed that Moulton “laid the cornerstone for an entirely new science of ballistics,” and rather grandiosely compared him to Alexander Graham Bell and Samuel Morse [Hoar, 1927a, 325]. Ironically, this review provoked anger from Moulton and began a nasty exchange, which we will discuss shortly.

An assessment by a former student of Moulton’s appeared in Army Ordnance and praises the book in more measured terms, providing a summary of his techniques [Guion, 1926]. The reviewer in the JUSA/CAJ acknowledged Moulton’s work and its importance, but noted that “the method of presentation is beyond the comprehension of the casual reader in ordinary artillery circles” [Anonymous, 1927]. The most interesting review came from Joseph Eugene Rowe [1928], a student of the geometer Frank Morley and one-time president of Clarkson College. Rowe, though crediting Moulton with putting ballistics on a sound mathematical basis, and listing his book among three outstanding mathematical “products” of the war (p. 232), called into question the admiration that the new theories were receiving. His objections were basically those recognized by Moulton: that the methods involved approximations, as must any ballistic method, that the resistance function itself was subject to revision, that the ballistic coefficient only approximates the behavior of the projectile. He found the method of numerical integration too difficult for a typical
Army officer “of any nation,” arguing that “Their training must be broad and practical in the extreme, and they want mathematics presented in as simple and usable form as possible” (p. 231). He found the chapter in Moulton [1926] on differential variations “interesting to the mathematician, but [it] is not elementary, and in my opinion, is not given in a sufficiently elementary and practical manner” (p. 231). Perhaps Rowe’s most generous overall comment was that “they [Moulton and his associates] must have credit for giving an impetus to mathematical research in the Army and Navy that has had excellent results” (p. 232). Moulton’s existence and convergence proof, on which he laid such importance, is not specifically mentioned in this or other reviews.

The opinion of Philip Schwartz, a young man about to embark on a 30-year military career about this time, is pertinent. Schwartz received an undergraduate degree in mathematics in 1917 from Columbia University and entered the Army that same year. He took part in Veblen’s range firing activities at Sandy Hook in 1918, as described in Grier [2001], and worked for several months as a computer under Moulton [Moulton, 1919a, 4]. He could speak therefore as one familiar with both the old and new methods but with no entrenched experience to defend. In a document written during a stay at the Aberdeen Proving Grounds he addressed the nagging concerns about the level of sophistication of Moulton’s work and its accessibility, and a defense of older methods, as well as other issues [Schwartz, 1920]. It is worth quoting a single paragraph from this 16-page paper. [Typing errors and punctuation of the original are kept.]

Ballistic tables based on short arc computations are now being computed, and when they are published will undoubtedly do away with the use of Ingalls’s tables. But until the time when the new tables are available for general use it is believed that Ingalls’s tables may be used to solve most of our problems even for fire up to 45 degrees elevation... The benefits of the results obtained from computations by short arc methods may be easily overestimated by those who have not looked into the matter critically. Mathematicians may say that the Siacci methods is based on poor and complicated methods and that Army officers have been kept away from the study of ballistics on this account [a reference to Veblen’s report], but a casual glance at the published papers on the short arc method and the method of computing differential correction, will make one think that the Siacci mathematics is much simpler. A complete understanding of the mathematics of the new methods including the differential corrections, involves an understanding of a great deal of higher mathematics. The opinion has been expressed that the pre-War methods were to make firings at all elevations and thus really not need any good mathematical theory—merely a table for interpolation purposes [Mouton’s criticism]; but this procedure is still being followed and will be followed even after the new tables are published... [Schwartz, 1920, 6]

The most heated confrontation came in the wake of Hoar’s review of Moulton’s ballistics book, which appeared in the American Mathematical Monthly [Hoar, 1927a]. He portrayed Moulton’s work in glowing terms, but then proceeded somewhat tactlessly to point out that, among other developments, in the years since Moulton’s original war work Bliss’s approach to the variations made Moulton’s “obsolete” (p. 325). He listed a reduction, due to T. H. Gronwall, of the adjoint system, the use of the circular coordinate system as opposed to Moulton’s rectangular one, and a relabeling of the axes as evidence of progress in the subject not touched upon by Moulton’s work. This review brought forth a furious response from Moulton, which appeared about a year later in the same journal [Moulton, 1928a]. Moulton again promoted his existence/convergence proof: “Unfortunately for those interested in the applications, the proof of the validity of the method involves many of the
niceties and much of the close reasoning characteristic of modern analysis, and it would be
wholly beyond the appreciation of a beginner” (p. 247), evidently referring to Hoar. Sarcas-
tically appraising Hoar’s own book, he said,

He [Hoar] says that his book is more up-to-date than the one that I have recently pub-
lished. The purposes of the two books are so different that it would be difficult even for
an impartial person to compare them justly. Mr. Hoar’s book was prepared for use in a
course on ballistics at the Aberdeen Proving Ground. In the introduction the author says,
‘It is assumed that the student is thoroughly grounded in algebra and plane trigonome-
try, and knows enough calculus to appreciate the meaning of a derivative, a differential,
and a definite integral.’ It seems to me that the sentence quoted excellently defines the
scientific level of the book. [P. 248]

He then sniped specifically at Hoar’s definition of the integral as an area under a curve,
low-level exposition to a follower of Weierstrass. Moulton used his treatment of differential
variations as an explicit point of comparison:

The theory of differential variations depends upon linear differential equations. The very
heart of any adequate theory of them obviously is attached to that of a fundamental set
of solutions of the differential equations. In my treatment of the problem I went straight-
way to the fundamental set of solutions and made essential use of them in most of the
subsequent discussions of the chapter. Mr. Hoar uses no such general and powerful
machinery—he does not have a syllable on it. [P. 249]

He dismisses Gronwall’s reduction as an obvious step based on a method that had long
been in existence, thus indirectly impugning Hoar’s mathematical knowledge.

Hoar had written another review of *New Methods* that was essentially a repetition of his
earlier one [Hoar, 1927b]; this appeared in *Army Ordnance*, another professional military
journal of the day. The review provoked a similarly blistering reply, also appearing in *Army
Ordnance*, in which Moulton repeated the attack on Hoar’s exposition [Moulton, 1928b]:

In my first paper, written shortly after I entered the service, I seized the problem of com-
puting differential variations and pulled it out of the fog in which it was involved and
placed it on a solid foundation... Mr. Hoar did not connect with the essence of the chap-
ter on Differential Variations, which is based on a use of the fundamental set of solu-
tions... In particular, such things as proving the validity of the methods employed,
the use of fundamental sets of solutions of systems of linear differential equations ... are apparently completely beyond his horizon. [P. 321]

He was obviously unimpressed by Hoar’s work, dismissive of Gronwall’s reduction, and
out of patience with the issues of coordinate systems and their labels. Hoar replied wound-
edly to Moulton’s comments [Hoar, 1928], making it clear that he held Moulton’s work in
the highest regard, but defending his own concern with his list of issues. In this exchange
can be seen the natural conflict between exposition at a theoretical and at a practical level,
which occurred for the first time in American ballistics. Moulton’s book was the first high-
level exposition of exterior ballistics for a university mathematical audience produced in the
United States, and as such stood alone in comparison with earlier efforts like those of Ing-
alls and Hamilton, as well as contemporary textbooks like Hoar’s and Jackson’s. From this
point of view Hoar’s book is the lesser, but Moulton did not have to contend with the issues
of teaching range firing data interpretation, range firing table construction, and detailed
computation of them in his University of Chicago course. No doubt Moulton would have
been annoyed to learn that Rowe, in his review of *New Methods*, considered Hoar’s book, in addition to Moulton’s, as one of the three worthwhile products of exterior ballistic efforts from the World War, the third being an extensive set of tables used in ballistic computation.

Bliss’s approach received similar criticism and praise in reviews and internal reports. Philip Schwartz commented in his memo that the adjoint system is “an ingenious system of computing differential corrections to Moulton’s trajectories . . . The mathematical theory involved in the derivation is of very high order” [Schwartz, 1920, 3]. As a measure of popular success, both techniques were included in an Encyclopaedia Britannica article on ballistics [Tschappat, 1922]. Veblen’s post-War tour of European countries to report on current procedures for range firing table construction concluded that “The most satisfactory general method [of dealing with differential corrections] seems to be the one devised by Professor Bliss at the Aberdeen Proving Ground. The English have arrived after a long process of evolution at what amounts to the same method . . . but their mathematical treatment of it is far from being as clear as that given by Bliss” [Veblen, 1919, 4]. Ironically, the adjoint system idea, which provided the basis for the ultimate ease of the calculations, proved to be somewhat of a drawback as well, at least from the point of view of the some of the users of the method. While a mathematician such as the reviewer of Bliss’s [1944] textbook [Milne, 1945] could enjoy the occurrence of a familiar concept in a new setting, the response of others was not so welcoming. “The adjoint system of differential equations is introduced as if it were such a familiar portion of mathematics that this could be done without explanation,” complained Rowe in his *New Methods* review [1928, 231]. More troubling for some was the fact that, as presented, the variables in the adjoint system, called “auxiliary variables,” have no physical interpretation. This could not be said of any portion of Moulton’s work, let alone that of his predecessors. The fact was noted by Dunham Jackson [1921, p. 27]: “In behalf of the more direct method [Moulton’s] it is to be said that it is somewhat easier to follow, in practice as well as in theory, because of the more obvious physical significance of the quantities involved.” A subtle but clear difference in the level of abstraction was thus introduced: a mathematician has no trouble dealing with quantities having no specific meaning; this is not necessarily so for others. A remedy for this problem was attempted by Hoar, whose text features an interpretation of the auxiliary variables in terms of rates.

Bliss himself addressed the ever-present economic concern with the remark that his technique “reduced by three fourths the labor in computing the most numerous corrections in the range tables . . . ,” adding that “An improvement in a range table which might reduce by one the number of shots fired for adjustment from a fourteen-inch gun in order to attain a target, would on each such occasion save one third or more of the average salary of a scientific expert at 1918 rates” [Bliss, 1927, 314].

How did Moulton himself come to view his wartime work? We have already seen that he had a rather casual attitude toward his introduction of numerical integration. In fact, he bristled at the idea that, as an astronomer, he would have special knowledge of this method:

> There seems to be a widespread belief that astronomers use it in determining the orbits of comets . . . Nothing could be more remote from the facts. Astronomers as a class are as innocent of any acquaintanceship with this method of solving differential equations as most modern mathematicians before the War. Probably not one out of twenty of them ever heard that such a method exists . . . So far as I am concerned, I did not learn the method from astronomy, but from a general survey of the various processes that are known for solving differential equations. [Moulton, 1928a, 246–247]
He included a generalization of his variational method in his differential equations book [Moulton, 1926, 236–245], in which the ballistic material is presented as an illustration of the general technique. It appears that he considered the last chapter of his ballistics book, which takes up nearly a third of the volume and contains a treatment of a rotating projectile, to be his most consequential contribution to ballistics [Moulton, 1928a]. Aside from these specifics, he did not regard ballistics as playing a major role in his career, as did Charbonnier, nor did he see ballistics as the major enterprise that Charbonnier did, as deeply connected to the progress of science and mathematics. Even if one dates his active involvement with ballistics from 1917 until the appearance of New Methods in 1926, it is clear that the bulk of his scientific career was bound up with astronomy. While clearly aware of the deficiencies in knowledge under which he worked, he attempted to address the problems with a critical spirit. In doing so he introduced approximation problems of his own—anyone who thinks of numerical integration as practiced at that time as a razor-sharp tool for cutting to the solution of a differential equation should look at the amount of guesswork involved in beginning the process, as described in the 1922 Smithsonian volume. But his Weierstrassian training made him more conscious of the possibilities and consequences of inevitable error.

8. Numerical integration after World War I until 1934

The introduction of numerical integration as a method of solving differential equations by Moulton and his associates contributed to the elevation of this subject in limited ways in the decades following the war. One must first acknowledge the general lack of interest on the part of American mathematicians in problems of applied mathematics during this era, even though the sophistication of approaches to such problems grew greatly. Garrett Birkhoff has stated bluntly that “During the 1920’s and 1930’s, few American mathematicians paid much attention to contemporary engineering developments . . . the solution of special problems bored them” [Birkhoff, 1977, 62]. Against this backdrop one could expect little curiosity about the issues surrounding numerical integration.

From the end of the war until roughly 1930, many contributors to the rise in visibility of numerical integration in American can be traced directly to Moulton’s working group, whose members not only wrote expositions for military use but also generated a stream of publications in journals and differential equations textbooks. The volumes by Jackson and Hoar were accompanied by a work of similar aim by Bennett [1921] for the field artillery. This volume contains a section on numerically computation of trajectories and demonstrates the technique for a simple first-order linear equation, with exercises and numerous comments (pp. 131–139). Here for the first time Moulton’s convergence proof is noted (“. . . a rigorous mathematical proof of this convergence under the restrictions occurring in ballistic theory is available in the literature” (p. 134)), but the emphasis is clearly on the practical. Lester Ford, a Ph.D. in mathematics from Harvard and also an associate of Moulton, followed up the convergence proof with a treatment of his own [Ford, 1924]; thus began in the mathematical community the acknowledgment of a theoretical aspect of numerical integration not present prior to the war. Ford also presented

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28 In general this volume shows clearly the style of a mathematician, especially in its treatment of probable error and the ballistic coefficient in appendixes.
numerical techniques in a textbook on differential equations published early in the next decade [Ford, 1933, 147–161]. Moulton’s treatment [1922] demonstrated the slow beginnings of institutional support for and acknowledgment of applied mathematics; the volume’s introduction was an “Advertisement” by the secretary of the Smithsonian Institution, which explained that the book “comprises the most important formulae of many branches of applied mathematics, an illustrated discussion of the methods of mechanical integration, and tables of elliptic functions” (p. iii).

The sources of information on numerical integration mentioned in Section 2 continued their influence. Runge’s method appeared in an appendix of a text on differential equations by an English mathematician [Ince, 1926, 540–547]. Whittaker and Robinson’s book on general numerical analysis came out in the mid-1920s [Whittaker and Robinson, 1924]; it contains a chapter on Adams’s method (pp. 363–367).29 Bateman’s text was reissued in 1926. But most influential for the development of the subject was the work of Milne. His contributions include several new ideas in numerical integration, one of which has become known as the fourth-order implicit Milne–Simpson formula. These appear in a paper based on “a long series of integrations carried on for several years” [Milne, 1926].30 The paper appeared in the American Mathematical Monthly, which would have made the methods described accessible to a wide audience. For the first time in an American publication, a modest list of sources on numerical integration was included. A glance at even a partial list of Milne’s publications31 shows a career-long devotion to the theoretical and practical aspects of differential equations, which presumably grew out of his war experiences; in 1953 he published a text on numerical solutions [Milne, 1953].32

In the 1930s, more differential equations texts included numerical material. Of greater importance was the publication of two books: Numerical Mathematical Analysis by James B. Scarborough [1930] and Numerical Integration of Differential Equations by Bennett, Milne, and Bateman [1933]. The former, by an associate professor of mathematics at the U.S. Naval Academy, raised the level of discourse of the subject, devoting nearly 300 of its 400+ pages to discussions of interpolation formulae, difference formulae, accuracy of calculations, and expositions of the methods of Adams, Runge–Kutta, and Milne. A novel feature of this book is the author’s own comparisons of the advantages and disadvantages of the various methods (pp. 267–283) in self-constructed examples.33 A nod to Moulton’s convergence proof was given (p. 218), but the book was intended for use by the practical computer. Moulton’s use of numerical integration for the calculation of trajectories is explicitly mentioned after a long sample calculation (pp. 244–253). The book became a

29 This work is cited in Hitchcock’s volume on range table construction [Hitchcock, 1934, 275].
30 For the relation of these methods to those of Moulton and Adams see [Gear and Skeel, 1990, 91].
31 Such a list is readily available in the electronic journal collection JSTOR.
32 In this text Milne expressed the opinion that this topic had yet to achieve status among American pure mathematicians: “perhaps because it involves much tedious calculation, the numerical solution of differential equations has never quite attained respectability among pure mathematicians. Otherwise surely the subject would have been much better standardized, the gaps in our present knowledge would have been filled, and arguments concerning which method is the best would have been settled” [Milne, 1953, 4].
33 At least one attempt at such a comparison probably predated this book: [Hitchcock, 1934] lists an undated technical report by L. S. Dederick, “Relative Merits of Various Formulas for the Numerical Integration of Differential Equations,” among its references (p. 276).
standard prior to the Second World War, being used by computers of range tables at that
time; it, too, is cited in Hitchcock [1934, 275].

The second volume was a report of a committee formed by the National Research Coun-
cil consisting of the authors and Lester Ford and is the first consideration of the subject
given by American research mathematicians. Each of the authors contributed a chapter,
and the work is characterized by lengthy surveys of the international literature, detailed
examples for study by computers, but above all the connection with the pure mathematical
domains of approximation theory and asymptotic expansions. Many of the contributions
mentioned in our survey are included, but they keep company with those of d’Alembert,
Greenhill, Féjèr, Prandtl, Sturm, Gronwall (cited for his work on the convergence of the
Laplace series), and mathematicians of similar stature. Again Moulton’s method and proof
are mentioned (p. 69). The difficulty of method comparison is both acknowledged as a seri-
ous problem and given a surprising rationale: “The discussion as to the relative excellence
of alternative methods has been sometimes marred in the past by claims due perhaps to
ignorance, personal taste or wartime emotion, and may be difficult of objective scientific
settlement” (p. 6).

In spite of the rise in visibility of numerical integration resulting from these works, it
appears that awareness of the subject and its place in the classroom in the 1920s and
1930s had not changed much from Lipka’s time. A quite favorable review [Longley,
1932] of Scarborough’s book notes the rapid growth of numerical mathematics at the time
but emphasizes the author’s stated goal of the use of his book for independent study. “The
book would serve well as a classroom text. At the present time no large number of students
pursue such a course . . . it seems likely that the book will be most widely used for individual
study” (p. 331). Indeed, when Herman Goldstine assisted Bliss in the latter’s early 1940s
course in exterior ballistics at the University of Chicago, he learned of the numerical meth-
ods from an astronomer [Goldstine, 1990, 8]. Frame [1943] reports the contemporary lim-
itation of numerical integration to a topic in advanced calculus, thus robbing it of wider
exposure. Anecdotal evidence suggests that numerical analysis was not universally taught
and thus that Scarborough’s text appeared formidable to some of the computers at the
ENIAC project;34 see [Shurkin, 1996, 126–127; Fritz, 1996, 16] for some first-hand accounts
of experiences of these computers, undergraduate mathematics students with bachelor’s
degrees from a local liberal arts college. Thus we are left to conclude that although certain
participants in Moulton’s ballistic research efforts, notably Bennett, Ford, Milne, and
Moulton himself, were responsible for propagating the ideas of numerical integration to
a wider audience, resulting in a degree of exposure in the succeeding decades greater than
that of pre-War times, such exposure had quite a limited effect on instruction of the
material.

9. Conclusion

In a Phi Beta Kappa address at the University of Virginia a few weeks prior to the Armis-
stice of the World War, J. S. Ames remarked,

When this country entered the war, it is true beyond any doubt that the American people
had a great expectation, nay a conviction, that with our so-called inventive genius we

34 This was the Electronic Numerical Integrator and Computer project at the University of
Pennsylvania in the 1940s, which computed trajectories electronically.
would seriously influence the war, perhaps stop it, by the epoch-making inventions which
our professional, highly advertised, inventors would quickly make. . . . There was a great
disappointment, almost a shock, as the days went by, the periods promised for great
accomplishments passed, and certain names almost disappeared from the public press
. . . the knowledge required [for the solution of war-related problems] is not that of the
trained engineer, but definitely that of the scientific investigator. . . . The point I wish to
emphasize is that the ability and knowledge required in waging this war successfully
are not those possessed by any body of men except those with a profound knowledge
of science and of the scientific method. The problems are too complicated. [Ames,
1918, 402–403]³⁵

There are several obvious points at which an attempt to apply this opinion to the work
on the differential variations of the ballistic trajectory described in this paper prove prob-
lematic. These efforts occurred too late to have any impact on the progress of the war. The
domain of mathematical modeling is not that of invention. And yet the events described in
this paper can be seen in a manner similar to that described by Ames, as the “rough, old,
formal” ballistics began to change.

The variability problem as described here was present ever since the guns were able to
fire with a reasonably sure trajectory, and from this time efforts were made to account
for effects of the wind, for example, and minute changes in initial conditions. The JUSA/
CAJ had published articles since its inception which dealt with the effects of the wind on
trajectory, using mathematical tools no more advanced than algebra, trigonometry, and
integral calculus; one such [Ruckman, 1892] appears in the first issue. The techniques used
may be described as “inventive,” and bore the stamp of no research laboratory or scientific
school.

It seems, however, that the methods used by the first wave of ballistic reformers came to
be seen as somewhat primitive, e.g., Hamilton’s use of differentials, and the loading of the
ballistic coefficient with empirically determined constants. The word “empirical” itself came
to have a negative connotation when describing their work: the JUSA/CAJ editorial intro-
ducing the series of articles on the New Ballistics proclaims that “There is still much to be
done to divorce ballistics from the realm of the empirical. Major Moulton and his col-
leagues have opened several new doors.”³⁶ “For his courage in setting aside the long-estab-
lished, revered, but rather empirical method in use in the War department, and in
introducing a logical, simple method of computing trajectories, and for his energy in initi-
ating and pushing through certain experimental projects, he deserves great commendation,”
the physicist Gordon Hull said of Moulton [Hull, 1920, 225].

Making allowances for the differences in kind, the story of the dispersion of numerical
integration can be viewed in somewhat the same way as that of the differential variations.
The isolated pre-War work of such varied individuals as the marine engineer Durand, the
physicist Webster, and theballistician Hamilton in effecting numerical solutions to differ-
ential equations was not a result of exposure to long-familiar techniques of settled accuracy.
With and after the war came not only new approaches and consolidation of existing ones
but the beginnings of the theoretical considerations of convergence, wider studies of error
analysis, a deeper appreciation of the comparative value of each method, and a specific

³⁵ This claim is discussed from a general point of view in Siegmund-Schultze [2003, 119 ff].
³⁶ The same editorial bids the reader not to forget that the “groundwork for American ballistic
practice was laid by Artillery officers, notably Ingalls and Hamilton,” whose work took place “in the
face of a discouraging paucity of means of experiment and practical research” [Anonymous, 1919].
connection with approximation theory. The basics of the subject were related for general use in academic textbooks and military texts, which devoted chapters to numerical integration without hinting at these deeper issues. The analogy with the treatment of differential variations is evident: the expositions of Hoar [1921], Jackson [1921], Bliss [1919], some of Moulton [1919b, 1922], and Hitchcock [1934] give the mechanics of differential variations without discussing their relation to deeper mathematical topics.

The work of the second wave workers was at best seen as providing a logical, solid foundation for the accounts of variability, a “theoretically correct foundation,” as Philip Schwartz said in an article that described experiments to test the wind corrections devised by Moulton and Bliss [Schwartz, 1924, 93]. Moulton also initiated many scientific investigations, such as using a wind tunnel to explore the effect of the shape of a projectile on its trajectory. This was not an isolated phenomenon: the rise of the scientific laboratory occurred not only at the Aberdeen Proving Grounds but throughout America in the first decades of the 20th century. Thornton Fry’s mathematical division at Bell Telephone laboratories, Stephen Timoshenko’s work on stress analysis at Westinghouse, Viktor K. La Mer’s Dupont-funded work on electrolytes, and the use of Saint-Venant’s elasticity theory in the steel industry all took place during this era, as mathematics of a more sophisticated nature was brought into industrial settings. In this respect the war efforts were part of a national trend.

We have chosen to focus on differential variations in part because they involved some advanced mathematical tools of the day, but also because they found theoretical and practical uses in many contexts by the second wave reformers. These uses highlight their concern with solving the problems of trajectory prediction while also seeing other applications for the variations in higher mathematics. The Moulton/Bliss differential variations were used to calculate effects of many factors, and these results appeared in the vastly expanded range-firing tables for contemporary guns. They were used in Hoar’s text to correct the results of actual firing for conditions that varied from standard. Moulton thought they could be used to adjust the general ballistic tables for any experimentally determined change in the resistance law, thus overcoming the obstacle of dependence on a given, possibly outdated law. He showed how they could serve another technical purpose, namely providing equal argument spacing, in those tables. They appear in Bliss’s investigations in two ways: they form a part of his general investigations of the differential of a function of a line, and are a part of his research into the variation of solutions of a differential equation as a function of its parameters [Bliss, 1920a,b]. They provided Moulton with a general variational technique, as explained in his text on differential equations.

The range-firing table has been chosen as the central material object of this paper for the obvious reason that it was the focus of the mathematical activity described; the computations for it were done by the new numerical methods and the differential variations accounted for the increased size. But it also highlights other aspects of our story, especially the economic ones. The technological progress that America underwent in the period considered, resulting in a larger variety of guns and ammunition, required more tables. The first wave of ballistic reform was hampered by small governmental appropriations in their efforts to test their theories, and thus the tables, but the massive investments of the World War enabled funding of the second wave work. Bliss noted that the accuracy of the firing, helped by more accurate tables, was important for economic reasons by making a pointed comparison of the amount saved to the salary of a contemporary technical worker. The expense of firing the new weapons was partly responsible for Moulton’s vision to conduct firing “by theory alone,” putting a great responsibility on the tables. Any mathematical
method, such as Gronwall’s reduction of the Bliss adjoint system, which eased the work of the computers, was hailed.

It is clear that table construction was an ongoing process. The production of the tables evolved in the larger picture from results achieved by firing alone prior to the first wave to the introduction of ballistic theories from abroad, particularly the Siacci theory, through the 60-page booklet produced by Moulton’s group. The general ballistic tables, a mathematical product in which any identification with a specific gun was absent, were begun after the World War. Many techniques, including the heavily criticized Siacci method, continued to be used alongside the newer approaches. Hoar’s textbook shows a method that used a combination of first and second wave ideas. Moulton’s numerical integration techniques were supplemented by those of Bennett, Arnaud, and Milne almost as soon as they were introduced; his computations of the differential variations were quickly enhanced by Milne and Bliss. The weighting factor curves used in the Siacci theory appeared in the JUSA/CAJ nearly simultaneously with descriptions of the newer ones [Alger, 1919]. In short, in table construction, the New Ballistics did not render previous work obsolete, and was itself subject to quick revisions.

The JUSA/CAJ is an ideal vehicle through which to view the changes we describe. The first published exposition of Moulton’s new method for numerical integration appeared there in 1919, and the New Ballistics was proclaimed and laid out in its pages. Itself established to promote, debate, and distribute information among a “progressive” group of officers in 1892, its editors had mixed feelings about the work of these men:

The editors of the Journal confess that the contents of some of these articles [the New Ballistics series] appear rather formidable to them ... but at the same time it is realized that there are readers who will distinctly relish these contributions ... it is believed that these articles will break the ground for a more general delving of the ballistic field by Coast Artillery officers ... the Journal suggests that, based on the thorough analytical discussion of this series, there is a demand for a comprehensive presentation of up-to-date ideas on ballistics and gunnery.... [Anonymous, 1919]

A consideration of the post-Armistice ballistics literature as presented in this paper leads to the conclusion that the years following the World War can best be described as a period of consolidation for the theories considered here. In these years work continued on all aspects of ballistic problems. The numerical techniques introduced by Moulton and his followers won a degree of international recognition by their inclusion, along with the methods of the French and English, in Charbonnier’s two-volume treatise [1927, 732–737]. Thus the “progressive” desire that the American military resemble those of foreign countries was achieved in miniature in these pages.

Forest Ray Moulton, our central figure, was responsible for even more ballistic-related work than we have mentioned: he supervised experiments to determine the position of shell bursts at night and experiments on the temperature of gun muzzles, read papers by Bennett on statistical treatment of dispersion of shots, and coordinated information exchanges with the French and English ballisticians. “Rapid progress has been made ... in the applications of mathematics to ballistic problems. The field is not exhausted; in fact, there is no reason to believe it is more than well begun,” he wrote near the end of his History [Moulton, 1919a, 89]. After leaving the University of Chicago in 1926 he was for the following decade the financial director of the Utilities Power and Light Corporation of Chicago. He did much

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37 A picture of developments can be found in Ballisticians in War and Peace [1957].
public lecturing on scientific topics during this time. From 1927 to 1946 he was the secretary of the American Association for the Advancement of Science, a position he likely achieved in part as a result of his efforts to communicate science to larger audiences, sometimes in radio broadcasts. He died in 1952. We hope that this paper has provided some insight into his key role in the formation and propagation of a new American vision of the science of ballistics, providing a stage for the exploration of the work of artillerymen and mathematicians in this era.

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Archival materials (abbreviations)

[NARA Aberdeen] Record Group 156–866, National Archives and Records Administration, College Park, Maryland.
[University of Chicago] Special Collections Research Center, University of Chicago Library, Chicago, Illinois.

References

Anonymous, 1892. A course of instruction in Ordnance and Gunnery, Henry Metcalf (Book Review), JUSA 1, pp. 74–76.
Bennett, A.A., 1921. Introduction to Ballistics. Technical Staff of the Ordnance Department, United States Army, Engineer Reproduction Plant, Washington, Barracks, DC.
Charbonnier, P., 1907. The Gavre commission: an historical sketch of the progress of exterior ballistics. JUSA 27, 238–262.
Craig, T., 1893. The applications of elliptic functions by Alfred G. Greenhill, JUSA 2, 298 (Review).


Moulton, F.R., 1919a. History of the Ballistics Branch of the Artillery Ammunition Section. Engineering Division of the Ordnance Department for the Period April 6, 1918 to April 2, 1919. War Department, Office of the Chief of Ordnance, Washington, DC.


Provisional Range Tables for the British 75-mm. Gun, Model 1917, 1918. Ordnance Department, U.S. Army, War Plans Division, Government Printing Office, Washington, DC.

Ruckman, J.W., 1892. The effect of wind on the motion of a projectile. JUSA 1, 5–24.
Schwartz, P., 1924. Experimental investigation of the effect of the wind upon the motion of a projectile. CAJ 60, 93–97.
Very, E.W., 1877. Computation of Range Tables. Annapolis, Md.