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Enumeration of generalized Young tableaux with bounded height

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Abstract

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We consider Young tableaux strictly increasing in rows, weakly increasing in columns. We show that the number of such tableaux with entries between 1 and n, with p columns having an odd number of elements and having at most $2k$ rows, is the product

$$\frac{\binom{n}{p}\binom{2k+p-1}{p}}{\binom{n+2k+p}{p}} \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{2k+i+j}{i+j}.$$

The proof is mainly bijective, using configurations of noncrossing paths.

1. Introduction

A *generalized Young tableau* of shape $(p_1, p_2, \ldots, p_m)$ is an array of $p_1 + p_2 + \cdots + p_m$ positive integers into m left-justified rows, with p_i elements in row i, and $p_1 \leqslant p_2 \leqslant \cdots \leqslant p_m$; the numbers in each column are in nondecreasing order from bottom to top, and the numbers in each row are in strictly increasing order from left to right. We consider generalized Young tableaux with entries from the set $\{1, 2, \ldots, n\}$. Figure 1 shows a generalized Young tableau whose shape is $(1, 3, 3, 5)$.

Gordon [6] proved that the number $a_{n,m}$ of such tableaux having at most m rows is given by the product

$$a_{n,m} = \prod_{1 \leqslant i \leqslant j \leqslant n} \frac{m+i+j-1}{i+j-1}. \tag{1}$$

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4				
2	3	5		
1	2	4		
1	2	3	4	5

Fig. 1.

Desainte-Catherine and Viennot [3] proved a companion formula for the number $b_{n,2k}$ of generalized Young tableaux with columns having only an even number of elements and bounded by height $q=2k$:

$$b_{n,2k}=\prod_{1\leqslant i\leqslant j\leqslant n}\frac{2k+i+j}{i+j}. \tag{2}$$

We prove here a second similar formula for the number $c_{n,2k,p}$ of generalized Young tableaux having p columns with an odd number of elements and bounded by height $q=2k$:

$$c_{n,2k,p}=\frac{\binom{n}{p}\binom{2k+p-1}{p}}{\binom{n+2k+p}{p}}\prod_{1\leqslant i\leqslant j\leqslant n}\frac{2k+i+j}{i+j}. \tag{3}$$

Our motivation, as with Desainte-Catherine–Viennot's, comes from the question of Stanley [11] about finding a "natural" proof of (1). In fact, the right-hand side of (1) is obtained by setting $q=1$ in the product giving the generating function of the corresponding *plane partitions*, proved by Gordon [6], and conjectured by Bender and Knuth [1].

The main purpose of this paper is to show that Desainte-Catherine–Viennot's methodology for proving (2) can be refined in order to take into account the number of odd-height columns (a classical parameter of Young tableaux connected with fixed points of involutions). Thus, we obtain $a_{n,2k}$ by summing $c_{n,2k,p}$ over all p: $a_{n,2k}=\sum_{p=0}^{n}c_{n,2k,p}$.

2. Bijection between configurations of Dyck paths and generalized Young tableaux

Let $M=(M_{i,j})_{1\leqslant i,j\leqslant n}$ be a symmetric matrix of nonnegative integers. A *decreasing subsequence* extracted from M is a sequence $\sigma=(M_{i_1,j_1},M_{i_2,j_2},\ldots,M_{i_q,j_q})$ of elements of M such that, for each r, $1\leqslant r\leqslant q-1$, $M_{i_{r+1},j_{r+1}}$ is located at the South–West of M_{i_r,j_r} (i.e. either $i_{r+1}>i_r$ and $j_{r+1}\leqslant j_r$ or $i_{r+1}=i_r$ and $j_{r+1}<j_r$). The *value* of a decreasing subsequence is the sum of its elements $M_{i_1,j_1}+M_{i_2,j_2}+\cdots+M_{i_q,j_q}$. The *depth* of a matrix M is the maximum value of its decreasing subsequence.

We can associate in an obvious way [7, 2] a symmetric matrix M of depth $2k$ with a *generalized involution*:

$$\begin{pmatrix} u_1 & u_2 & \dots & u_m \\ v_1 & v_2 & \dots & v_m \end{pmatrix}, \tag{4}$$

where there are exactly $M_{i,j}$ occurrences of the pair (i, j) and where $u_k \leqslant u_{k+1}$ and when $u_k = u_{k+1}$ then $v_k \geqslant v_{k+1}$.

We can easily verify that the length of the longest nonincreasing subsequence of $v_1, v_2, \dots, v_m$ is $2k$. If we consider the value of the first element of all the nonincreasing subsequences of $v_1, v_2, \dots, v_m$ having length $2k$, we define the *root of the involution* as the smallest of these values.

We denote by $T_{n,h,p}$ the set of generalized Young tableaux of height h having exactly p odd-height columns and with entries in $\{1, 2, \dots, n\}$. $T_{n,h}$ is the subset of $T_{n,h,p}$ whose tableaux have only columns with an even number of elements.

Theorem 2.1 (Knuth [7] and Burge [2]). *The set $T_{n,2k,p}$ of generalized Young tableaux is in bijection with the set of $[n] \times [n]$ symmetric matrices of nonnegative integers having depth $2k$ and p odd elements on the main diagonal. Moreover, if $P \in T_{n,2k,p}$ corresponds to the symmetric matrix M and if the $2k$th element of the first column of P contains the value x, then the involution corresponding to M has a root equal to x.*

2.1. *The insertion and deletion algorithms*

The Knuth–Robinson–Schensted insertion procedure [7, 9, 10], the so-called bumping process, is usually applied to rows of a tableau (i.e. an element is extracted from the current row and displaced into the upper row). We consider a version of the algorithm where the bumping process is applied to columns and not to rows. Under some conditions this version of the algorithm will insert a supplementary element in a tableau without increasing its height.

This new algorithm is called "INSERT-1".

In the sequel we number rows from bottom to top and columns from left to right. Let δT be the tableau obtained from T by deleting the first column. The recursive insertion algorithm is as follows:

INSERT-1(T, a)
- *If* a is greater or equal to all the elements of the first column *then* add a at the top of the first column and stop
- *else* let x be the first element of the first column such that $x > a$ when we examine this column from bottom to top. Put a in place of x and perform INSERT-1$(\delta T, x)$.

Theorem 2.2. *Assume that we insert x' immediately after inserting x. Assume also that the insertion algorithm $INSERT$-1(T, x)* (resp. *$INSERT$-1$(INSERT$-1$(T, x), x')$*) *defines* *a position* (l, m) (resp. (l', m')) *at which the original tableau* T (resp. $INSERT\text{-}1(T, x)$) *has been extended. Then* $x > x'$ *if and only if* $l \geqslant l'$ *and* $m < m'$.

Proof. A symmetric proof can be found in [7] for the equivalent theorem in the classical case. □

The algorithm INSERT-1 has an inverse called "EXTRACT-1". We can delete from the tableau T the element placed in position (l, m) (l is the height of the column m). EXTRACT-1 produces the resulting tableau T' and the deleted value a.

EXTRACT-1$(T,(l, m))$
- $a \leftarrow$ the value placed in position (l, m)
- we delete from T the square in (l, m)
- *For i varying from* $m-1$ *down to* 1 *by step of* -1
- *Begin*
- Let (l', i) be the position of the topmost square in the ith column which contains a value x such that $x < a$
- place the value a in position (l', i)
- $a \leftarrow x$
- *End*
- T' is the resulting tableau and a is the deleted value

2.2. *Bijection between generalized Young tableaux and generalized fans*

A *Dyck path* of length $2n$ is a sequence $(s_0, s_1, \ldots, s_{2n})$ of points in $N \times N$ such that $s_0 = (0, 0)$, $s_{2n} = (2n, 0)$ and, for $0 \leqslant i \leqslant 2n-1$, if $s_i = (x, y)$ then $s_{i+1} = (x+1, y+1)$ (North–East step) or $s_{i+1} = (x+1, y-1)$ (South–East step).

When the end of a path is not on the horizontal axis but is at the point (a, b), we have then a *left factor* of a Dyck path of length a ending at height b.

For a *left factor* of a Dyck path of length $2n+1$ ending at $(2n+1, 2p+1)$, we define the *terminal point* as the first point belonging to the line with equation $y = -x + 2n + 2$, when we move along the increasing abscissae. If the coordinates of the end point are $(2n+2-h, h)$, we deduce that $p < h \leqslant n+1$.

A *fan* of k paths of length l is a k-tuple $(C_1, C_2, \ldots, C_k)$ of left factors of Dyck paths satisfying the following two conditions:

(1) each path starts from $(0, 0)$,

(2) for every j, $1 \leqslant j \leqslant k-1$, C_j is *under* C_{j+1}, that is, for every i, $0 \leqslant i \leqslant l$, the ordinate of the ith vertex of C_j is less than or equal to the ordinate of the ith vertex of C_{j+1}.

A *generalized fan* of k paths of length $2n+1$ and of *deviation* p is a fan of k paths of length $2n+1$, where the $k-1$ first paths go from $(0, 0)$ to $(2n+1, 1)$ and where the kth path goes from $(0, 0)$ to $(2n+1, 2p+1)$ (see Fig. 2).

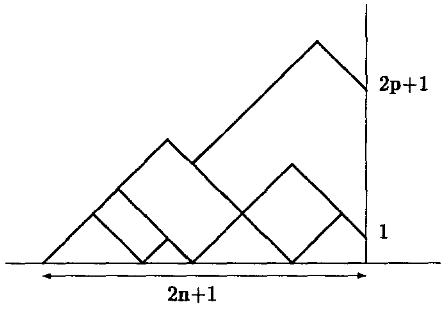

Fig. 2.

Lemma 2.3. *For* $0 \leqslant p < h \leqslant n+1$, *there exists a bijection between the set of generalized fans of* k *paths of length* $2n+1$ *and of deviation* p, *the* kth *path having* $(2n+2-h,\ h)$ *as terminal point, and the set of pairs* (E, R), *where*

- E *is a fan of* k *paths of length* $2n+1$ *and of deviation* 0, *where the* kth *path terminates with a North–East step coming before* $h-1$ *South–East steps,*
- R *is a p-element subset of* $\{1, 2, \ldots, h-1\}$.

Proof. The kth path of the fan splits into two parts: a path from (0,0) to $(2n+2-h,\ h)$ and a path from $(2n+2-h,\ h)$ to $(2n+1,\ 2p+1)$.

By adding the $h-1$ South–East steps to the first path, we obtain E. The other path contains p North–East steps and $h-p-1$ South–East steps. Thus, it can be associated to a p-element subset R of $\{1, 2, \ldots, h-1\}$.

It is obvious that this correspondence is a bijection (see Fig. 3). □

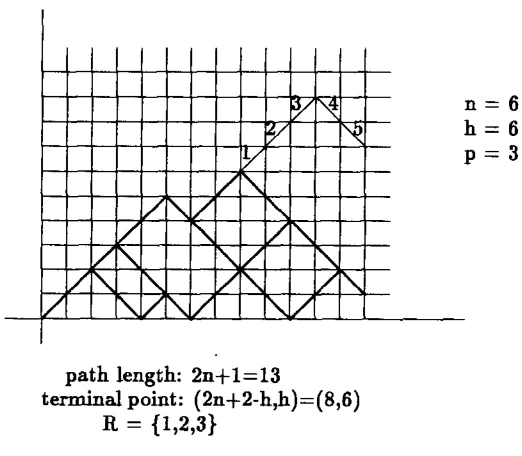

Fig. 3.

For a left factor of Dyck path, a *peak* (*valley*) is a point s_j such that (s_{j-1}, s_j) is a North–East (South–East) step and (s_j, s_{j+1}) is a South–East (North–East) step. A generalized fan of k paths and of length $2n+1$ is *exact* if the first $(n+1)$ steps of the kth path contain at least one peak.

Lemma 2.4. *There exists a bijection between the set of exact generalized fans of* k *paths of length* $(2n+1)$ *and of deviation* 0, *where the* kth *path terminates with one North–East step followed by* $(h-1)$ *South–East steps and the set of* $[n]\times[n]$ *symmetric matrices of depth* $2k$ *with no odd element on the main diagonal and whose root of the corresponding involution is equal to* h.

Proof. The matrix can be built with the following algorithm:

- *Begin*
- Set the matrix M to 0
- *For p varying from* 1 *to k*
-     *Begin*
-         *For* every valley c of the pth path
-             *Begin*
-                 Let (x, y) be the coordinates of c
-                     $i \leftarrow n+1-(x-y)/2$
-                     $j \leftarrow n+1-(x+y)/2$
-                     $M_{i,j} \leftarrow M_{i,j}+1$
-                     $M_{j,i} \leftarrow M_{j,i}+1$
-             *End*
-     *End*
- *End*

We can easily verify that a decreasing subsequence extracted from M corresponds to a path using only North–East and North–West steps on the grid where the fan is drawn. The value of the decreasing subsequence is twice as big as the number of valleys encountered by this path.

If we consider two noncrossing left factors of length $2n+1$, for every valley c on the upper path there exists a valley c' of the lower path such that c' can be connected to c by a path using only North–East and North–West steps. By iterating this remark, we easily verify that the depth of the matrix is $2k$ and that the corresponding involution has a root equal to h. $\square$

Remark. Let M be a $[n]\times[n]$ symmetric matrix of depth $2k$ with only even numbers on the diagonal. This matrix corresponds to a generalized Young tableau of height $2k$ having 0 odd-height columns and with entries in $1, 2, \ldots, n$ [7]. The number of such tableaux is $\prod_{1\leqslant i\leqslant j\leqslant n} 2k+i+j/i+j$ [3].

If we add one unit to p elements of the main diagonal of M, we obtain a matrix M' of depth at most $2k+1$. So, M' corresponds to a generalized Young tableau of height $2k$ or $2k+1$ having p odd-height columns and with entries in $1, 2, \ldots, n$.

Thus, the number of generalized Young tableaux of height at most $2k+1$ having p odd-height columns is $\binom{n}{p}\prod_{1\leqslant i\leqslant j\leqslant n}(2k+i+j)/(i+j)$.

This result was also proved by Désarménien [4], Stembridge [12] and Proctor [8].

Theorem 2.5. *There exists a bijection between $T_{n,2k,p}$ and the set of exact generalized fans of k paths, of length $2n+1$ and of deviation p.*

Proof. Let F be an exact generalized fan of k paths, of length $2n+1$ and of deviation p. It can be interpreted, using Lemma 2.3, as a couple (E, R), where

- E is an exact fan of k paths of length $2n+1$ and of deviation 0, where the kth path terminates with a North–East step followed by $h-1$ South–East steps,
- R is a p-element subset of $\{1, 2, \ldots, h-1\}$.

But E can also be interpreted, using Lemma 2.4, as an $[n]\times[n]$ symmetric matrix M of depth $2k$ with no odd element on the main diagonal. Moreover, the corresponding involution has a root equal to h.

To M, by Theorem 2.1, corresponds a generalized Young tableau T of height $2k$ whose columns have only an even number of elements. Moreover, the value of the $2k$th square of the first column of T is h.

If we insert in T the p elements of R in decreasing order using the INSERT-1 algorithm, Theorem 2.2 claims that the resulting generalized Young tableau has p odd-height columns and its height is equal to $2k$.

All the steps of this proof being bijective, the one-to-one correspondence is established. $\square$

3. Enumeration

In this section we enumerate generalized Young tableaux having at most $2k$ rows, with entries in $\{1, 2, \ldots, n\}$ and with p columns with an odd number of elements. In the previous section we have studied exact fans and tableaux with exactly $2k$ rows. Thus, we have to examine here exact or nonexact fans because no peak in the first $(n+1)$ steps of the upper path means that the height of the corresponding tableau will be less than $2k$.

3.1. Determinant of ballot numbers

We consider the infinite matrix A built with ballot numbers (or Delannoy numbers):

$$A=(a_{i,j})_{i\geqslant 0,\, j\geqslant 0},$$

where if i and j have same parity and if $i \geqslant j$, then $a_{i,j} = (j+1)i!/((i+j)/2+1)!((i-j)/2)!$; else $a_{i,j} = 0$.

We denote by

$$A\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix}$$

the determinant corresponding to the minor of A, where $0 \leqslant \alpha_1 < \alpha_2 < \cdots < \alpha_k$ are the row indices and $0 \leqslant \beta_1 < \beta_2 < \cdots < \beta_k$ are the column indices.

Element (i, j) of the matrix A is either the number of Dyck left factors going from the point $(\alpha_k - \alpha_i, 0)$ to the point (α_k, β_j) or the number of Dyck left factors going from the point $(0, 0)$ to the point (α_i, β_j).

An *open fan* of k noncrossing paths is a k-tuple $(C_1, C_2, \dots, C_k)$ of Dyck left factors that satisfy the following two conditions:

(1) for each pair of paths (C_i, C_j), $1 \leqslant i < j \leqslant k$, the paths C_i and C_j have no common vertices,

(2) for $1 \leqslant i \leqslant k$, C_i goes from $(a_i, 0)$ to (a, b_i), with $0 \leqslant a_k < a_{k-1} < \cdots < a_1 \leqslant a$ and $0 \leqslant b_1 < b_2 < \cdots < b_k$.

Using the Gessel–Viennot methodology [5], the determinant

$$A\begin{pmatrix} \alpha_1 & \alpha_2 & \dots & \alpha_k \\ \beta_1 & \beta_2 & \dots & \beta_k \end{pmatrix}$$

can be interpreted as the number of open fans of k noncrossing paths, the ith path, $1 \leqslant i \leqslant k$, connecting $(\alpha_k - \alpha_i, 0)$ to (α_k, β_j).

We denote by $D_{n,k}$ the determinant

$$A\begin{pmatrix} 2n+1 & 2n+3 & \dots & 2n+2k-1 \\ 1 & 3 & \dots & 2k-1 \end{pmatrix}.$$

$D_{n,k}$ enumerates fans of k noncrossing paths where the ith path, $1 \leqslant i \leqslant k$, connects $(2k-2i, 0)$ to $(2n+2k-1, 2i-1)$ (see Fig. 4). For any open fan in $D_{n,k}$, if we move the ith path, $1 \leqslant i \leqslant k$, by $2k-2$ steps to the West and the same path by $2i-2$ steps to the South and if we delete $2i-2$ North–East step from the beginning, we obtain a generalized fan of k paths of length $2n+1$ and ending at 0. Thus, $D_{n,k}$ also enumerates generalized fans of k paths of length $2n+1$ and of deviation 0.

We denote by $D_{n,k,p}$ the determinant

$$A\begin{pmatrix} 2n+1 & \dots & 2n+2k-3 & 2n+2k-1 \\ 1 & \dots & 2k-3 & 2k+2p-1 \end{pmatrix}.$$

Hence, $D_{n,k,p}$ enumerates open fans of k noncrossing paths where the ith path, $1 \leqslant i \leqslant k-1$, connects $(2k-2i, 0)$ to $(2n+2k-1,\ 2i-1)$ and where the kth path connects $(0, 0)$ to $(2n+2k-1,\ 2k+2p-1)$ (see Fig. 5). As above, $D_{n,k,p}$ also enumerates generalized fans of k paths of length $2n+1$ and of deviation p.

To compute the $D_{n,k,p}$, we use Hankel determinants of Catalan numbers and the Viennot combinatorial interpretation of the so-called qd-algorithm from Padé approximants theory [13].

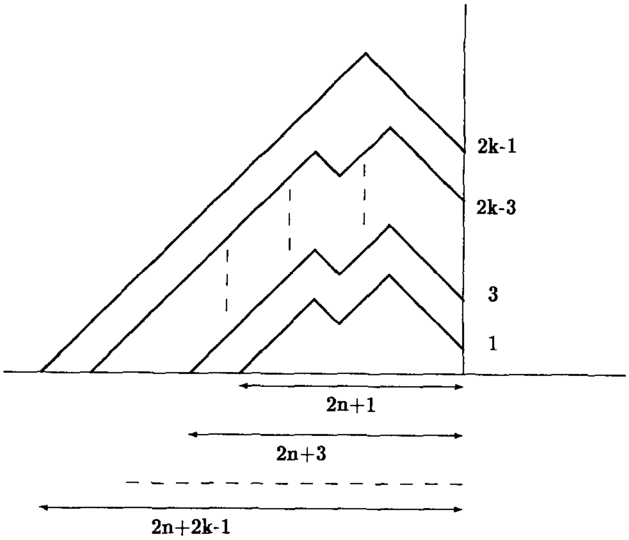

Fig. 4.

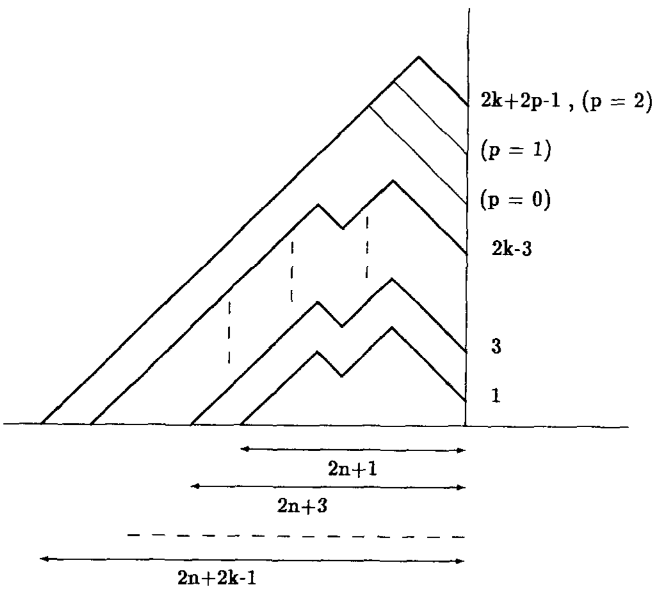

Fig. 5.

3.2. Hankel determinants and Catalan numbers

Let $\mu=\{\mu_n\}_{n\geqslant 0}$ be a sequence of real numbers. We consider the infinite matrix $H(\mu)=(\mu_{i+j})_{0\leqslant i,j}$. A Hankel determinant is any minor of $H(\mu)$. Such a determinant will be denoted by

$$H\begin{pmatrix}\alpha_1 & \alpha_2 & \dots & \alpha_k\\ \beta_1 & \beta_2 & \dots & \beta_k\end{pmatrix},$$

where $0\leqslant\alpha_1<\alpha_2<\cdots<\alpha_k$ are the row indices and $0\leqslant\beta_1<\beta_2<\cdots<\beta_k$ the column indices of the extracted minor.

We choose as sequence μ the Catalan numbers $C=\{C_n\}_{n\geqslant 0}$, where C_m is the mth Catalan number $(2m)!/m!(m+1)!$.

We denote by $H_{n,k,p}(C)$ the Hankel determinant extracted from $H(C)$:

$$H\begin{pmatrix}n+1 & n+2 & \dots & n+k-1 & n+k\\ 0 & 1 & \dots & k-2 & k-1+p\end{pmatrix}.$$

As in the previous section, we can interpret $H_{n,k,p}(C)$, using Gessel–Viennot methodology [5] as a fan of noncrossing paths. More precisely, $H_{n,k,p}(C)$ is equal to the number of k-tuples $(w_1, w_2, \dots, w_k)$ of Dyck left factors, being disjoint with each other (no common vertices), such that, for i, $1\leqslant i\leqslant k-1$, w_i connects $(-n-2i+1,0)$ to $(n+2i-1,0)$ and w_k connects $(-n-2k+1,0)$ to $(n+2k+2p-1,0)$. For $1\leqslant i\leqslant k-1$, the length of w_i is $2n+4i-2$ and the length of w_k is $2n+4k+2p-2$ (see Fig. 6).

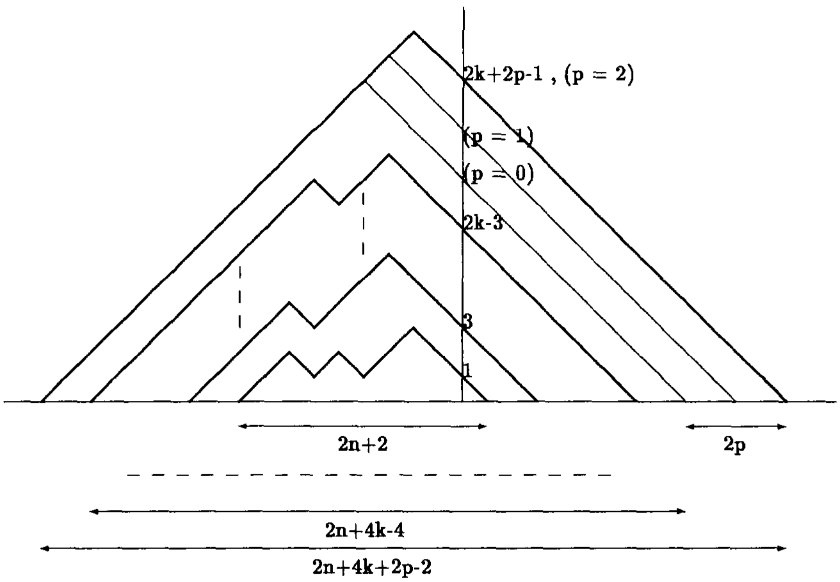

Fig. 6.

We denote $H_{n,k}(C)$ as the determinant $H_{n,k,0}(C)$. We can easily verify that $H_{n,k}(C)=D_{n,k}$.

To evaluate $H_{n,k,p}(C)$, we "compress", using the qd-algorithm methodology, the corresponding configurations into configurations easier to enumerate.

3.3. *qd-algorithm*

We give here an overview of the combinatorial interpretation of the qd-algorithm (for more details, see [13]).

Let $r=\{r_k\}_{k\geqslant 1}$ be a sequence of real numbers. This sequence allows us to valuate elementary steps of any Dyck path $w=(s_0,s_1,\ldots,s_{2n})$ in the following way:

- a South–East step connecting level k at level $k-1$, $1\leqslant k\leqslant n$, is weighted by r_k,
- a North–East step is weighted by 1.

The weight (or valuation) $V(w)$ of the path w is the product of the valuations of the elementary steps $\prod_{0\leqslant i\leqslant 2n-1} V(s_i,s_{i+1})$. We introduce the generating function of weighted Dyck paths (according to the valuation r) $s(t;r)=\sum_{n\geqslant 0} b_n t^n$, where b_n is equal to $\sum_{|w|=2n} V(w)$.

The sequence $r'=\{r'_k\}_{k\geqslant 1}$ is the qd-transform of the sequence r (denoted by $r'=\mathrm{qd}(r)$) iff $s(t;r)=1+r_1ts(t;r')$.

We can compute r' by recurrence using the following relations ("rhombus rules"):

$$r_{2k+1}+r_{2k+2}=r'_{2k}+r'_{2k+1},$$

$$r_{2k}r_{2k+1}=r'_{2k-1}r'_{2k}.$$

Thus, we can compute the coefficient of t^n in the generating function $s(t;r)$ by adding all the valuations over all the Dyck paths of length $2n-2$ (according to valuation r') or by adding all the valuations over all the Dyck paths of length $2n$ (according to valuation r). The idea is to change scale. Using r' rather than r, we replace Dyck paths by shorter (and less numerous) Dyck paths. In this compression process the weight of the shorter paths is changed from r into r'.

The qd-algorithm is obtained by applying recursively the qd-transform to the sequence $r=\{r_k^{(0)}\}_{k\geqslant 1}$. For $n\geqslant 0$, we obtain $r^{(n+1)}$ from $r^{(n)}$ with the relation $r^{(n+1)}=\mathrm{qd}(r^{(n)})$, where $r^{(n)}$ denotes the sequence $\{r_k^{(n)}\}_{k\geqslant 1}$.

Note that $s(t;r)=1+\sum_{n\geqslant 1}(\prod_{0\leqslant j\leqslant n-1} r_1^{(j)})t^n$ and, hence, that $b_n=\prod_{0\leqslant j\leqslant n-1} r_1^{(j)}$. We can also verify that $s(t;r^{(n)})=\sum_{k\geqslant 0} b_{k+n}/b_n t^k$.

For our purpose, we use the sequence $\{1\}$ as sequence $\{r\}$ (all the elements are equal to 1). In this case, we state:

$$r_{2k}^{(n)}=\frac{2k(2k+1)}{(n+2k)(n+2k+1)} \quad \text{and} \quad r_{2k+1}^{(n)}=\frac{(2n+2k-1)(2n+2k)}{(n+2k-1)(n+2k)}.$$

We denote $C^{(n)}$ as the sequence $\{C_i^{(n)}\}_{i\geqslant 0}$ and $C_i^{(n)}$ as the number C_{i+n}.

Following [13], if we apply $n+1$ qd-transforms to the configurations of paths corresponding to $H_{n,k,p}(C)$, we obtain the relations

$$H_{n,k,p}(C)=H_{-1,k,p}(C^{(n+1)}), \tag{5}$$

$$H_{n+1,k,p}(C)=H_{0,k,p}(C^{(n+1)}), \tag{6}$$

and if we denote $\bar{C}^{(n)}$ as the sequence $\{\bar{C}_i^{(n)}\}_{i\geqslant 0}$, where $\bar{C}_i^{(n)}=C_{i+n}/C_n$, we also obtain

$$H_{n,k,p}(C)=(C_{n+1})^k H_{-1,k,p}(\bar{C}^{(n+1)}), \tag{7}$$

$$H_{n+1,k,p}(C)=(C_{n+1})^k H_{0,k,p}(\bar{C}^{(n+1)}). \tag{8}$$

$H_{-1,k,p}(\bar{C}^{(n+1)})$ can be interpreted as the number of k-tuples $(w_1,w_2,\ldots,w_k)$ of Dyck paths, being disjoint each other, such that, for $1\leqslant i\leqslant k-1$, w_i connects $(2-2i,0)$ to $(-2+2i,0)$ and w_k connects $(2-2k,0)$ to $(-2+2k+2p,0)$. They can be seen as compressed configurations with steps differently weighted (weighted by $r^{(n+1)}$) (see Figs. 7 and 8).

When p is equal to 0, there is a single configuration of paths for $H_{-1,k}(\bar{C}^{(n+1)})$. The valuation of this configuration has been computed by Desainte-Catherine and Viennot [3]:

$$\begin{aligned} D_{n,k}=H_{n,k}(C)=(C_{n+1})^k H_{-1,k}(\bar{C}^{(n+1)})&=(C_{n+1})^k(r_1^{(n+1)}r_2^{(n+1)})^{k-1}\cdots(r_{2k-3}^{(n+1)}r_{2k-2}^{(n+1)})\\ &=\prod_{1\leqslant i\leqslant j\leqslant n}\frac{2k+i+j}{i+j}. \end{aligned} \tag{9}$$

But, in the general case where $p\neq 0$, there are several configurations for $H_{-1,k,p}(\bar{C}^{(n+1)})$.

Theorem 3.1. *$H_{n,k,p}(C)$ is equal to $D_{n,k}\prod_{i=0}^{p-1}(2k+2i)(2n+2k+2i+1)/(i+1)(n+2k+i+1)$.*

Proof. We denote $\alpha_{n,k,p}$ as the ratio $H_{n,k,p}(C)/H_{n,k}(C)$.

We have seen above that this ratio is nothing but $\alpha_{n,k,p}=H_{-1,k,p}(\bar{C}^{(n+1)})/H_{-1,k}(\bar{C}^{(n+1)})$.

Then $\alpha_{n,k,p}(\prod_{i=1}^{2k-2}r_i^{(n+1)})=\sum_w V(w)$, where the summation is over all the Dyck paths going from $(0,2k-2)$ to $(2k-2+2p,0)$ and where South–East steps from level k to level $k-1$ are valuated by $r_k^{(n+1)}$.

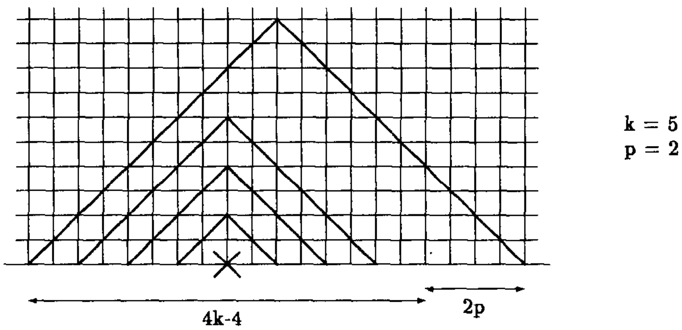

Fig. 7.

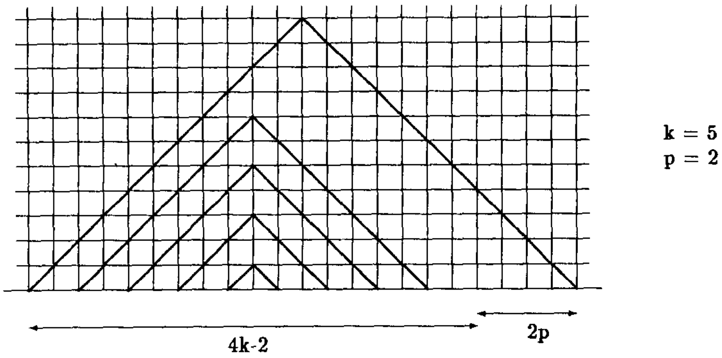

Fig. 8.

Using the same argument, we obtain that

$$\alpha_{n+1,k,p}\left(\prod_{i=1}^{2k-1} r_i^{(n+1)}\right)=\sum_w V(w)=\frac{H_{-1,k,p}(\bar{C}^{(n+2)})}{H_{-1,k}(\bar{C}^{(n+2)})}=\frac{H_{0,k,p}(\bar{C}^{(n+1)})}{H_{0,k}(\bar{C}^{(n+1)})},$$

where the summation is over all the Dyck paths going from $(0,2k-1)$ to $(2k-1+2p,0)$ and where South–East steps from level k to level $k-1$ are valuated by $r_k^{(n+1)}$.

To compute the valuation of a Dyck path of length $2k+2p-2$ connecting $(0,2k-2)$ to $(2k-2+2p,0)$, we consider the first step of this path:

(1) either the path begins with one North–East step valuated by 1, followed by a path of length $2k+2p-3$ going from $(1,2k-1)$ to $(2k-2+2p,0)$,

(2) or the path begins with one South–East step valuated by $r_{2k-2}^{(n+1)}$, followed by a path of length $2k+2p-3$ going from $(1,2k-3)$ to $(2k-2+2p,0)$.

We assume, by induction on l, that the total valuation of the Dyck paths going from $(0,2h+1)$ to $(2l+1,0)$, a South–East step from level k to level $k-1$ being valuated by $r_k^{(n+1)}$, is equal to

$$\prod_{i=0}^{l-h-1}\frac{(2h+2i+2)(2n+2h+2i+5)}{(i+1)(n+2h+i+4)}\prod_{j=1}^{2h+1} r_j^{(n+1)}.$$

For the first class of paths, the total valuation is

$$\prod_{i=0}^{p-2}\frac{(2k+2i)(2n+2k+2i+3)}{(i+1)(n+2k+i+2)}\left(\prod_{j=1}^{2k-1} r_j^{(n+1)}\right)$$

and for the second class

$$r_{2k-2}^{(n+1)}\prod_{i=0}^{p-1}\frac{(2k+2i-2)(2n+2k+2i+1)}{(i+1)(n+2k+i)}\left(\prod_{j=1}^{2k-3} r_j^{(n+1)}\right).$$

So we state that the valuation is

$$\prod_{i=0}^{p-2}\frac{(2k+2i)(2n+2k+2i+3)}{(i+1)(n+2k+i+2)}\left(\prod_{j=1}^{2k-1} r_j^{(n+1)}\right)$$

$$+r_{2k-2}^{(n+1)}\prod_{i=0}^{p-1}\frac{(2k+2i-2)(2n+2k+2i+1)}{(i+1)(n+2k+i)}\left(\prod_{j=1}^{2k-3} r_j^{(n+1)}\right)$$

$$=\prod_{j=1}^{2k-2} r_j^{(n+1)}\left(\frac{(2n+2k+1)(2n+2k+2)}{(n+2k)(n+2k+1)}\prod_{i=0}^{p-2}\frac{(2k+2i)(2n+2k+2i+3)}{(i+1)(n+2k+i+2)}\right.$$

$$\left.+\prod_{i=0}^{p-1}\frac{(2k-2+2i)(2n+2k+2i+1)}{(i+1)(n+2k+i)}\right)$$

$$=\prod_{j=1}^{2k-2} r_j^{(n+1)}\prod_{i=0}^{p-1}\frac{(2k+2i)(2n+2k+2i+1)}{(i+1)(n+2k+i+1)}.$$

Hence, this valuation is also equal to $\alpha_{n,p,k}\prod_{j=1}^{2k-2} r_j^{(n+1)}$.

For odd lengths, we have the same proof, but the inductive hypothesis becomes as follows: the total valuation of the Dyck paths going from $(0,2h+2)$ to $(2l+1,0)$, where South-East steps from level k to level $k-1$ are valuated by $r_k^{(n+1)}$, is equal to

$$\prod_{i=0}^{l-h-1}\frac{(2h+2i+4)(2n+2h+2i+5)}{(i+1)(n+2h+i+5)}\prod_{j=1}^{2h+2} r_j^{(n+1)}.$$

We next obtain

$$H_{n,k,p}(C)=H_{n,k}(C)\,\alpha_{n,p,k}=D_{n,k}\alpha_{n,p,k}. \qquad \square$$

Theorem 3.2. *$D_{n,k,p}$ is equal to $D_{n,k}\prod_{i=0}^{p-1}(2k+i)(n-i)/(i+1)(n+2k+i+1)$.*

Proof. If we remark that a configuration corresponding to $H_{n,k,p}(C)$ is a pair composed of a configuration enumerated by $D_{n,k,l}$, $0\leqslant l\leqslant p$, and of a Dyck left factor of length $2k+2p-1$ going from $(0,0)$ to $(2k+2p-1,\, 2k+2l-1)$, we obtain the relation

$$H_{n,k,p}(C)=\sum_{l=0}^{p} a_{2k+2p-1,2k+2l-1}D_{n,k,l}.$$

With the value of $H_{n,k,p}(C)$ and with the value of $D_{n,k,l}$ given by inductive hypothesis, for $0\leqslant l\leqslant p$, we can compute $D_{n,k,p}$ using the relation

$$D_{n,k,p}=H_{n,k,p}(C)-\sum_{l=0}^{p-1}\frac{(2k+2l)(2k+2p-1)!}{(2k+p+l)!(p-l)}D_{n,k,l}.$$

If we assume that

$$D_{n,k,l}=D_{n,k}\prod_{i=0}^{l-1}\frac{(2k+i)(n-i)}{(i+1)(n+2k+i+1)},$$

we need to verify the identity

$$\prod_{i=0}^{p-1} \frac{(2k+i)(n-i)}{(i+1)(n+2k+i+1)} = \prod_{i=0}^{p-1} \frac{(2k+2i)(2n+2k+2i+1)}{(i+1)(n+2k+i+1)} - \sum_{l=0}^{p-1} \frac{(2k+2l)(2k+2p-1)!}{(2k+p+l)!(p-l)!} \prod_{i=0}^{l-1} \frac{(2k+i)(n-i)}{(i+1)(n+2k+i+1)}. \tag{10}$$

Verification of identity (10) has easily been performed with the aid of the symbolic manipulation system Maple. □

With the five previous theorems, we can establish the announced result.

Theorem 3.3.

$$D_{n,k,p} = c_{n,2k,p} = \frac{\binom{n}{p}\binom{2k+p-1}{p}}{\binom{n+2k+p}{p}} \prod_{1 \le i \le j \le n} \frac{2k+i+j}{i+j}.$$

Remark. C. Krattenthaler (Universität Wien) pointed out that determinant $D_{n,k,p}$ can be computed by a more direct methodology using Vandermonde determinants (private communication).

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