Enumeration of labelled quasi-initially connected digraphs

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Abstract

A new class of connected digraphs is introduced — the class of quasi-initially connected digraphs. They are enumerated in the labelled case. Using the apparatus developed for labelled quasi-initially connected digraphs respective results for other classes of labelled connected digraphs were obtained. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

In the literature, for example, in [1], several classes of connected digraphs are defined. A digraph is weakly connected if ignoring all orientations of its arcs we get a connected multigraph. A digraph is strongly connected if every two of its points are mutually reachable (point v is reachable from point u if there exists a directed path from point u to point v). A digraph is unilaterally connected if for any two of its points at least one is reachable from the other. In an initially connected digraph there is a point (at least one) called a source, from which every point is reachable. Similarly, a digraph is finally connected if there exists a point (at least one) called a sink which is reachable from every point. The class of initially–finally connected digraphs is naturally defined, too, namely, as a class of digraphs with at least one source and one sink. In the literature the term ‘digraph with a source (sink)’ is also used instead of the term ‘initially (finally) connected digraph’, we can also say ‘a digraph with a source and a sink’ instead of ‘an initially–finally connected digraph’. Here we follow the notation of Liskovec (for example [6]).

In his list of unsolved graph enumeration problems Harary [1] mentioned the problems of enumerating the above-defined classes of connected digraphs with n points,
both for labelled and unlabelled digraphs. Note that Harary did not mention (we do not know why) certain results of Liskovec referring to these classes. Liskovec [3] counted labelled strongly connected digraphs, and his formulas were simplified by Wright [11]. The cases of unlabelled strongly connected digraphs and unlabelled rooted digraphs with a source were also solved by Liskovec [4–7]. In [8] Robinson gave a combinatorial proof of the formulas of Wright and counted labelled initially and unilaterally connected digraphs. In his research announcement Robinson [9] outlined a method of counting the class of all unlabelled strongly connected, initially and unilaterally connected digraphs with \( n \) points.

In [2,10] the problem of counting labelled initially–finally connected digraphs is solved.

Here we introduce the class of quasi-initially connected digraphs — the class of digraphs \( G = (V,E) \) having a point \( u \in V \) such that for every point \( v \in V \) it is satisfied that \( u \) is reachable from \( v \) or \( v \) is reachable from \( u \). This kind of connectedness is a new natural generalization of the notion of strong connectedness.

Let us denote by \( W, S, U, I, F, IF (=I \cap F) \) and \( QI \) the classes of labelled weakly, strongly, unilaterally, initially, finally, initially–finally and quasi-initially connected digraphs with \( n \) points. The diagram shows the interrelationship between these classes (here \( \rightarrow \) denotes the relation of inclusion \( \subseteq \). For \( n \geq 3 \) we can take \( \subset \) instead of \( \subseteq \)).

Here we enumerate labelled quasi-initially connected digraphs. Using the apparatus developed for these digraphs we give formulas for calculating the cardinalities of all the above-mentioned classes depending on the number of points \( n \). Liskovec’s result related to the class of labelled strongly connected digraphs was obtained as a special case. The formula for labelled weakly connected digraphs can be found, for example, in [1,3].

With the obtained recurrence formulas we calculated on the computer the cardinalities of the above-mentioned classes of labelled connected digraphs (and also with the fixed number of arcs) for \( n \leq 30 \). The corresponding tables for labelled quasi-initially connected digraphs are given at the end of the paper.

2. The main notions and auxiliary results

In what follows, all digraphs will be labelled digraphs, if not stated otherwise.
Let \( \mathcal{G}_0, \mathcal{G}_1, \ldots, \mathcal{G}_n \) be finite classes of digraphs. We say that \( \mathcal{G}_0 \) is of the type \( \{ \mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_n \} \) if:

1. every digraph \( G_0 \) from \( \mathcal{G}_0 \) can be represented in the form \( G_1 \cup G_2 \cup \cdots \cup G_n \), where \( G_i \in \mathcal{G}_i \) for each \( i = 1, \ldots, n \);
2. for each collection of digraphs \( G_i \in \mathcal{G}_i, i = 1, \ldots, n \), the digraph \( G_1 \cup G_2 \cup \cdots \cup G_n \) belongs to the class \( \mathcal{G}_0 \);
3. for two different collections of digraphs, \( G_i \in \mathcal{G}_i \) and \( G'_i \in \mathcal{G}_i \), \( i = 1, \ldots, n \), the digraphs \( G_1 \cup G_2 \cup \cdots \cup G_n \) and \( G'_1 \cup G'_2 \cup \cdots \cup G'_n \) are different.

(Here the operation \( \cup \) is defined in such a way that for every two digraphs \( G_1 = (V_1, E_1) \) and \( G_2 = (V_2, E_2) \) the digraph \( G_1 \cup G_2 \) is a digraph with the set of points \( V_1 \cup V_2 \) and the set of the arcs \( E_1 \cup E_2 \).) Then it is clear that

\[
|\mathcal{G}_0| = |\mathcal{G}_1| \cdot |\mathcal{G}_2| \cdot \cdots \cdot |\mathcal{G}_n|.
\]

Let us denote by \( O_n \) the empty digraph of the order \( n \) (without arcs and with \( n \) points). We denote by \( B_{m,n} \) the class of all bipartite digraphs whose parts contain \( m \) and \( n \) points, respectively, and whose arcs can only have beginning in the first and end in the second part. Let us designate by \( R \) the class of all digraphs with \( n \) points.

By \( \mathbb{N}_0 \) we mean the set of all non-negative integers.

Let \( G = (V, E) \) be a digraph, and \( U, W \subseteq V \). Denote by \( \text{Rch}(U, W; G) \) the set of all \( v \in W \) for which there exists at least one \( u \in U \) such that in the digraph \( G \) there exists a directed path from \( u \) to \( v \), i.e., point \( v \) is reachable from point \( u \) or, which is the same, point \( u \) is antireachable from point \( v \). Let us assume here that \( U \cap W \subseteq \text{Rch}(U, W; G) \) (the paths of the length 0 are also taken into account). Also, designate by \( \text{Rch}(U, W; G) \) the set of all points from \( W \) antireachable from some point from \( U \). Next, let \( V_1 \) be a subset of \( V \). Then, designate \( E(V_1, G) = (V_1 \times V_1) \cap E \). If it is clear from the context what graph is meant, then instead of \( E(V_1, G) \) we write \( E(V_1) \).

Let disjoint sets \( V_1 \) and \( V_2 \) be given, where \( |V_1| = m \) and \( |V_2| = n \); \( m \in \mathbb{N}, n \in \mathbb{N}_0 \). Let \( G = (V, E) \), where \( V = V_1 \cup V_2 \), be a digraph such that \( E(V_1) = \emptyset \), \( \text{Rch}(V_1, V_2; G) = V_2 \), and \( (V_2 \times V_1) \cap E = \emptyset \). We denote the set of all such digraphs by \( A(V_1; V_2) \); if it is clear from the context which sets \( V_1 \) and \( V_2 \) are meant, then instead of \( A(V_1; V_2) \) we use the notation \( A_m(n) \). It is clear that the number of elements in \( A(V_1; V_2) \) does not depend on the sets \( V_1 \) and \( V_2 \), but only on their cardinalities. We denote \( \lambda_m(n) = |A_m(n)| \).

**Lemma 1.** The recurrence formula

\[
\lambda_m(n) = 2^{m(n+m-1)} - \sum_{k=0}^{n-1} C_n^k 2^{(n-k)(n-1)} \lambda_m(k),
\]

with the initial condition \( \lambda_m(0) = 1 \), holds for every \( m \in \mathbb{N} \).
Proof. Let us fix sets $V_1$ and $V_2$, where $|V_1|=m$ and $|V_2|=n$, such that $V_1 \cap V_2=\emptyset$; $m \in \mathbb{N}, n \in \mathbb{N}$. Let $\tilde{A}_m(n)$ be the set of all digraphs $G=(V,E)$, where $V=V_1 \cup V_2$, such that $\mathbf{E}(V_1)=\emptyset$ and $(V_2 \times V_2 \cap E=\emptyset$. Denote $\hat{\lambda}_m(n)=|\tilde{A}_m(n)|$. Also, let $V_3(G)=\text{Rch}(V_1,V_2;G)$ for $G \in \tilde{A}_m(n)$. Denoted by $\tilde{A}_m(n,k)$, $0 \leq k \leq n-1$, the class of all $G \in \tilde{A}_m(n)$ such that $|V_3(G)|=k$, and let $\hat{\lambda}_m(n,k)=|\tilde{A}_m(n,k)|$. Then

$$\hat{\lambda}_m(n) = \hat{\lambda}_m(n) = \sum_{k=0}^{n-1} \hat{\lambda}_m(n,k).$$ \hspace{1cm} (1)

Let us show that

$$\hat{\lambda}_m(n,k) = C_n^k 2^{(n-k)(n-1)} \lambda_m(k).$$ \hspace{1cm} (2)

If $k=0$, we have that $\hat{\lambda}_m(n,k) = 2^{n(n-1)}$, and it is obvious that (2) holds. So, we can suppose that $k > 0$. Let us fix a set $V_3$, $V_3 \subseteq V_2$, such that $|V_3|=k$, $0 < k \leq n-1$. Let $\tilde{A}_0$ be the set of all digraphs $G \in \tilde{A}_m(n)$ such that $V_3(G)=V_3$, and let $\hat{\lambda}_0 = |\tilde{A}_0|$. It is clear that $\lambda_m(n,k) = C_n^k \lambda_0$. Let $G = (V,E)$ be a digraph from $\tilde{A}_0$. Put $E_1(G) = [(V_2 \setminus V_3) \times V_3] \cap E$. It is obvious that $[V_3 \setminus (V_2 \setminus V_3)] \cap E = \emptyset$. It is easy to see that the digraphs $G_1 = (V_1 \cup V_3,E(V_1 \cup V_3))$, $G_2 = (V_2 \setminus V_3,E(V_2 \setminus V_3))$ and $G_3 = (V_2,E_1(G))$ are from the classes $A_m(k), D_n-k$ and $B_{n-k,k}$, respectively. It is clear that $\lambda_0$ is of the type $\{A_m(k), D_{n-k}, B_{n-k,k}\}$. Then we have

$$\hat{\lambda}_m(n,k) = C_n^k \lambda_0 = C_n^k \hat{\lambda}_m(k) 2^{(n-k)(n-1)} 2^{(n-k)k} = C_n^k 2^{(n-k)(n-1)} \lambda_m(k).$$

Also, it is obvious that

$$\hat{\lambda}_m(n) = 2^{n(n-1)} 2^{mn} = 2^{n(m-1)},$$

therefore from (1) and (2) follows the statement of the lemma.

Let $V_1$, $V_2$ and $V_3$ be pairwise disjoint sets, where $|V_1|=k$, $|V_2|=m$ and $|V_3|=n$; $k \in \mathbb{N}, n \in \mathbb{N}, m \in \mathbb{N}$. Let $A_{V_1,V_3}(V_2)$ be a class of all digraphs $G=(V,E)$, where $V=V_1 \cup V_2 \cup V_3$, such that $E(V_1)=E(V_2)=\emptyset$, every point from the set $V_2$ is reachable from some point from the set $V_1$ and antireachable from some point from the set $V_3$, and $[(V_2 \cap V_3) \times V_3] \cap E = (V_3 \times V_2) \cap E = (V_1 \times V_3) \cap E = \emptyset$. If it is clear from the context what sets $V_1$, $V_2$ and $V_3$ are meant, then instead of the notation $A_{V_1,V_3}(V_2)$ we use the notation $A_{k,n}(m)$. It is clear that the number of elements of the set $A_{V_1,V_3}(V_2)$ depends only on the cardinalities of the sets $V_1$, $V_2$ and $V_3$. Therefore we denote $\hat{\lambda}_{k,n}(m) = |A_{k,n}(m)|$.

Lemma 2. The recurrence formula

$$\hat{\lambda}_{k,n}(m) = \hat{\lambda}_{k,n}(m) = \sum_{i=0}^{m-1} C_n^i \hat{\lambda}_{k+i}(m-i) \hat{\lambda}_{k,n}(i),$$

with the initial condition $\hat{\lambda}_{k,n}(0)=1$, holds for every $k,n \in \mathbb{N}$.

Proof. Let us fix pairwise disjoint finite sets $V_1$, $V_2$ and $V_3$, where $|V_1|=k$, $|V_2|=m$, and $|V_3|=n$ for some $k,m,n \in \mathbb{N}$. We denote by $\tilde{A}_{k,n}(m)$ the set of all digraphs
$G = (V,E)$, where $V = V_1 \cup V_2 \cup V_3$, such that $E(V_1) = E(V_2) = \emptyset$, $V_2 = \text{Rch}(V_1,V_2;G)$, and $[(V_2 \cup V_3) \times V_1] \cap E = (V_3 \times V_2) \cap E = (V_1 \times V_3) \cap E = \emptyset$. Also let $V'_2(G) = \overline{\text{Rch}}(V_3,V_2;G)$ for every $G \in \hat{A}_{k,n}(m)$. We denote by $\hat{A}_{k,n}(m,i)$, $0 \leq i \leq m - 1$, the class of all $G \in \hat{A}_{k,n}(m)$ such that $|V'_2(G)| = i$. It is clear that $|\hat{A}_{k,n}(m)| = j_k(m) 2^m$. Then

$$\lambda_k(m) = |\hat{A}_{k,n}(m)| - \sum_{i=0}^{m-1} |\hat{A}_{k,n}(m,i)| = j_k(m) 2^m - \sum_{i=0}^{m-1} |\hat{A}_{k,n}(m,i)|. \quad (3)$$

Let us calculate the number $|\hat{A}_{k,n}(m,i)|$. Let $V'_2$ be a subset of $V_2$ such that $|V'_2| = i$. Let $\hat{A}_1$ be the set of all digraphs $G \in \hat{A}_{k,n}(m)$ such that $V'_2(G) = V'_2$, and let $\hat{A}_1 = |\hat{A}_1|$. It is clear that $|\hat{A}_{k,n}(m,i)| = C_m^{\hat{A}_1} j_i(1)$, Let $G = (V,E)$ be a digraph from the set $\hat{A}_1$. It is clear that $[(V_2 \backslash V'_2) \times V'_2] \cap E = \emptyset$. Let us consider the digraphs $\hat{G} = (V_1 \cup V_2, E(V_1 \cup V_2) \cup E(V_1 \cup V'_2))$ and $\hat{G} = (V_1 \cup V'_2 \cup V_3, E(V_1 \cup V'_2 \cup V_3))$. It is obvious that the digraphs $\hat{G}$ and $\hat{G}$ belong to classes $A_{k,2}(m-i)$ and $A_{k,n}(i)$, respectively, and that the class of digraphs $\hat{A}_1$ is of the type \{ $A_{k,2}(m-i), A_{k,n}(i)$\}. It means that $|\hat{A}_{k,n}(m,i)| = C_m^{\hat{A}_1} j_k(m-i) \lambda_k(i)$. Then the statement of the lemma follows from (3).

3. Strongly, initially, initially–finally and unilaterally connected digraphs

The classes of strongly, initially and unilaterally connected digraphs were already enumerated (see, for example, [3,8,11]). For the sake of completeness, we give the formulas for counting them. Here they are obtained using the numbers $\lambda_m(n)$ and $\lambda_k(m)$. Certainly, the formulas can be improved as it is pointed out in the Introduction. The class of initially–finally connected digraphs was enumerated in [2] and [10].

Let $G = (V,E)$ be a digraph. We say that $G$ is an initially connected digraph if there is $u \in V$ such that $\text{Rch}(\{u\},V;G) = V$; such point $u$ is called a source. We say that $G$ is a strongly connected digraph if for every $(u,v) \in V^2$ there is a directed path in $G$ from $u$ to $v$, that is, all the points in the digraph are sources. We denote by $\hat{I}(n)$ the class of all initially connected digraphs with $n$ points and with a fixed source, and let $\hat{n}(n) = |\hat{I}(n)|$. Also denote by $S(n)$ the class of all strongly connected digraphs with $n$ points, and let $s(n) = |S(n)|$. From the above, it is easy to see that $\hat{n}(n) = 2^{n-1} s_1(n-1)$, that is, the recurrence formula

$$\hat{n}(n) = 2^{n(n-1)} - \sum_{j=1}^{n-1} C_{n-1}^{j-1} 2^{n-j(n-1)} \hat{n}(j), \quad n > 1, \quad (4)$$

holds, with the initial condition $\hat{n}(1) = 1$.

From the above given lemmas and from (4) immediately follows the result for the class of strongly connected digraphs, which was first proved in [3].
Theorem 1. For the number \( s(n) \) of strongly connected digraphs with \( n \) points the following recurrence formula:

\[
s(n) = \tilde{i}(n) - \sum_{j=1}^{n-1} C_{n-1}^{j-1} \lambda_j(n-j)s(j), \quad n > 1,
\]

holds, with the initial condition \( s(1) = 1 \).

Proof. Let us fix a set \( V, |V| = n \), and let \( v \in V \). Let \( G \) be a digraph from \( S(n) \), with \( V \) as the set of its points. Replace \( v \) by two new points \( v_1 \) and \( v_2 \), where \( v_1, v_2 \notin V \). All the arcs in \( G \) that went out from point \( v \) now go out from point \( v_1 \), and all the arcs that entered point \( v \), now enter point \( v_2 \). There are no other arcs incident with points \( v_1 \) and \( v_2 \). Thus, we have obtained a graph which belongs to the class \( A_{1,1}(n-1) \). In this way, as it is easy to see, we can define a bijection between the classes \( S(n) \) and \( A_{1,1}(n-1) \), so that \( s(n) = \tilde{\lambda}_{1,1}(n-1) \). By Lemma 2 we have

\[
\tilde{\lambda}_{1,1}(n-1) = 2^{n-1} \tilde{\lambda}_1(n-1) - \sum_{i=0}^{n-2} C_{n-1}^i \tilde{\lambda}_{i+1}(n-i-1) \lambda_{1,1}(i).
\]

Using the relation \( \tilde{i}(n) = 2^{n-1} \tilde{\lambda}_1(n-1) \) we get the statement of the theorem.

Let \( G = (V, E) \) be a digraph, and \( 1 \leq l \leq |V| \). We say that \( G \) is \( l \)-initially (\( l \)-finally) connected if:

1. there exists \( V_1 \subseteq V, |V_1| = l \), such that for every \( v \in V \) and for every \( u \in V_1 \) there is a directed path in \( G \) from \( u \) to \( v \) (a path from \( v \) to \( u \)), that is \( Rch(\{u\}, V; G) = V \) (\( Rch(\{u\}, V; G) = V \)) for every \( u \in V_1 \);
2. there does not exist a subset of \( V \) which has more than \( l \) elements and has a property of the set \( V_1 \) from (1).

We call the points from the set \( V_1 \) sources (sinks). In other words, the digraph \( G \) is \( l \)-initially (\( l \)-finally) connected if it has exactly \( l \) sources (sinks). We denote the class of all \( l \)-initially (\( l \)-finally) connected digraphs with \( n \) points by \( I_l(n) \) (\( F_l(n) \)). Digraphs with \( n \) points which contain at least one source (sink) constitute class \( I(n) \) (\( F(n) \)) of all initially (finally) connected digraphs. Let us also consider the class \( IF_{k,l}(n) \), where \( k \in \mathbb{N}, l \in \mathbb{N}, \) and \( k + l \leq n \), of all digraphs with \( n \) points, which have exactly \( k \) sources and exactly \( l \) sinks. It is clear that if the intersection of the set of all sources and of the set of all sinks in a digraph is not empty, then it is strongly connected. We denote by \( IF(n) \) the class of all digraphs with \( n \) points which have at least one source and at least one sink. Let \( i_l(n) = |I_l(n)|, \ i(n) = |I(n)|, \ f_l(n) = |F_l(n)|, \ f(n) = |F(n)|, \ if_{k,l}(n) = |IF_{k,l}(n)| \) and \( if(n) = |IF(n)| \). It is obvious that

\[
i_l(n) = \sum_{j=1}^{n} i_j(n), \quad f_l(n) = \sum_{j=1}^{n} f_j(n), \quad if(n) = s(n) + \sum_{i=1}^{n-1} \sum_{j=1}^{n-i} if_{i,j}(n).
\]
Let
\[ \tilde{\lambda}_{i,j}(k) = \begin{cases} 2^j \lambda_{i,j}(k), & k \neq 0, \\ 2^j - 1, & k = 0, \end{cases} \]
for \( i \in \mathbb{N}, j \in \mathbb{N}, k \in \mathbb{N}_0 \). The following statement holds [2].

**Theorem 2.** For the number \( i_k(n) (f_k(n)) \) of \( k \)-initially (\( k \)-finally) connected digraphs with \( n \) points and the number \( i_{k,l}(n) \) of initially–finally connected digraphs with \( n \) points which have exactly \( k \) sources and exactly \( l \) sinks, the following formulas hold:

\[
i_k(n) = f_k(n) = C^k_n \lambda_k(n - k) s(k) \quad (k \leq n),
\]

\[
i_{k,l}(n) = C^k_n C^{l-k}_n s(k) s(l) \tilde{\lambda}_{k,l}(n - k - l) \quad (k + l \leq n).
\]

We say that a digraph \( G = (V,E) \) is unilaterally connected if for every \((u,v) \in V^2\) there exists in \( G \) a path from \( u \) to \( v \) or a path from \( v \) to \( u \). For \( v \in V \), let us denote by \( G - v \) a digraph which is obtained from the digraph \( G \) when all the arcs incident with the point \( v \) as well as the point \( v \) itself are removed. Consider the condensation \( G^* = (V^*,E^*) \) of the digraph \( G = (V,E) \): the set \( V^* \) is the set of all components of strong connectedness of the digraph \( G \), and \((u^*,v^*) \in E^* \), where \( u^*,v^* \in V^* \), iff there exists \( u' \in u^* \) and \( v' \in v^* \) such that \((u',v') \in E \) (here, the component of strong connectedness which contains \( v \) is denoted by \( v^* \)). Let us define an ordering \( \leq \) on the set \( V^* \) in such a way that \( u^* \leq v^* \) iff \( u^* \) is reachable in \( G^* \) from the point \( v^* \). It is easy to prove the following statement.

**Lemma 3.** Let \( G = (V,E) \) be a unilaterally connected digraph. Then the condensation \( G^* \) is also a unilaterally connected digraph, and the pair \((G^*, \leq)\) is a linearly ordered set.

It follows from the lemma that in \((G^*, \leq)\) there exist the maximum and the minimum element; denote them by \( v^*_\text{max} \) and \( v^*_\text{min} \), respectively. It is clear that the components of strong connectedness determined by the points \( v^*_\text{max} \) and \( v^*_\text{min} \) of the condensation \( G^* \) consist, respectively, of the set of all sources and the set of all sinks of the digraph \( G \). Denote by \( U(n) \) the class of all unilaterally connected digraphs with \( n \) points, and let \( u(n) = |U(n)| \). From the above it is clear that \( U(n) \subseteq UF(n) \). Then it is clear that the following statement holds.

**Lemma 4.** If \( G = (V,E) \) is a unilaterally connected digraph then the digraph \( G^*-v^*_\text{max} \) is also unilaterally connected.

Denote by \( U(n,k) \) the class of all \( G \in U(n) \) for which \(|v^*_\text{max}| = k \), and let \( u(n,k) = |U(n,k)| \). By Lemmas 3 and 4 it is easy to prove that the following statement holds.
Theorem 3. The numbers \( u(n,k) \) and \( u(n) \) can be calculated by the following formulas:

\[
\begin{align*}
    u(n,n) &= s(n), \\
    u(n,k) &= C^k_{n-k}(s(k) \sum_{j=1}^{n-k} 2^{k(n-k-j)}(2^{kj} - 1)u(n-k,j), \quad k = 1, \ldots, n-1, \\
    u(n) &= \sum_{i=1}^{n} u(n,i).
\end{align*}
\]

4. Quasi-initially connected digraphs

Let a digraph \( G = (V,E) \) be given. We say that the point \( v \in V \) is a quasi-source of the digraph \( G \) if for every \( u \in V \) there is a directed path in \( G \) from \( u \) to \( v \) or a directed path from \( v \) to \( u \). Let us call a digraph with at least one quasi-source, a quasi-initially connected digraph. Denote by \( QI(n) \) the set of all digraphs with \( n \) points with a quasi-source, and let \( q(n) = |QI(n)| \). It is obvious that the following statement is valid.

Lemma 5. The condensation \( G^* = (V^*, E^*) \) of a quasi-initially connected digraph \( G = (V,E) \) is a quasi-initially connected digraph, too. Also, if the point \( v^* \in V^* \) contains a quasi-source of the digraph \( G \), then \( v^* \) is a quasi-source of the digraph \( G^* \).

Denote by \( V^*_{q} \) the set of all quasi-sources of the digraph \( G^* \). It is clear that for every pair of elements \( u^*, v^* \in V^*_{q} \) only one of the following two possibilities in \( G^* \) takes place: either \( u^* \) is reachable from \( v^* \) or \( v^* \) is reachable from \( u^* \). If now we introduce an ordering \( \leq \) on \( V^*_{q} \) in such a way that \( v^* \leq u^* \) iff \( v^* \) is reachable in \( G^* \) from \( u^* \), then we get a linearly ordered set. Denote the maximum element in this set by \( v^*_{\text{max}} \). Consider now in \( G \) the sets: \( W_0(G) = v^*_{\text{max}}, W_1(G) = \overline{\text{Rch}}(v^*_{\text{max}}, V ; G), \overline{\text{Rch}}(v^*_{\text{max}}, V ; G) \) and \( W_2(G) = \overline{\text{Rch}}(v^*_{\text{max}}, V ; G) \). It is clear that these sets define a partition of the set \( V \).

Denote by \( \Omega(n,i,j) \), where \( i \in \mathbb{N} \) and \( j \in \mathbb{N}_0 \), the set of all digraphs \( G \in QI(n) \) with a fixed set of points \( V \), \( |V| = n \), for which \( W_0(G) = V_1 \) and \( W_2(G) = V_2 \), where \( V_1 \) and \( V_2 \) are fixed subsets of the set \( V \) such that \( |V_1| = i \) and \( |V_2| = j \). Let \( \omega(n,i,j) = |\Omega(n,i,j)| \). It can be easily seen that the following statement holds.

Lemma 6. \( q(n) = \sum_{i=1}^{n} \sum_{j=0}^{n-i} C^i_n C^n_{n-i} \omega(n,i,j) \).

Let \( G = (V,E) \) be a digraph, and let \( V_1, V_2 \subseteq V \). We say that \( V_1 \) cover (anticover) \( V_2 \) if \( \text{Rch}(V_1, V_2; G) = V_2 \) (\( \overline{\text{Rch}}(V_1, V_2; G) = V_2 \)).

Denote by \( A(n,i) \), \( 1 \leq i \leq n \), the set of all digraphs \( G \) with a fixed set of points \( V \), \( |V| = n \), and with a fixed component of strong connectedness \( V_1 \subseteq V \), \( |V_1| = i \), such that \( V_1 \) anticovers \( V \) and the digraph \( (V \setminus V_1, E(V \setminus V_1, G)) \) has no quasi-source. Let \( \pi(n,i) = |A(n,i)| \). It should be mentioned that if above we had taken the word ‘cover’
instead of the word ‘anticover’, then the respective set of digraphs would have had the same number of elements. By the definition it follows that $\alpha(n, n) = s(n)$.

**Lemma 7.** $\omega(n, i, j) = 2^{\hat{\lambda}(n - i - j)_{\lambda}(n - i - j)\alpha(i + j, i)}$.

**Proof.** Let $G = (V, E) \in \Omega(n, i, j)$. If $i + j = n$, then it is clear that $\omega(n, i, j) = \alpha(i + j, i) = 2^{\hat{\lambda}(n - i - j)_{\lambda}(n - i - j)\alpha(i + j, i)}$. So we can suppose that $i + j < n$. Denote $W_0 = W_0(G)$, $W_1 = W_1(G)$ and $W_2 = W_2(G)$. Consider digraphs $G_1 = (W_0 \cup W_1, E(W_0 \cup W_1) \cup E(W_0))$, $G_2 = (W_2 \cup W_0, E(W_2 \cup W_0))$ and $G_3 = (W_2 \cup W_1, E_1)$, where $E_1 = E \cap (W_2 \times W_1)$. Let us note that $(W_1 \times W_2) \cap E = \emptyset$. Now, it is clear that the graphs $G_1, G_2$ and $G_3$ determine a ‘complete partition’ of the given graph, and that $G_1 \in A_i(n - i - j)$, $G_2 \in A(i + j, i)$ (the set $W_2$ does not contain a quasi-source), and $G_3 \in \mathcal{B}_{j,n-i-j}$. Therefore, the class $\Omega(n, i, j)$ is of the type \{\[A_i(n - i - j), A(i + j, i), \mathcal{B}_{j,n-i-j}\]\}, and the statement of the lemma holds.

**Lemma 8.** The recurrence formula

$$
\alpha(n, i) = s(i) \cdot \left[ \hat{\lambda}(n - i) - \sum_{j=1}^{n-i} \sum_{k=0}^{n-i-j} 2^{2(n-j-k)}C_{n-j}^jC_{n-i-j}^k \right] \\
\cdot \alpha(j, i) \alpha(n - i - j - k) \alpha(j + k, j)
$$

holds for $i < n$.

**Proof.** Denote by $\tilde{A}(n, i, j, k)$, where $i \in \mathbb{N}$, $j \in \mathbb{N}$, $k \in \mathbb{N}_0$, and $i + j + k \leq n$, a set of all digraphs $G$ with a fixed set of points $V$, $|V| = n$, and with a fixed component of strong connectedness $S(G) \subseteq V$, $|S(G)| = i$, which anticovers $V$, such that $W_0(G') = V_1$ and $W_2(G') = V_2$, where $|V_1| = j$, and $|V_2| = k$, are fixed subsets of the set $V \setminus S(G)$, and $G' = (V \setminus S(G), E(V \setminus S(G), G'))$. Let $\tilde{\alpha}(n, i, j, k) = |\tilde{A}(n, i, j, k)|$. Then

$$
\alpha(n, i) = \lambda(n - i)\alpha(i) - \sum_{j=1}^{n-i} \sum_{k=0}^{n-i-j} C_{n-j}^jC_{n-i-j}^k \tilde{\alpha}(n, i, j, k).
$$

(5)

Let us calculate the number $\tilde{\alpha}(n, i, j, k)$ in the following way. Let $G = (V, E) \in \tilde{A}(n, i, j, k)$. Consider digraphs

- $G_1 = (W_0(G') \cup W_1(G') \cup S(G), E(W_0(G') \cup W_1(G') \cup S(G)) \setminus (E(W_0(G')) \cup E(S(G)) \cup ((W_0(G') \times S(G)) \cap E))$,
- $G_2 = (W_0(G') \cup S(G), (W_0(G') \times S(G)) \cap E)$,
- $G_3 = (W_2(G') \cup W_0(G'), E(W_2(G') \cup W_0(G')))$,
- $G_4 = (W_2(G') \cup W_1(G') \cup S(G), E \cap (W_2(G') \times (W_1(G') \cup S(G))))$.

Let us note that the bipartite digraph $G_2$ has at least one arc if $n - i - j - k = 0$. Because of that we denote by $\mathcal{B}_{j,n-i-j}^b$ the class of bipartite digraphs which is equal to
### Table 1
The number of labelled quasi-initially connected digraphs with \( n \leq 20 \) points.

<table>
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<th>( q(n) )</th>
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**Lemma 2.** If \( k' \neq 0 \), and equal to \( B_{k',j} \) if \( k' = 0 \), where \( O_{k',j} \) is a bipartite digraph from the class \( B_{k',j} \) which has not any arc. Now it is obvious that \( G_2 \in B_{k,j;i} \). Note that \([W_1(G')] \times (W_0(G') \cup W_2(G'))) \cap E = \emptyset \) and \([S(G') \times (W_0(G') \cup W_1(G') \cup W_2(G'))]) \cap E = \emptyset \). Also note that the set \( W_2(G') \) does not contain a single quasi-source of the digraph \( G \). The digraphs \( G_1, G_2, G_3, G_4 \) and \( S(G) \) define a `complete partition' of the digraph \( G \), i.e. the class \( \tilde{A}(n, i, j, k) \) is of the type \( \tilde{A}(n, i, j, k) = \tilde{A}(n, i, j, k, j); S(i) \). Then we obtain

\[
\exists(n, i, j, k) = 2^{\delta(n-i-j-k)}(i)\frac{C^i}{C^j_{i}(n-i-j-k)}(i+j, i).
\]

The proof of the lemma follows from equalities (5) and (6).

Let us sum up the results of Lemmas 6–8 into the main theorem.

**Theorem 4.** The number \( q(n) \) of all quasi-initially connected digraphs with \( n \) points can be calculated by the formula

\[
q(n) = \sum_{i=1}^{n} \sum_{j=0}^{n-i} C^i_n C^j_n 2^{\delta(n-i-j)}(i, i+j, i).
\]
Table 2
The number of labelled quasi-initially connected digraphs with $l$ arcs and with $n$ points, $n \leq 5$.

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where

\[
\varphi(n, i) = s(i) \left[ \lambda_i(n) - \sum_{j=1}^{n-i} \sum_{k=0}^{n-j} 2^{k(n-j-k)} C_{n-i}^j C_{n-i-j}^k \cdot \tilde{\lambda}_{j,j}(n-i-j-k) \varphi(j+k, j) \right] \quad (i < n),
\]

\[
\varphi(n, n) = s(n).
\]

5. Tables

Using the obtained recurrence formulas the cardinalities of all above-mentioned classes of labelled connected digraphs with $n$ points, $n \leq 30$, were calculated on a computer. Here is given only the number of all labelled quasi-initially connected digraphs with $n$ points, $n \leq 20$ (Table 1).

From the above given formulas it is easy to obtain the formulas for labelled quasi-initially connected digraphs with the fixed number of arcs (Table 2).
Table 3
The number of unlabelled quasi-initially connected digraphs with \( \ell \) arcs and with \( n \) points, \( n \leq 5 \).

<table>
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\( \Sigma \) | 1  | 2  | 13 | 197 | 9312 |

Table 3 gives the numbers of all unlabelled quasi-initially connected digraphs, which are calculated by computer generative enumeration (the problem of their enumeration has not been solved yet).

It is well known that \( s(n) \sim 2^{n(n-1)} \) [3], i.e. almost all labelled digraphs are strongly connected. Then the above given asymptotic evaluation holds for labelled quasi-initially connected digraphs, too. The class of quasi-initially connected digraphs is ‘very close’ to the class of weakly connected digraphs. For \( n = 4 \) there exist only two unlabelled weakly connected digraphs that are not quasi-initially connected, and for \( n = 5 \) — only 52. For comparision purposes, there are 116 unlabelled weakly connected digraphs with 4 points that are not strongly connected, and 4316 for \( n = 5 \).
References