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Error bounds of certain Gaussian quadrature formulae☆

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ABSTRACT

We study the kernel of the remainder term of Gauss quadrature rules for analytic functions with respect to one class of Bernstein–Szegö weight functions. The location on the elliptic contours where the modulus of the kernel attains its maximum value is investigated. This leads to effective error bounds of the corresponding Gauss quadratures.

1. Introduction

We study the kernels $K_n(z)$ in the remainder terms $R_n(f)$ of the Gaussian quadrature formula

$$\int_{-1}^{1} f(t)w(t)\,dt = G_n[f] + R_n(f), \quad G_n[f] = \sum_{\nu=1}^{n} \lambda_\nu f(\tau_\nu)\ (n \in \mathbb{N}) \tag{1.1}$$

for analytic functions on elliptical contours with foci at ∓ 1 and the sum of semi-axes $\rho > 1$,

$$\mathcal{E}_\rho = \left\{ z \in \mathbb{C} \,\middle|\, z = \frac{1}{2}\left(\xi + \xi^{-1}\right),\ 0 \le \theta \le 2\pi \right\}, \quad \xi = \rho\, e^{i\theta}, \tag{1.2}$$

where w is a nonnegative and integrable function on the interval $(-1,1)$, which is exact for all algebraic polynomials of degree at most $2n-1$. The nodes τ_ν in (1.1) are zeros of the orthogonal polynomials π_n with respect to the weight function w.

When $\rho \to 1$ the ellipse (1.2) shrinks to the interval $[-1, 1]$, while with increasing ρ it becomes more and more circle-like. The advantage of the elliptical contours, compared to the circular ones, is that such a choice needs the analyticity of f in a smaller region of the complex plane, especially when ρ is near 1.

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In this paper w represents the class of symmetric weight functions of Bernstein–Szegö type

$$w(t) \equiv w_\gamma(t) = \frac{\sqrt{1-t^2}}{1-\frac{4\gamma}{(1+\gamma)^2}t^2}, \quad t \in (-1,1),\ \gamma \in (-1,0). \tag{1.3}$$

The weight functions under consideration are special cases of the (more general) Bernstein–Szegö weight functions

$$w(t) = \frac{\sqrt{1-t^2}}{\beta(\beta-2\alpha)t^2 + 2\delta(\beta-\alpha)t + \alpha^2 + \delta^2}, \quad t \in (-1,1), \tag{1.4}$$

where $0 < \alpha < \beta$, $\beta \neq 2\alpha$, $|\delta| < \beta - \alpha$, having in the denominator an arbitrary polynomial of exact degree 2 that remains positive on $[-1,1]$. Namely, if we set $\alpha = 1$, $\beta = 2/(1+\gamma)$, $-1 < \gamma < 0$ and $\delta = 0$, (1.4) reduces to (1.3). The weight functions (1.4) have been studied extensively in [1], and therefore the results obtained there can be specialized in the case of (1.3).

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and $\mathcal{D} = \mathrm{int}\Gamma$ its interior. If the integrand f is analytic in $\mathcal{D}$ and continuous on $\overline{\mathcal{D}}$, then the remainder term $R_n(f)$ in (1.1) admits the contour integral representation

$$R_n(f) = \frac{1}{2\pi i}\oint_\Gamma K_n(z) f(z)\mathrm{d}z. \tag{1.5}$$

The *kernel* is given by

$$K_n(z) \equiv K_n(z,w) = \frac{\varrho_n(z)}{\pi_n(z)}, \quad z \notin [-1,1],$$

where

$$\varrho_n(z) \equiv \varrho_{n,w}(z) = \int_{-1}^{1} \frac{\pi_n(t)}{z-t} w(t)\mathrm{d}t.$$

The modulus of the kernel is symmetric with respect to the real axis, i.e., $|K_n(\overline{z})| = |K_n(z)|$. If the weight function w in (1.1) is even, the modulus of the kernel is symmetric with respect to both axes, i.e., $|K_n(-\overline{z})| = |K_n(z)|$ (see [2]).

The integral representation (1.5) leads to the error estimate

$$|R_n(f)| \le \frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_n(z)|\right)\left(\max_{z\in\Gamma}|f(z)|\right), \tag{1.6}$$

where $\ell(\Gamma)$ is the length of the contour Γ. For a different approach to the estimation of $R_n(f)$, see [3].

In this paper we take $\Gamma = \mathcal{E}_\rho$, where the ellipse $\mathcal{E}_\rho$ is given by (1.2). The estimate (1.6) reduces to

$$|R_n(f)| \le \frac{\ell(\mathcal{E}_\rho)}{2\pi}\left(\max_{z\in\mathcal{E}_\rho}|K_n(z)|\right)\left(\max_{z\in\mathcal{E}_\rho}|f(z)|\right). \tag{1.7}$$

The derivation of adequate bounds for $|R_n(f)|$ on the basis of (1.7) is possible only if good estimates for $\max_{z\in\mathcal{E}_\rho}|K_n(z)|$ are available, especially if we know the location of the extremal point $\eta \in \mathcal{E}_\rho$ at which $|K_n|$ attains its maximum. In such a case, instead of looking for upper bounds for $\max_{z\in\mathcal{E}_\rho}|K_n(z)|$ one can simply try to calculate $|K_n(\eta,w)|$. In general, this may not be an easy task, but in the case of the Gauss-type quadrature formula (1.1) there exist effective algorithms for calculation of $K_n(z)$ at any point z outside $[-1,1]$ (see Gautschi and Varga [2]).

So far, this approach (cf. (1.7)) has been discussed for Gaussian quadrature rules (1.1) with respect to the Chebyshev weight functions (see [2,4])

$$w_1(t) = \frac{1}{\sqrt{1-t^2}}, \qquad w_2(t) = \sqrt{1-t^2}, \qquad w_3(t) = \sqrt{\frac{1+t}{1-t}}, \qquad w_4(t) = \sqrt{\frac{1-t}{1+t}},$$

and later it was extended by Schira to symmetric weight functions under the restriction of monotonicity type (either $w(t)\sqrt{1-t^2}$ is increasing on (0,1) or $w(t)/\sqrt{1-t^2}$ is decreasing on $(0,1)$), including certain Gegenbauer weight functions (see [5]). For the Chebyshev weight function of the second kind $w_2(t)$, which is the special case of our weight (1.3) with $\gamma = 0$, this approach has been considered by Gautschi et al., first for n odd in [2], and then for n even in [4]. The weight function (1.3), which we consider here, belongs to the class considered by Schira [5] ($w_\gamma(t)/\sqrt{1-t^2}$ is decreasing on $(0,1)$). He proved (see [5, Th. 3.2.(b) on p. 302]) that the kernel K_n of a Gaussian quadrature rule with respect to a symmetric weight function w on $(-1,1)$ satisfies: If $w(t)/\sqrt{1-t^2}$ is decreasing on $(0,1)$, then

$$\max_{z\in\mathcal{E}_\rho}|K_n(z)| = \left|K_n\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right| \quad \text{for } \rho \ge \rho_n^*,$$

where $\rho_n^* := 1+\sqrt{2}$ if $n \ge 1$ is odd, and if $n \ge 2$ is even, ρ_n^* is the greatest zero of

$$d_n(\rho) := (\rho-\rho^{-1})^2 - 4 - (\rho^2-\rho^{-2})^2\left(\frac{(n+1)^2}{(\rho^{n+1}-\rho^{-n-1})^2} + \frac{(n+3)^2}{(\rho^{n+3}-\rho^{-n-3})^2}\right).$$

On the basis of the displayed values for n even in [5, Table 1 on p. 302], ρ_n^* converges rapidly towards $1+\sqrt{2}\,(+0)$ with increasing n.

In this paper, with respect to the weight function (1.3), sufficient conditions are found ensuring that there exists a $\rho^* = \rho_n^*\ (= \rho^*(n,\gamma))$ such that for each $\rho \ge \rho_n^*$ the kernel attains its maximal absolute value at the intersection points of the ellipse with the imaginary axis. For this specialized case, we obtained much smaller values for $\rho = \rho_n^*$ than the ones obtained by Schira (except for γ close to 0 and n even), especially for large values of n.

2. Maximum of the modulus of kernel of the Gauss quadrature formula with the weight function $w_\gamma(t)$ ($\gamma \in (-1, 0)$)

For the weight function under consideration, the corresponding (monic) orthogonal polynomial $\pi_n(t) = \pi_{n,\gamma}(t)$ of the degree n has the form (see [1])

$$\pi_n(t) = \pi_{n,\gamma}(t) = \frac{1}{2^n}[U_n(t) - \gamma U_{n-2}(t)], \quad n \ge 1, \tag{2.1}$$

where U_n denotes the Chebyshev polynomial of the second kind, characterized by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}.$$

As usual we use the substitution

$$z = \frac{1}{2}(\xi+\xi^{-1}), \quad \xi = \rho\,e^{i\theta}.$$

Using the well-known facts (cf. [2])

$$U_n(z) = \frac{\xi^{n+1}-\xi^{-(n+1)}}{\xi-\xi^{-1}}, \quad z = \frac{1}{2}(\xi+\xi^{-1}),$$

and

$$\int_{-1}^{1}\frac{U_n(t)}{z-t}\sqrt{1-t^2}\,dt = \int_0^{\pi}\frac{\sin(n+1)\theta\sin\theta}{z-\cos\theta}\,d\theta = \frac{\pi}{\xi^{n+1}},$$

on the basis of direct calculation, we obtain that the kernel can be expressed ($\gamma \in (-1,0),\ n \in \mathbb{N}$) in the following way:

$$K_{n,\gamma}(z) = \frac{\pi(1+\gamma)^2(1-\gamma\xi^2)(\xi-\xi^{-1})}{\xi^{n+1}\left[(1+\gamma)^2-\gamma(\xi+\xi^{-1})^2\right]\left[(\xi^{n+1}-\xi^{-(n+1)})-\gamma(\xi^{n-1}-\xi^{-(n-1)})\right]}. \tag{2.2}$$

Namely,

$$\varrho_n(z) = \varrho_{n,\gamma}(z) = \int_{-1}^{1}\frac{2^n\pi_{n,\gamma}(t)}{z-t}\frac{\sqrt{1-t^2}}{1-\frac{4\gamma}{(1+\gamma)^2}t^2}\,dt.$$

We use the decomposition

$$\frac{1}{(z-t)\left(1-\frac{4\gamma}{(1+\gamma)^2}t^2\right)} = \frac{A_1}{z-t} + \frac{A_2t+A_3}{1-\frac{4\gamma}{(1+\gamma)^2}t^2}, \tag{2.3}$$

where

$$A_1 = \frac{(1+\gamma)^2}{(1+\gamma)^2-4\gamma z^2}, \qquad A_2 = \frac{-4\gamma}{(1+\gamma)^2-4\gamma z^2}, \qquad A_3 = \frac{-4\gamma z}{(1+\gamma)^2-4\gamma z^2}.$$

Multiplying (2.3) by $2^n\pi_{n,\gamma}(t)\sqrt{1-t^2}$ and integrating over the interval $[-1, 1]$, we obtain

$$\varrho_{n,\gamma}(z) = A_1\int_{-1}^{1}\frac{U_n(t)-\gamma U_{n-2}(t)}{z-t}\sqrt{1-t^2}\,dt + 2^n\int_{-1}^{1}\pi_{n,\gamma}(t)(A_2t+A_3)\frac{\sqrt{1-t^2}}{1-\frac{4\gamma}{(1+\gamma)^2}t^2}\,dt.$$

If $n \ge 2$, in the last equality the second integral is equal to zero; then the formula (2.2) can be derived easily.

If $n = 1$, we have

$$\varrho_{1,\gamma}(z) = A_1 \int_{-1}^{1} \frac{U_1(t)}{z-t}\sqrt{1-t^2}\,\mathrm{d}t + \frac{1}{2}A_2 \int_{-1}^{1} \frac{U_1^2(t)\sqrt{1-t^2}}{1-\frac{4\gamma}{(1+\gamma)^2}t^2}\,\mathrm{d}t,$$

where $U_1(t) = 2t$. On the basis of [1, Eq. (3.22)], we have

$$\int_{-1}^{1} \frac{U_1^2(t)\sqrt{1-t^2}}{1-\frac{4\gamma}{(1+\gamma)^2}t^2}\,\mathrm{d}t = \frac{(1+\gamma)^2\pi}{2}.$$

Finally, using the representation $U_1(z) = 2z = \xi + \xi^{-1}$, we obtain

$$K_{1,\gamma}(z) = \frac{\pi(1+\gamma)^2(1-\gamma\xi^2)}{\xi^2[(1+\gamma)^2 - \gamma(\xi+\xi^{-1})^2](\xi+\xi^{-1})},$$

which represents (2.2) for $n = 1$.

From here on, we use the usual notation (see, for example, [2])

$$a_j = \frac{1}{2}(\rho^j + \rho^{-j}), \quad j \in \mathbb{N}.$$

Using (2.2), and on the basis of

$$\begin{aligned}
&|1-\gamma\xi^2| = \left(1+\gamma^2\rho^4 - 2\gamma\rho^2\cos 2\theta\right)^{1/2}, \\
&|\xi - \xi^{-1}| = \sqrt{2}\,(a_2 - \cos 2\theta)^{1/2}, \\
&\left|(1+\gamma)^2 - \gamma(\xi+\xi^{-1})^2\right| = \left[(1+\gamma^2)^2 - 4\gamma(1+\gamma^2)a_2\cos 2\theta + 2\gamma^2(a_4 + \cos 4\theta)\right]^{1/2}, \\
&\left|\xi^{n+1} - \xi^{-(n+1)} - \gamma(\xi^{n-1} - \xi^{-n+1})\right| = \sqrt{2}\left[a_{2n+2} - \cos(2n+2)\theta + \gamma^2(a_{2n-2} - \cos(2n-2)\theta)\right. \\
&\qquad \left. -\ 2\gamma(a_{2n}\cos 2\theta - a_2\cos 2n\theta)\right]^{1/2},
\end{aligned}$$

we obtain (for $n \in \mathbb{N}$)

$$\begin{aligned}
\left|K_{n,\gamma}(z)\right| = {} & \frac{\pi(1+\gamma)^2(a_2 - \cos 2\theta)^{1/2}\left(1+\gamma^2\rho^4 - 2\gamma\rho^2\cos 2\theta\right)^{1/2}}{\rho^{n+1}\left[(1+\gamma^2)^2 - 4\gamma(1+\gamma^2)a_2\cos 2\theta + 2\gamma^2(a_4+\cos 4\theta)\right]^{1/2}} \\
& \times \frac{1}{\left[a_{2n+2} - \cos(2n+2)\theta + \gamma^2(a_{2n-2} - \cos(2n-2)\theta) - 2\gamma(a_{2n}\cos 2\theta - a_2\cos 2n\theta)\right]^{1/2}}.
\end{aligned} \tag{2.4}$$

Numerical experiments showed us that there exists a $\rho^* = \rho_n^* = \rho^*(n,\gamma) > 1$ so that $\left|K_{n,\gamma}(z)\right|$ attains its maximum value on the imaginary axis, i.e., at $\theta = \pi/2$, for each $\rho \geq \rho^*$. For the proof of this assertion we need the following lemma.

Lemma 2.1. *For $\rho > 1$, $\gamma \in (-1, 0)$ and $\theta \in [0, \pi/2]$, we have that*

$$\frac{1+\gamma^2\rho^4 - 2\gamma\rho^2\cos 2\theta}{(1+\gamma^2)^2 - 4\gamma(1+\gamma^2)a_2\cos 2\theta + 2\gamma^2(a_4+\cos 4\theta)} \leq \frac{1+\gamma^2\rho^4 + 2\gamma\rho^2}{(1+\gamma^2)^2 + 4\gamma(1+\gamma^2)a_2 + 2\gamma^2(a_4+1)}. \tag{2.5}$$

Proof. Let us denote

$$\begin{aligned}
&A = 1+\gamma^2\rho^4 + 2\gamma\rho^2 \ (\geq 0), \qquad A_1 = -4\gamma\rho^2\cos^2\theta, \\
&B = (1+\gamma^2)^2 + 4\gamma(1+\gamma^2)a_2 + 2\gamma^2(a_4+1) \ (\geq 0), \\
&B_1 = -8\gamma(1+\gamma^2)a_2\cos^2\theta - 4\gamma^2\sin^2 2\theta.
\end{aligned}$$

The inequality (2.5) can now be written in the form

$$\frac{A+A_1}{B+B_1} \leq \frac{A}{B},$$

that is

$$AB_1 - BA_1 \geq 0,$$

i.e.,

$$\begin{aligned}
&(1+\gamma^2\rho^4 + 2\gamma\rho^2)\left(-8\gamma(1+\gamma^2)a_2\cos^2\theta - 4\gamma^2\sin^2 2\theta\right) \\
&\quad - \left((1+\gamma^2)^2 + 4\gamma(1+\gamma^2)a_2 + 2\gamma^2(a_4+1)\right)\cdot(-4\gamma\rho^2\cos^2\theta) \geq 0.
\end{aligned}$$

The last inequality is satisfied for $\theta = \pi/2$. Therefore, let us consider the case when $\theta \in [0, \pi/2)$, and divide the last inequality by $\cos^2\theta$. We obtain

$$(1+\gamma^2\rho^4+2\gamma\rho^2)\left(-8\gamma(1+\gamma^2)a_2-16\gamma^2\sin^2\theta\right)+4\gamma\rho^2\left((1+\gamma^2)^2+4\gamma(1+\gamma^2)a_2+2\gamma^2(a_4+1)\right)\geq 0.$$

The last inequality holds if

$$(1+\gamma^2\rho^4+2\gamma\rho^2)\left((1+\gamma^2)a_2+2\gamma\right)-\frac{1}{2}\rho^2\left((1+\gamma^2)^2+4\gamma(1+\gamma^2)a_2+2\gamma^2(a_4+1)\right)\geq 0.$$

Now, we use $a_4 = 2a_2^2 - 1$, and conclude that the last inequality holds, since the left-hand side of it becomes

$$\begin{aligned}
&\left[(1+\gamma^2)a_2+2\gamma\right](1+\gamma\rho^2)^2-\frac{1}{2}\rho^2\left[(1+\gamma^2)+2\gamma a_2\right]^2\\
&=\frac{1}{2}\left[(1+\gamma^2)(\rho^2+\rho^{-2})+4\gamma\right](1+\gamma\rho^2)^2-\frac{1}{2}\rho^2\left[(1+\gamma\rho^2)+\gamma(\gamma+\rho^{-2})\right]^2\\
&=\frac{1}{2}\left\{\left[(1+\gamma^2)(\rho^2+\rho^{-2})+4\gamma\right](1+\gamma\rho^2)^2\right.\\
&\quad\left.-\ \rho^2(1+\gamma\rho^2)^2-2\gamma\rho^2(1+\gamma\rho^2)(\gamma+\rho^{-2})-\gamma^2\rho^2(\gamma+\rho^{-2})^2\right\}\\
&=\frac{1}{2}\left\{\left[(1+\gamma^2)(\rho^2+\rho^{-2})+4\gamma\right](1+\gamma\rho^2)^2\right.\\
&\quad\left.-\ \rho^2(1+\gamma\rho^2)^2-2\gamma(1+\gamma\rho^2)^2-\rho^{-2}\gamma^2(1+\gamma\rho^2)^2\right\}\\
&=\frac{1}{2}(1+\gamma\rho^2)^2(\gamma^2\rho^2+2\gamma+\rho^{-2})\\
&=\frac{1}{2}(1+\gamma\rho^2)^2(\gamma\rho+\rho^{-1})^2. \quad\square
\end{aligned}$$

Theorem 2.2. *For the Gauss quadrature formula (1.1), $n \in \mathbb{N}$, with the weight function (1.3), $\gamma \in (-1, 0)$, there exists a $\rho^* \in (1, +\infty)$ ($\rho^* = \rho_n^* = \rho^*(n, \gamma)$) such that for each $\rho \geq \rho^*$ the modulus of the kernel $\left|K_{n,\gamma}(z)\right|$ attains its maximum value on the imaginary axis ($\theta = \pi/2$), i.e.,*

$$\max_{z\in\mathcal{E}_\rho}\left|K_{n,\gamma}(z)\right| = \left|K_{n,\gamma}\left(\frac{i}{2}(\rho-\rho^{-1})\right)\right|.$$

Proof. On the basis of (2.4), Lemma 2.1 and the inequality $a_2 - \cos 2\theta \leq a_2 + 1$, it is sufficient to prove the following inequality for $\theta \in [0, \pi/2]$:

$$\begin{aligned}
&\frac{1}{a_{2n+2}-\cos(2n+2)\theta+\gamma^2(a_{2n-2}-\cos(2n-2)\theta)-2\gamma(a_{2n}\cos 2\theta-a_2\cos 2n\theta)}\\
&\leq\frac{1}{a_{2n+2}+(-1)^n+\gamma^2(a_{2n-2}+(-1)^n)+2\gamma(a_{2n}+(-1)^n a_2)}.
\end{aligned}\tag{2.6}$$

First, let n be EVEN. We put

$$C + C_1 = a_{2n+2}-\cos(2n+2)\theta+\gamma^2(a_{2n-2}-\cos(2n-2)\theta)-2\gamma(a_{2n}\cos 2\theta-a_2\cos 2n\theta),\tag{2.7}$$

where

$$\begin{aligned}
C &= a_{2n+2}+1+\gamma^2(a_{2n-2}+1)+2\gamma(a_{2n}+a_2),\\
C_1 &= -2\cos^2(n+1)\theta-2\gamma^2\cos^2(n-1)\theta-4\gamma a_{2n}\cos^2\theta-4\gamma a_2\sin^2 n\theta.
\end{aligned}$$

For the second fraction in (2.4) it holds that

$$\frac{1}{C+C_1}\leq\frac{1}{C},$$

if $C_1 \geq 0$. This is satisfied if $\theta = \pi/2$, so we consider the case when $\theta \in [0, \pi/2)$. Using the well-known inequality

$$\left|\frac{\cos(n+1)\theta}{\cos\theta}\right|\leq n+1,\quad n\text{ even},$$

we conclude that

$$\frac{C_1}{\cos^2\theta}\equiv -2\frac{\cos^2(n+1)\theta}{\cos^2\theta}-2\gamma^2\frac{\cos^2(n-1)\theta}{\cos^2\theta}-4\gamma a_{2n}-4\gamma a_2\frac{\sin^2 n\theta}{\cos^2\theta}\geq 0,$$

Table 3.1
The values of ρ^* for some $n \in 2\mathbb{N}$ and $\gamma \in (-1, 0)$.

(n, γ)	ρ^*	(n, γ)	ρ^*	(n, γ)	ρ^*
(2, −0.001)	9.741	(2, −0.1)	3.081	(2, −0.2)	2.593
(2, −0.3)	2.346	(2, −0.5)	2.073	(2, −0.7)	1.917
(2, −0.8)	1.86	(2, −0.9)	1.814	(2, −0.99)	1.778
(10, −0.001)	1.796	(10, −0.1)	1.427	(10, −0.2)	1.38
(10, −0.3)	1.354	(10, −0.5)	1.327	(10, −0.7)	1.313
(10, −0.8)	1.309	(10, −0.9)	1.306	(10, −0.99)	1.305
(30, −0.001)	1.259	(30, −0.1)	1.166	(30, −0.2)	1.153
(30, −0.3)	1.146	(30, −0.5)	1.139	(30, −0.7)	1.135
(30, −0.8)	1.134	(30, −0.9)	1.134	(30, −0.99)	1.134
(50, −0.001)	1.16	(50, −0.1)	1.108	(50, −0.2)	1.1
(50, −0.3)	1.096	(50, −0.5)	1.092	(50, −0.7)	1.09
(50, −0.8)	1.09	(50, −0.9)	1.09	(50, −0.99)	1.089
(100, −0.001)	1.085	(100, −0.1)	1.06	(100, −0.2)	1.056
(100, −0.3)	1.054	(100, −0.5)	1.052	(100, −0.7)	1.052
(100, −0.8)	1.051	(100, −0.9)	1.051	(100, −0.99)	1.051

if it holds that

$$-2(n+1)^2 - 2\gamma^2(n-1)^2 - 4\gamma a_{2n} \geq 0,$$

i.e., after dividing the last inequality by 2,

$$-(n+1)^2 - \gamma^2(n-1)^2 - 2\gamma a_{2n} \geq 0. \tag{2.8}$$

This is satisfied for each $\rho \geq \rho_E$ (> 1).

Now, let n be ODD. In (2.7) we now take that

$$C = a_{2n+2} - 1 + \gamma^2(a_{2n-2} - 1) + 2\gamma(a_{2n} - a_2),$$
$$C_1 = 2\sin^2(n+1)\theta + 2\gamma^2\sin^2(n-1)\theta - 4\gamma a_{2n}\cos^2\theta + 4\gamma a_2\cos^2 n\theta.$$

For the second fraction in (2.4) it holds that $1/(C + C_1) \leq 1/C$, if $C_1 \geq 0$. This is satisfied if $\theta = \pi/2$, so we consider the case when $\theta \in [0, \pi/2)$. Similarly as in the previous case we conclude that

$$\frac{C_1}{\cos^2\theta} \equiv 2\frac{\sin^2(n+1)\theta}{\cos^2\theta} + 2\gamma^2\frac{\sin^2(n-1)\theta}{\cos^2\theta} - 4\gamma a_{2n} + 4\gamma a_2\frac{\cos^2 n\theta}{\cos^2\theta} \geq 0,$$

if it holds that

$$-4\gamma a_{2n} + 4\gamma n^2 a_2 \geq 0,$$

i. e., after dividing it by -4γ, if it holds that

$$a_{2n} - n^2 a_2 \geq 0. \tag{2.9}$$

If $n = 1$, then the expression $a_{2n} - n^2 a_2$ is equal to zero. If $n > 1$, let us write it in the form $h(x) = \cosh(nx) - n^2\cosh(x)$, where $x = \ln\rho^2$. We have that $h'(x) = ng(x)$, where $g(x) = \sinh(nx) - n\sinh(x)$. Since $g'(x) = n(\cosh(nx) - \cosh(x)) > 0$ for $x > 0$, $g(0) = 0$, we conclude that the function g is positive for $x > 0$. For the function h we conclude that it is strongly increasing for $x > 0$, $h(0) < 0$. Therefore, the inequality (2.9) holds for each $\rho \geq \rho_O$ (> 1), $n = 3, 5, \ldots,$ and $\rho > 1$ for $n = 1$. Observe that (2.9) does not depend on γ.

Taking $\rho^* = \rho_E$ for n even and $\rho^* = \rho_O$ for n odd, the inequality (2.6) holds on the interval $[\rho^*, +\infty)$. □

3. Numerical results

The proof of Theorem 2.2 is of practical importance. Namely, on the basis of the conditions (2.8) and (2.9), we can determine the intervals $[\rho^*, +\infty)$ on which the modulus of the kernel $K_{n,\gamma}$ attains its maximum value on the imaginary axis. For some values of n, γ the values of ρ^* are displayed in Tables 3.1 and 3.2. Observe that the results become very satisfactory when n increases.

Remainder terms for quadrature formulae are traditionally expressed in terms of some high-order derivative of the function involved. This is a serious disadvantage, if such derivatives are not known, do not exist or are too complicated to be handled.

Let us consider numerical calculation of the integral

$$I(f) = \int_{-1}^{1} f(t)\frac{\sqrt{1-t^2}}{1 - \frac{4\gamma}{(1+\gamma)^2}t^2}\,\mathrm{d}t, \tag{3.1}$$

Table 3.2
The values of ρ^* for some $n \in 2\mathbb{N}+1$ and $\gamma \in (-1, 0)$.

n	ρ^*	n	ρ^*	n	ρ^*
3	1.774	5	1.528	7	1.41
9	1.339	13	1.256	15	1.23
25	1.155	35	1.119	45	1.097
55	1.083	65	1.073	75	1.065
85	1.059	95	1.053	145	1.038

with

$$f(t) = \frac{e^{e^t}}{(a+t)^k (b+t)^\ell (c+t)^m},$$

where $c \le b \le a < -1$; $k \in \mathbb{N}$, $\ell, m \in \mathbb{N}_0$.

Under the assumption that f is analytic inside $\mathcal{E}_{\rho_{\max}}$, from (1.7) we obtain the error bound

$$|R_n(f)| \le \tilde{r}_n(f), \tag{3.2}$$

where

$$\tilde{r}_n(f) = \inf_{\rho_n^* < \rho < \rho_{\max}} \left[\frac{\ell(\mathcal{E}_\rho)}{2\pi} \left(\max_{z \in \mathcal{E}_\rho} |K_n(z)| \right) \left(\max_{z \in \mathcal{E}_\rho} |f(z)| \right) \right],$$

and ρ_n^* is defined by Theorem 2.2. In the case under consideration, $|a| = \frac{1}{2}(\rho_{\max} + \rho_{\max}^{-1})$.

The length of the ellipse $\mathcal{E}_\rho$ can be estimated by (see [6, Eq. (2.2)])

$$\ell(\mathcal{E}_\rho) \le 2\pi a_1 \left(1 - \frac{1}{4} a_1^{-2} - \frac{3}{64} a_1^{-4} - \frac{5}{256} a_1^{-6} \right), \tag{3.3}$$

where $a_1 = (\rho + \rho^{-1})/2$.

For $z \in \mathcal{E}_\rho$, we have

$$e^{e^z} = e^{e^{a_1 \cos\theta} \cdot \cos\left(\frac{1}{2}(\rho - \rho^{-1}) \sin\theta\right)} \cdot e^{i e^{a_1 \cos\theta} \cdot \sin\left(\frac{1}{2}(\rho - \rho^{-1}) \sin\theta\right)},$$

and from this it follows that

$$\max_{z \in \mathcal{E}_\rho} |e^{e^z}| = e^{e^{a_1}}. \tag{3.4}$$

The above maximum is attained at $\theta = 0$.

Further, we have

$$\frac{1}{|a+z|} = \frac{1}{\sqrt{a^2 + \frac{1}{2}(a_2 - 1) + 2 a a_1 \cos\theta + \cos^2\theta}} \le \frac{1}{|a + a_1|},$$

where the equality holds for $\theta = 0$. We have used the facts that the function under the squared root has minimum at $\theta = 0$ and $a_2 = 2a_1^2 - 1$.

On the basis of the above analysis and (3.4), we have

$$\max_{z \in \mathcal{E}_\rho} \left| \frac{e^{e^z}}{(a+z)^k (b+z)^\ell (c+z)^m} \right| = \frac{e^{e^{a_1}}}{|a+a_1|^k |b+a_1|^\ell |c+a_1|^m},$$

where the maximum is attained at $\theta = 0$. Now, $r_n(f)$ $(\ge \tilde{r}_n(f))$ has the form

$$r_n(f) = \inf_{\rho_n^* < \rho < \rho_{\max}} \left\{ \pi a_1 \left(1 - \frac{1}{4} a_1^{-2} - \frac{3}{64} a_1^{-4} - \frac{5}{256} a_1^{-6} \right) \frac{e^{e^{a_1}}}{|a+a_1|^k |b+a_1|^\ell |c+a_1|^m} \right.$$
$$\times \frac{(1+\gamma)^2 (a_2+1)^{1/2} \left(1 + \gamma \rho^2\right)}{\rho^{n+1} \left[(1+\gamma^2)^2 + 4\gamma(1+\gamma^2) a_2 + 2\gamma^2 (a_4 + 1)\right]^{1/2}}$$
$$\left. \times \left[a_{2n+2} + (-1)^n + \gamma^2 (a_{2n-2} + (-1)^n) + 2\gamma (a_{2n} + (-1)^n a_2) \right]^{-1/2} \right\}.$$

Table 3.3
The values of $\hat{r}_n^{(\mathrm{Sten})}(f)$, $\hat{r}_n^{(\mathrm{Syd})}(f)$, $\hat{r}_n^{(\mathrm{Not})}(f)$, $r_n(f)$ for $n = 15, 35$ and some $\gamma \in (-1, 0)$.

γ	$\hat{r}_{15}^{(\mathrm{Sten})}(f)$	$\hat{r}_{15}^{(\mathrm{Syd})}(f)$	$\hat{r}_{15}^{(\mathrm{Not})}(f)$	$r_{15}(f)$	$\hat{r}_{35}^{(\mathrm{Sten})}(f)$	$\hat{r}_{35}^{(\mathrm{Syd})}(f)$	$\hat{r}_{35}^{(\mathrm{Not})}(f)$	$r_{35}(f)$
−0.1	1.63(−6)	1.61(−6)	2.64(−7)	3.66(−7)	5.04(−21)	4.84(−21)	8.31(−22)	1.11(−21)
−0.2	1.45(−6)	1.43(−6)	2.01(−7)	2.30(−7)	4.48(−21)	4.30(−21)	6.34(−22)	9.04(−22)
−0.3	1.27(−6)	1.26(−6)	1.48(−7)	2.41(−7)	3.92(−21)	3.76(−21)	4.69(−22)	7.19(−22)
−0.4	1.09(−6)	1.08(−6)	1.05(−7)	1.86(−7)	3.36(−21)	3.23(−21)	3.33(−22)	5.49(−22)
−0.5	9.04(−7)	8.93(−7)	6.97(−8)	1.35(−7)	2.80(−21)	2.69(−21)	2.24(−22)	3.97(−22)
−0.6	7.23(−7)	7.15(−7)	4.30(−8)	9.01(−8)	2.24(−21)	2.15(−21)	1.39(−22)	2.64(−22)
−0.7	5.42(−7)	5.36(−7)	2.34(−8)	5.32(−8)	1.68(−21)	1.62(−21)	7.54(−23)	1.55(−22)
−0.8	3.62(−7)	3.58(−7)	9.99(−9)	2.48(−8)	1.12(−21)	1.08(−21)	3.25(−23)	7.17(−23)
−0.9	1.81(−7)	1.79(−7)	2.42(−9)	6.51(−9)	5.60(−22)	5.37(−22)	7.87(−24)	1.88(−23)

Let $-\sqrt{2} < a < -1$, $c \le b \le a$. This condition means that the function f is analytic inside the elliptical contour $\mathcal{E}_{\rho_{\max}}$, where $\rho_{\max} = 1 + \sqrt{2}$. Therefore, the results obtained by Schira [5] cannot be used here. Also, the classical error bound in this case is difficult to determine, since the derivatives $f^{(2n)}(t)$ for higher values n are too complicated to be handled. However, we can use the error bound (3.2) based on the results of Theorem 2.2.

The error bound (3.2) is valid for integrands that are analytic on a neighborhood of the interval of integration and should be compared with other error bounds intended for the same class of integrands. There are several classical error bounds for Gaussian quadrature rules of analytic functions. See Theorem 4 in [7] or Theorem 1 in [8], where the contour Γ is the ellipse $\mathcal{E}_\rho$ given by (1.2). We also take into account the error bounds appearing in [9], where the contour Γ is the circumference $C_r = \{z \in \mathbb{C} : |z| = r\}$ $(r > 1)$.

Therefore, the error bound $\hat{r}_n(f)$ $(|R_n(f)| \le \hat{r}_n(f))$ of the Gauss quadrature formula (1.1) with respect to the weight function (1.3), for the integrand f under consideration, can be given by (see Stenger [7, Eq. (38)])

$$\hat{r}_n(f) = \hat{r}_n^{(\mathrm{Sten})}(f) = \inf_{1<\rho<\rho_{\max}} \left\{ \frac{16\mu_0}{\pi\rho^{2n}} \cdot \frac{e^{e^{a_1}}}{|a+a_1|^k |b+a_1|^\ell |c+a_1|^m} \right\},$$

where $\mu_0 = \pi(1+\gamma)/2$ (cf. [1, Eqs. (2.24),(2.27)]), or by (see von Sydow [8, Th. 1])

$$\hat{r}_n(f) = \hat{r}_n^{(\mathrm{Syd})}(f) = \inf_{1<\rho<\rho_{\max}} \left\{ \frac{4\mu_0}{(1-\rho^{-2})\rho^{2n}} \cdot \frac{e^{e^{a_1}}}{|a+a_1|^k |b+a_1|^\ell |c+a_1|^m} \right\},$$

or by (see Notaris [9, Eq. (3.28)])

$$\hat{r}_n(f) = \hat{r}_n^{(\mathrm{Not})}(f) = \inf_{1<r<r_{\max}} \left\{ \frac{2\pi(1+\gamma)^2\tau^{2n+2} r\sqrt{r^2-1}}{(1-\gamma\tau^2)[1-\tau^{2n+2}-\gamma\tau^2(1-\tau^{2n-2})]} \frac{e^{e^r}}{|a+r|^k |b+r|^\ell |c+r|^m} \right\},$$

where $\tau = r - \sqrt{r^2-1}$ and $r_{\max} = |a|$.

Let the integrand f be specialized by $k = 1$, $\ell = 5$, $m = 10$, and

$$a = -1.408333333333333, \qquad b = -1.892857142857143, \qquad c = -2.408695652173913,$$

which means that $\rho_{\max} = 2.4$.

We have calculated the values of $\hat{r}_n^{(\mathrm{Sten})}(f)$, $\hat{r}_n^{(\mathrm{Syd})}(f)$, $\hat{r}_n^{(\mathrm{Not})}(f)$, $r_n(f)$ for the corresponding integral $I(f)$ given by (3.1). The results show the effectiveness of the error bound (3.2) compared to, for instance, the error bounds given by $\hat{r}_n^{(\mathrm{Sten})}(f)$, $\hat{r}_n^{(\mathrm{Syd})}(f)$. For some values of γ and $n = 15, 35$, the results obtained are displayed in Table 3.3. (Numbers in parentheses indicate decimal exponents.)

Finally, let us consider numerical calculation of the integral (3.1), with

$$f(t) = \bar{f}(t) = e^{\cos t}.$$

The function $\bar{f}(z) = e^{\cos z}$ is entire, and it is easy to see that

$$\max_{z\in C_r} \left|e^{\cos z}\right| = e^{\cosh(r)} \quad \text{and} \quad \max_{z\in\mathcal{E}_\rho} \left|e^{\cos z}\right| = e^{\cosh(b_1)},$$

where $b_1 = \frac{1}{2}(\rho - \rho^{-1})$.

For some values of γ, the results obtained for $\hat{r}_9^{(\mathrm{Sten})}(\bar{f})$, $\hat{r}_9^{(\mathrm{Syd})}(\bar{f})$, $\hat{r}_9^{(\mathrm{Not})}(\bar{f})$, $r_9(\bar{f})$ and the actual error are displayed in Table 3.4. The true value of the integral was evaluated by the Gauss–Chebyshev quadrature formula of the second kind.

Table 3.4
The values of $\hat{r}_9^{(\text{Sten})}(\bar{f})$, $\hat{r}_9^{(\text{Syd})}(\bar{f})$, $\hat{r}_9^{(\text{Not})}(\bar{f})$, $r_9(\bar{f})$ and the actual error for some $\gamma \in (-1, 0)$.

γ	$\hat{r}_9^{(\text{Sten})}(\bar{f})$	$\hat{r}_9^{(\text{Syd})}(\bar{f})$	$\hat{r}_9^{(\text{Not})}(\bar{f})$	$r_9(\bar{f})$	Error
−0.1	4.300(−10)	3.500(−10)	2.788(−10)	7.923(−11)	7.787(−12)
−0.2	3.822(−10)	3.111(−10)	2.187(−10)	6.305(−11)	6.191(−12)
−0.3	3.345(−10)	2.722(−10)	1.662(−10)	4.861(−11)	4.773(−12)
−0.4	2.867(−10)	2.334(−10)	1.212(−10)	3.597(−11)	3.524(−12)
−0.5	2.389(−10)	1.945(−10)	8.353(−11)	2.516(−11)	2.468(−12)
−0.6	1.911(−10)	1.556(−10)	5.307(−11)	1.622(−11)	1.588(−12)
−0.7	1.434(−10)	1.167(−10)	2.964(−11)	9.187(−12)	8.993(−13)
−0.8	9.555(−11)	7.778(−11)	1.308(−11)	4.113(−12)	4.021(−13)
−0.9	4.778(−11)	3.889(−11)	3.245(−12)	1.036(−12)	1.015(−13)

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