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Analytically unramified one-dimensional semilocal rings and their value semigroups

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Abstract

In a one-dimensional local ring R with finite integral closure each nonzerodivisor has a value in $\mathbb{N}^d$, where d is the number of maximal ideals in the integral closure. The set of values constitutes a semigroup, the value semigroup of R. We investigate the connection between the value semigroup and the ring. There is a particularly close connection for some classes of rings, e.g. Gorenstein rings, Arf rings, and rings of small multiplicity. In many respects, the Arf rings and the Gorenstein rings turn out to be opposite extremes. We give applications to overrings, intersection numbers, and multiplicity sequences in the blow-up sequences studied by Lipman. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

A one-dimensional local domain R is analytically irreducible, i.e., the completion $\hat{R}$ is a domain, if and only if the integral closure $\bar{R}$ of R is a DVR, finite over R. In this case, since $\bar{R}$ is a DVR, every nonzero element of R has a value in $\mathbb{N}$, and the set of values $v(R)$ constitutes a numerical semigroup, i.e., an additive submonoid of $\mathbb{N}$ with finite complement to $\mathbb{N}$.

For a one-dimensional reduced Noetherian local ring R, its integral closure $\bar{R}$ is finite over R if and only if the completion $\hat{R}$ is reduced (cf. [12, Ch. 10]). Such a ring is called *analytically unramified* and these rings are the basic objects of our study in this paper. An important class of examples of such rings are the local rings of an algebraic

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curve. For analytically unramified rings the set of values of nonzerodivisors constitutes a subsemigroup S of $\mathbb{N}^d$, where d is the number of maximal ideals of the integral closure (cf. [7]).

These semigroups satisfy some fundamental conditions that follow from their definition by valuations. We develop a theory on subsemigroups of $\mathbb{N}^d$ satisfying these conditions and we call them good semigroups. Although the theory of these good semigroups is parallel to that of rings, the value semigroup of rings are not characterized as being good.

Some of our constructions lead us to semilocal rings and to the corresponding semigroups. Thus, we have found it natural to consider semilocal rings and their value semigroups throughout the paper.

Since semigroups are simpler objects than rings, we want to see what a semigroup of a ring can reveal about the ring. The connection between rings and their value semigroups is particularly strict for some special classes of rings, like almost Gorenstein, and Arf rings. We also show this strict relation between rings and their semigroups when the multiplicity is small. In this case the class of value semigroups of rings coincides with the class of good semigroups.

A sequence of overrings consisting of consecutive blowing-ups from R to $\bar{R}$ is studied by Lipman in [11]. We construct a "dual" sequence from $\bar{R}$ to R, giving the Lipman sequence in reverse order when R is an Arf ring. Considering the localizations of the rings in the Lipman sequence, their multiplicities give precise geometric information about the singularity of R, when R is local. Using the connection with the semigroups, we are able to give a numerical characterization of the multiplicity sequences of any semilocal ring R.

1.1. Preliminaries

Let R be a Noetherian, reduced, one-dimensional, semilocal ring such that its integral closure $\bar{R}$ in the total ring of fractions $Q(R)$ of R is a finite R-module. If $P_1,\dots,P_n$ are the minimal primes of R, then $\bar{R}\simeq\prod_{i=1}^{n}\overline{R/P_i}$, where $\overline{R/P_i}$ is the integral closure of R/P_i in its quotient field $Q(R/P_i)$. Since $\overline{R/P_i}$ is a one-dimensional, Noetherian, integrally closed domain, it is the intersection of the DVRs obtained by localizing $\overline{R/P_i}$ at its maximal ideals. Let $\overline{R/P_i}=V_{i,1}\cap\cdots\cap V_{i,h_i}$ and denote by $v_{i,j}$ the valuation associated to $V_{i,j}$. For each $r\in Q(R)\simeq Q(R/P_1)\times\cdots\times Q(R/P_n)$ we can consider the *value* of r, $v(r)=(v_{1,1}(r),\dots,v_{n,h_n}(r))$. The set $v(R)=\{v(r)\mid r\in R, r \text{ nonzerodivisor}\}$ is a subset of $\mathbb{N}^H$, $H=\sum_{i=1}^{n}h_i$. In the same way the ring R_{m_i} (where $m_1,\dots,m_r$ are the maximal ideals of R) has a value semigroup $v_i(R_{m_i})$. The limit cases of this general situation are when R is a domain, i.e., when there is only one minimal prime, and when each R/P_i is analytically irreducible, i.e., when each $\overline{R/P_i}$ is a DVR. In the first case $R=\bigcap_{i=1}^{r}R_{m_i}$ and in the second case $R=\prod_{i=1}^{r}R_{m_i}$ (as we show in Proposition 3.1), but we want to notice that, always in our hypotheses $v(R)=\prod_{i=1}^{r}v_i(R_{m_i})$. It is clear that $v(R)\subseteq\prod_{i=1}^{r}v_i(R_{m_i})$. Conversely, if $(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_r)\in\prod_{i=1}^{r}v_i(R_{m_i})$ ($\boldsymbol{\alpha}_i$ is in general a vector), then there exist elements $x_1,\dots,x_r\in R$ such that $v_i(x_i)=\boldsymbol{\alpha}_i$ in R_{m_i}. For each

i pick an element $y_i \in (\bigcap_{j \neq i} m_j) \backslash m_i$. Then $v_i(y_i) = \mathbf{0}$ in R_{m_i} and $v_j(y_i)$ has only positive coordinates if $j \neq i$. Hence, taking sufficiently large powers of y_i, the element $\sum_{i=1}^{r} x_i y_i^{N_i} \in R$ has value $(\boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_r)$.

We will all through the paper assume that R is residually rational, i.e., that all localizations at maximal ideals of R and $\bar{R}$ have the same residue field, and we will assume that the cardinality of this field is at least equal to the number of maximal ideals of $\bar{R}$. This *is* a restriction, but it is necessary to get a close correspondance between R and its semigroup of values $v(R)$ (and it is fulfilled e.g. if R is local with algebraically closed residue field).

In order to get simpler notation and to make the arguments easier, we will assume the following additional hypothesis: $\bar{R}$ is a product of DVRs. This assumption is motivated by the following remarks. If we consider the completion $\hat{R}$ of R with respect to its Jacobson radical, then $\hat{R} = \widehat{R_{m_1}} \times \cdots \times \widehat{R_{m_r}}$, cf. [13, Theorem 17.7]. Hence $\hat{R}$ is a one-dimensional, Noetherian, semilocal ring and, by the fact that $\bar{R}$ is finite over R, it follows that $\hat{R}$ is reduced. More precisely $\bar{R}$ finite over R implies that $\overline{R_{m_i}}$ is finite over R_{m_i}, and this fact is equivalent to the fact that $\widehat{R_{m_i}}$ is reduced, cf. [12, Theorem 10.2], and therefore $\hat{R}$ is reduced. Now the integral closure of $\hat{R}$ in its total quotient ring is a product of DVRs, since this is true for each $\widehat{R_{m_i}}$, cf. e.g. [8, Section 1]. Moreover, in the local case we have also $v_i(R_{m_i}) = \hat{v}_i(\widehat{R_{m_i}})$, cf. e.g. [8, Section 1]. It follows that $v(R) = \prod_{i=1}^{r} v_i(R_{m_i}) = \prod_{i=1}^{r} \hat{v}_i(\widehat{R_{m_i}}) = \hat{v}(\hat{R})$. If $\bar{R}$ is a product of DVRs we can use the fact that R is a product of local rings (cf. Theorem 5.11) to simplify our arguments, but all other results are true without this assumption.

We repeat our standing hypotheses and fix the following notation. In this paper we study semilocal, one-dimensional, residually rational rings with not too small residue field, with d minimal primes, with finite integral closure which is a product of DVRs, $\bar{R} = V_1 \times \cdots \times V_d$, and with value semigroup $S = v(R) \subseteq \mathbb{N}^d$.

1.2. Description of the contents

In Section 2 we produce a theory for good semigroups, i.e., the semigroups satisfying the conditions listed in Proposition 2.1. Since we are interested in semilocal rings, our setting, also at the semigroup level, is a bit more general than in previous works. However it is natural to define local semigroups and we show in Theorem 2.5 that any good semigroup is a direct product of local ones. Using this result, it is possible to use the $d(\_\backslash\_)$ function as in [8], which translates to the numerical level the length function for R-modules, and to reduce computations to the local case (cf. Proposition 2.12). In Example 2.16, we give an example of a good semigroup which is not the value semigroup of a ring.

In Sections 3 and 4 we investigate what can be said about the ring from its semigroup of values. As we have noticed in Section 1.1, the semigroup of values of a semilocal ring R is the direct product of the corresponding semigroups of the localizations of R at its maximal ideals. Section 3 begins with the more precise result that, under the additional hypothesis that $\bar{R}$ is a product of DVRs, the semilocal ring itself is the direct product of its localizations. We then show that, for the connection between the ring and its semigroup, we get satisfying results for three classes of rings. Almost Gorenstein rings and rings of maximal embedding dimension (in particular Arf rings) are characterized in terms of their value semigroups in both the local (Propositions 3.7 and 3.19, Corollary 3.15) and the semilocal case. The rings of small multiplicity are treated in Section 4. We classify completely the good semigroups of multiplicity less than or equal to 3 (cf. Proposition 4.1 and Theorem 4.6) and we show that, in this case, the class of good semigroups coincides with the class of value semigroups of rings (cf. Example 4.12).

In Section 5 we construct the sequence of overrings dual to the Lipman sequence for an Arf ring, make a similar construction for semigroups, and study multiplicity sequences. We show that any Arf ring (semigroup, resp.) can be obtained by our constructions, cf. Theorem 5.5(ii) (Theorem 5.7(ii), resp.). As a consequence, we obtain the result that any Arf semigroup is the value semigroup of an Arf ring (cf. Corollary 5.8). We introduce the concept of multiplicity forest of a ring. The multiplicity forest of a ring R contains much information about the ring, e.g. all its multiplicity sequences along the branches and also the conductor $R:\bar{R}$. Moreover the multiplicity forest of a ring R is the same as the multiplicity forest of R', the smallest Arf overring of R (cf. Proposition 5.3) and describes completely the semigroup $v(R')$. By this non severe restriction to the Arf case, all possible multiplicity forests of rings are characterized (cf. Theorem 5.11). As an example we give all possible multiplicity forests (in this case trees) for local rings of multiplicity 3 (cf. Example 5.16).

2. Generalities on semigroups

In this section we want to see what are the properties of the class of semigroups, we are interested in, that we can conclude intrinsically. The semigroup of values of a ring has been studied by many authors, and many results can be found in [7]. Since they study rings which are not necessarily residually rational, we prefer to refer to [8] for precise known results, even if some can be found in [7]. The scope in [8] is semigroups of *local* rings. It turns out that the semilocal case is not much different, and we will refer to [8] even in the more general case if the proof is identical. Let $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d)$ denote an element in $\mathbb{N}^d$. If $\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}^d$, we set $\boldsymbol{\alpha}\geq\boldsymbol{\beta}$ if $\alpha_i\geq\beta_i$ for all i, and $\boldsymbol{\alpha}>\boldsymbol{\beta}$ if $\boldsymbol{\alpha}\geq\boldsymbol{\beta}$ and $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$.

The set of values of a semilocal ring R constitutes a subsemigroup containing 0 of $\mathbb{N}^d$ which satisfies some good properties, cf. [7, 8].

Proposition 2.1. *Let* $S=v(R)\subseteq\mathbb{N}^d$. *Then*

(1) *If* $\boldsymbol{\alpha},\boldsymbol{\beta}\in S$, *then* $\min(\boldsymbol{\alpha},\boldsymbol{\beta})=(\min\{\alpha_1,\beta_1\},\dots,\min\{\alpha_d,\beta_d\})\in S$.

(2) *If* $\boldsymbol{\alpha},\boldsymbol{\beta}\in S$, $\boldsymbol{\alpha}\neq\boldsymbol{\beta}$ *and* $\alpha_i=\beta_i$ *for some* $i\in\{1,\dots,d\}$, *then there exists* $\boldsymbol{\epsilon}\in S$ *such that* $\varepsilon_i>\alpha_i=\beta_i$ *and* $\varepsilon_j\geq\min\{\alpha_j,\beta_j\}$ *for each* $j\neq i$ (*and if* $\alpha_j\neq\beta_j$, *the equality holds*).

(3) *There exists* $\boldsymbol{\delta}\in\mathbb{N}^d$ *such that* $S\supseteq\boldsymbol{\delta}+\mathbb{N}^d$.

A subsemigroup of some $\mathbb{N}^d$ that satisfies (1)–(3) above will be called a *good semigroup*. Notice that conditions (1) and (2) vanish when $d=1$.

First we note that the class of good semigroups is closed under projection. The proof is immediate.

Proposition 2.2. *Let $S\subseteq\mathbb{N}^d$ be a good semigroup. If $I=\{i_1,\dots,i_l\}\subseteq\{1,\dots,d\}$, then $\pi_I(S)=\{(\alpha_{i_1},\dots,\alpha_{i_l})\in\mathbb{N}^l \mid$ there is a $\boldsymbol{\beta}\in S$ such that $\beta_{i_j}=\alpha_{i_j}$ for $j=1,\dots,l\}$ is a good semigroup.*

Also the following statement is easily proved:

Proposition 2.3. *If $S=S_1\times\cdots\times S_r$ is a semigroup, then S is good if and only if all S_i are good.*

Let S be a good semigroup. If $E\subseteq\mathbb{Z}^d$ is such that $E+S\subseteq S$ and $\boldsymbol{\alpha}+E\subseteq S$ for some $\boldsymbol{\alpha}\in S$, then E is called a *relative ideal* of S. For any $\boldsymbol{\alpha}\in\mathbb{Z}^d$ we denote by $\Delta(\boldsymbol{\alpha})=\{\beta\in\mathbb{Z}^d \mid \beta_i=\alpha_i$ for some $i\in\{1,\dots,d\}$ and $\beta_j>\alpha_j$ if $j\neq i\}$ (notice that if $d=1$, $\Delta(\alpha)=\{\alpha\}$). If E is a relative ideal of S we set $\Delta^E(\boldsymbol{\alpha})=\Delta(\boldsymbol{\alpha})\cap E$. If E,F are relative ideals of S we set $E-_{\mathbb{Z}^d}F$ (or simply $E-F$) $=\{\boldsymbol{\alpha}\in\mathbb{Z}^d \mid \boldsymbol{\alpha}+F\subseteq E\}$. This last is also a relative ideal of S. If $\boldsymbol{\delta}$ is the minimum element satisfying $\boldsymbol{\delta}+\mathbb{N}^d\subseteq S$, cf. Proposition 2.1(3), (such an element exists by Proposition 2.1(1)) then we will denote $\boldsymbol{\delta}+\mathbb{N}^d$ by $C(S)$ and call it the conductor of S. We have $C(S)=S-\mathbb{N}^d$. The element $\boldsymbol{\gamma}=\boldsymbol{\delta}-(1,\dots,1)$ is called the *Frobenius vector* of S. This vector will play an important role in the theory of good semigroups.

Lemma 2.4. *Let $S\subseteq\mathbb{N}^d$ be a good semigroup and let $I=\{i_1,\dots,i_l\}\subseteq\{1,\dots,d\}$. We have*:

(i) *If $\boldsymbol{\alpha}\in S$ and $\alpha_i>\gamma_i$ if $i\in I$, then every element ϵ of $\mathbb{N}^d$ such that $\varepsilon_i>\gamma_i$ if $i\in I$ and $\varepsilon_h=\alpha_h$ for each $h\notin I$ is an element of S.*

(ii) *$\Delta^S(\boldsymbol{\gamma})=\emptyset$.*

Proof. (i) This follows from properties (1) and (2) of good semigroups.

(ii) This follows from (i) and from the fact that $\boldsymbol{\gamma}+(1,\dots,1)$ is the minimum element such that $S\supseteq\boldsymbol{\gamma}+(1,\dots,1)+\mathbb{N}^d$.

A good semigroup S is said to be *local* if $\mathbf{0}$ is the only element in S which has some coordinate equal to 0. We have chosen this name because if R is a ring, then $v(R)$ is local if and only if R is local. □

Theorem 2.5. *Every good semigroup is a direct product of good local semigroups.*

Proof. Consider the subsets $A\subseteq\{1,\dots,d\}$ with the following property: (P) There is an $\boldsymbol{\alpha}\in S$ such that $\alpha_j=0$ if $j\notin A$ and $\alpha_j\neq 0$ if $j\in A$.

Denote the minimal nonempty subsets verifying (P) by $A_1,\dots,A_r$. We claim that $A_1,\dots,A_r$ is a partition of $\{1,\dots,d\}$. If $A_i\cap A_j\neq\emptyset$, then choose $\boldsymbol{\alpha}\in S$ such that $\alpha_h=0$

if $h \notin A_i$ and $\alpha_h \neq 0$ if $h \in A_i$, and $\boldsymbol{\beta} \in S$ such that $\beta_h = 0$ if $h \notin A_j$, $\beta_h \neq 0$ if $h \in A_j$. Since $\boldsymbol{\delta} = \min(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S$ and $\delta_h = 0$ if $h \notin A_i \cap A_j$, $\delta_h \neq 0$ if $h \in A_i \cap A_j$, we get by the minimality of A_i and A_j that $A_i = A_i \cap A_j = A_j$. To prove that $\bigcup_{i=1}^{r} A_i = \{1,\dots,d\}$, choose $\boldsymbol{\alpha}^i \in S$, such that $\alpha_j^i = 0$ if $j \notin A_i$, $\alpha_j^i \neq 0$ if $j \in A_i$. Assume that $\bigcup_{i=1}^{l} A_i \neq \{1,\dots,d\}$ and let h be an index such that $h \in \{1,\dots,d\} \setminus \bigcup_{i=1}^{r} A_i$. Applying property (2) of good semigroups to the elements $\mathbf{0}$ and $\boldsymbol{\alpha} = \boldsymbol{\alpha}^1 + \cdots + \boldsymbol{\alpha}^l$, with respect to the h'th coordinate, we get that there exists $\boldsymbol{\beta} \in S$ such that $\beta_j = 0$ if $j \in \bigcup_{i=1}^{r} A_i$ and $\beta_h > 0$. Set $B = \{i \in \{1,\dots,d\} \mid \beta_i \neq 0\}$. Then $B \cap A_j = \emptyset$ for $j = 1,\dots,r$. It follows that there exists a minimal subset of $\{1,\dots,d\}$ veryfying (P) and different from $A_1,\dots,A_r$, which is a contradiction. We have proved the claim.

Now, by Proposition 2.2 the projections S_{A_i} are good semigroups, and by minimality of A_i they are local. Clearly $S \subseteq S_{A_1} \times \cdots \times S_{A_r}$. Let $T_{A_j} = \{\pi_{A_j}(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in S, \alpha_h = 0$ if $h \notin A_j\}$. Since S is a semigroup we have $T_{A_1} \times \cdots \times T_{A_r} \subseteq S$. Now let $\pi_{A_j}(\boldsymbol{\alpha})$ be an element of S_{A_j}. By definition of A_j we can find an element $\boldsymbol{\beta} \in S$ such that $\beta_i = 0$ if $i \notin A_j$, $\beta_i > 0$ if $i \in A_j$; hence $\pi(\boldsymbol{\beta}) \in T_{A_j}$. By replacing $\boldsymbol{\beta}$ with a large multiple of $\boldsymbol{\beta}$ we get $\beta_i = 0$ if $i \notin A_j$, $\beta_i > \alpha_i$ if $i \in A_j$. Then $\boldsymbol{\epsilon} = \min(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in S$ and $\varepsilon_i = 0$ if $i \notin A_j$, $\varepsilon_i = \alpha_i$ if $i \in A_j$. Thus $\pi_{A_j}(\boldsymbol{\alpha}) \in T_{A_j}$ so $S_{A_j} = T_{A_j}$ and $S = S_{A_1} \times \cdots \times S_{A_r}$. $\square$

Remark 2.6. The representation of a good semigroup as a product of good local semigroups is unique. Since the local components of S determine a partition $A_1,\dots,A_r$ of $\{1,\dots,d\}$, we will denote this unique representation by $S = S_{A_1} \times \cdots \times S_{A_r}$, when we want to emphasize this partition.

2.1. Relative ideals and the function $d(\_\backslash\_)$

In the following, S will always be a good semigroup. A relative ideal of S need not satisfy the properties (1) and (2) of good semigroups (cf. Proposition 2.1), as the following example shows. (However (3) is always satisfied.)

Example 2.7. Let $S = \{(0,0)\} \cup ((1,1) + \mathbb{N}^2)$. The set $E = \{(1,1)\} \cup \{(1,n) \mid n \geq 3\} \cup ((2,2) + \mathbb{N}^2)$ is a relative ideal of S. But $(1,3),(2,2) \in E$ and $(1,2) = \min((1,3),(2,2)) \notin E$. Moreover $(1,1) \in E$ and $(1,3) \in E$, but $(m,1) \notin E$ for every $m \geq 2$.

A relative ideal E that does satisfy properties (1) and (2) will be called a *good relative ideal*. If $E = v(I)$ is the value set of a fractional ideal I of a ring R, then E is a good relative ideal. By this fact it follows that it is possible to compute $l_R(I/J)$, where $I \supseteq J$ are fractional ideals of R, in terms of the sets of values $v(I)$ and $v(J)$, cf. [7, 8]. More precisely, we have (cf. [8, Proposition 2.3]):

Proposition 2.8. *If E is a good relative ideal and $\boldsymbol{\alpha}, \boldsymbol{\beta} \in E$, $\boldsymbol{\alpha} < \boldsymbol{\beta}$, then all saturated chains $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(0)} < \boldsymbol{\alpha}^{(1)} < \cdots < \boldsymbol{\alpha}^{(n)} = \boldsymbol{\beta}, \boldsymbol{\alpha}^{(i)} \in E$, have the same length. (A chain $\boldsymbol{\alpha} = \boldsymbol{\alpha}^{(0)} < \boldsymbol{\alpha}^{(1)} < \cdots < \boldsymbol{\alpha}^{(n)} = \boldsymbol{\beta}$ is called saturated if it cannot be extended to a longer chain between $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in E.)*

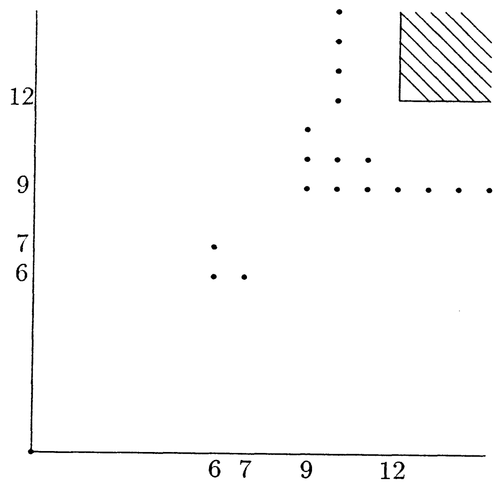

Fig. 1. A picture of S.

There are relative ideals E which are not good, but satisfy the conclusion of Proposition 2.8.

Example 2.9. Let $S=\{(0,0)\}\cup((1,1)+\mathbb{N}^2)$ and $E=\{(1,1)\}\cup((1,2)+\mathbb{N}^2)$. Then E satisfies the conclusion of Proposition 2.8, but clearly E does not satisfy (2).

Notice also that, if E,F are good relative ideals of S, the relative ideal $E-F$ is not necessarily good.

Example 2.10. Let S be the semigroup consisting of the dots and let $C(S)=\{\boldsymbol{\alpha}\in\mathbb{Z}^2 \mid \boldsymbol{\alpha} \geq (12,12)\}$ in Fig. 1. It is easily checked that S is a good semigroup and the subset $M=S\backslash\{\mathbf{0}\}$ is a good relative ideal of S, but $S-M$ (that is depicted in Fig. 2) is not a good relative ideal of S, because for $\boldsymbol{\alpha}=(4,3)$ and $\boldsymbol{\beta}=(5,3)$, property (2) of Proposition 2.1 does not hold.

If E is a good relative ideal of S and if $\boldsymbol{\alpha},\boldsymbol{\beta}\in E$, $\boldsymbol{\alpha}<\boldsymbol{\beta}$, we let $d_E(\boldsymbol{\alpha},\boldsymbol{\beta})$ denote the common length of a saturated chain in E from $\boldsymbol{\alpha}$ to $\boldsymbol{\beta}$. If $\boldsymbol{\alpha}=\boldsymbol{\beta}$ we set $d_E(\boldsymbol{\alpha},\boldsymbol{\beta})=0$. If $E\supseteq F$ are good relative ideals and $\mathbf{m}_E,\mathbf{m}_F$ are the minimal elements in E and F respectively, then for any sufficiently large $\boldsymbol{\alpha}\in F$ (it suffices to take $\boldsymbol{\alpha}\in F-_{\mathbb{Z}^d}\mathbb{N}^d$) we set $d(E\backslash F)=d_E(\mathbf{m}_E,\boldsymbol{\alpha})-d_F(\mathbf{m}_F,\boldsymbol{\alpha})$. It is shown in [8] that this definition is independent of the choice of $\boldsymbol{\alpha}$. The function $d(\_\backslash\_)$ has the following properties, cf. [8, Proposition 2.7, Corollary 2.5].

Proposition 2.11. (i) *If $E\supseteq F\supseteq G$ are good relative ideals of S, then we have $d(E\backslash G)=d(E\backslash F)+d(F\backslash G)$.*

(ii) *If $E\supseteq F$ are good relative ideals of S, then $d(E\backslash F)=0$ if and only if $E=F$.*

(iii) *If R is a ring, $I\supseteq J$ fractional ideals of R, then $l_R(I/J)=d(v(I)\backslash v(J))$.*

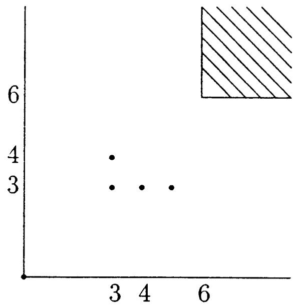

Fig. 2. A picture of $S - M$.

The following proposition shows that in many cases we can restrict ourselves to local semigroups.

Proposition 2.12. *If $S = S_{A_1} \times \cdots \times S_{A_r}$ is the representation of S in its local components and if E is a good relative ideal of S, then, if $E_{A_i} = \pi_{A_i}(E)$, we have $E = E_{A_1} \times \cdots \times E_{A_r}$. Moreover*

(i) *$\pi_{A_i}(E)$ is a good relative ideal of S_{A_i}.*

(ii) *If $\boldsymbol{\alpha}, \boldsymbol{\beta} \in E$, $\boldsymbol{\alpha} < \boldsymbol{\beta}$, then $d_E(\boldsymbol{\alpha}, \boldsymbol{\beta}) = \sum_{i=1}^r d_{E_{A_i}}(\pi_{A_i}(\boldsymbol{\alpha}), \pi_{A_i}(\boldsymbol{\beta}))$.*

(iii) *If $E \supseteq F$ are good relative ideals of S, $d(E \backslash F) = \sum_{i=1}^r d(E_{A_i} \backslash F_{A_i})$.*

Proof. Without loss of generality, we can assume that $\mathbf{0}$ is the smallest element in E. Obviously, $E \subseteq E_{A_1} \times \cdots \times E_{A_r}$. Let $G_{A_i} = \{\pi_{A_i}(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in E, \alpha_h = 0 \text{ if } h \notin A_i\}$. Since E is a relative ideal of S we have $E \supseteq S_{A_1} \times \cdots \times S_{A_{i-1}} \times G_{A_i} \times S_{A_{i+1}} \times \cdots \times S_{A_r}$, for each $i = 1, \ldots, r$. It follows, by property (1) of good relative ideals, that $E \supseteq G_{A_1} \times \cdots \times G_{A_r}$. Now, as in the proof of Theorem 2.5, we can conclude that $G_{A_i} = E_{A_i}$ and that $E = E_{A_1} \times \cdots \times E_{A_r}$. The remaining statements are easy to prove. □

The following lemma gives a method to compute the function $d(\_\backslash\_)$, that we will use later (cf. [8, Corollary 2.6]).

Lemma 2.13. *Let E be a good relative ideal of S, $\boldsymbol{\alpha} \in \mathbb{Z}^d$, and set $E(\boldsymbol{\alpha}) = \{\boldsymbol{\beta} \in E \mid \boldsymbol{\beta} \geq \boldsymbol{\alpha}\}$. If $\boldsymbol{\alpha}^i = \boldsymbol{\alpha} + \mathbf{e}^i$, where $e_j^i = 0$ if $j \neq i$ and $e_i^i = 1$, then $d(E(\boldsymbol{\alpha}) \backslash E(\boldsymbol{\alpha}^i)) \leq 1$ with equality if and only if $\{\boldsymbol{\beta} \in E \mid \beta_i = \alpha_i$ and $\beta_j \geq \alpha_j$, if $j \neq i\} \neq \emptyset$.*

2.2. The canonical relative ideal K

The *canonical ideal* $K = K(S)$ of S is defined as $\{\boldsymbol{\alpha} \in \mathbb{Z}^d \mid \Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset\}$, cf. [8]. The following is shown in [8], cf. [8, Proposition 3.2, Theorems 3.6, 4.1] in the local case. Notice that, in our hypotheses, it is always possible to find a canonical ideal between R and $\bar{R}$.

Proposition 2.14. (i) *K is a good relative ideal of S.*
(ii) $\Delta^K(\gamma)=\emptyset$.
(iii) *If S is local, then $\gamma\in K$.*
(iv) *If R is a ring and ω a fractional ideal of R, $R\subseteq\omega\subseteq\bar{R}$, then ω is a canonical ideal of R if and only if $v(\omega)=K$.*

Also here we can restrict to local semigroups. The proof of the following proposition is immediate.

Proposition 2.15. *If $S=S_{A_1}\times\cdots\times S_{A_r}$ is the representation of S in its local components, then $K(S)=K(S_{A_1})\times\cdots\times K(S_{A_r})$.*

We now give an example to show that not all good semigroups are semigroups of rings.

Example 2.16. Let S be the semigroup consisting of the dots and $C(S)=\{\boldsymbol{\alpha}\in\mathbb{Z}^2\mid\boldsymbol{\alpha}\geq(25,27)\}$ in Fig. 3. It is easily checked that S is a good semigroup. The semigroup ideal K consists of S and the circles in Fig. 3. We will show that S is not the semigroup of values for any ring. So suppose that $S=v(R)$ for some ring R and fix a canonical ideal ω, $R\subseteq\omega\subseteq\bar{R}$. Then $v(\omega)=K$. Since $l_R(\omega/R)=d(K\backslash S)=d_K(\mathbf{0},\gamma+(1,\dots,1))-d_S(\mathbf{0},\gamma+(1,\dots,1))=27-25=2$, there exists a fractional ideal I of R strictly between R and ω. It follows that $v(I)$ is a good relative ideal of S such that $v(I)$ is strictly between S and K (cf. Proposition 2.11(ii)). But it is not difficult to see that if we add to S any point of K, we have to add all the points of $K\backslash S$ in order to make properties (1) and (2) (of Proposition 2.1) hold. This fact means that there is no good relative ideal of S strictly between S and K, and this is a contradiction.

We conclude this section with a partial numerical analogue of the equality $l_R(\bar{R}/R)=l_R(\omega/C)$, where $C=R:\bar{R}$ (cf. e.g. [5, Lemma 19c)] for the local case).

Proposition 2.17. *Let S be a semigroup and K its canonical ideal. Then we have $d(\mathbb{N}^d\backslash S)\geq d(K\backslash C(S))$.*

Proof. Using Lemma 2.13 and the additivity of the function $d(\_\backslash\_)$ we can compute $d(\mathbb{N}^d\backslash S)=d(\mathbb{N}^d\backslash C(S))-d(S\backslash C(S))$ and $d(K\backslash C(S))$ moving from $\mathbf{0}$ to $\gamma+(1,\dots,1)$ adding step by step $\mathbf{e}^i$ for some i. Hence, for $\mathbf{0}\leq\boldsymbol{\alpha}<\gamma+(1,\dots,1)$, $d(\mathbb{N}^d\backslash S)$ increases with 1 if and only if $d(S(\boldsymbol{\alpha})\backslash S(\boldsymbol{\alpha}^i))=0$ and $d(K\backslash C(S))$ increases with 1 if and only if $d(K(\boldsymbol{\alpha})\backslash K(\boldsymbol{\alpha}^i))=1$. Fix any path from $\mathbf{0}$ to $\gamma+(1,\dots,1)$ and consider the path obtained taking the points $\gamma+(1,\dots,1)-\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is in the fixed path. Now if $d(K(\boldsymbol{\alpha})\backslash K(\boldsymbol{\alpha}^i))=1$, there exists $\boldsymbol{\beta}\in K$, $\beta_i=\alpha_i$ and $\beta_j\geq\alpha_j$ if $i\neq j$. Hence $\Delta^S(\gamma-\boldsymbol{\beta})=\emptyset$ and therefore $d(S(\gamma+(1,\dots,1)-\boldsymbol{\alpha}^i)\backslash S(\gamma+(1,\dots,1)-\boldsymbol{\alpha}))=0$, since any $\epsilon\in\mathbb{N}^d$ with $\varepsilon_i=\gamma_i+1-\alpha_i-1=\gamma_i-\alpha_i$ and $\varepsilon_j\geq\gamma_j+1-\alpha_j$ belongs to $\Delta(\gamma-\boldsymbol{\beta})$. $\square$

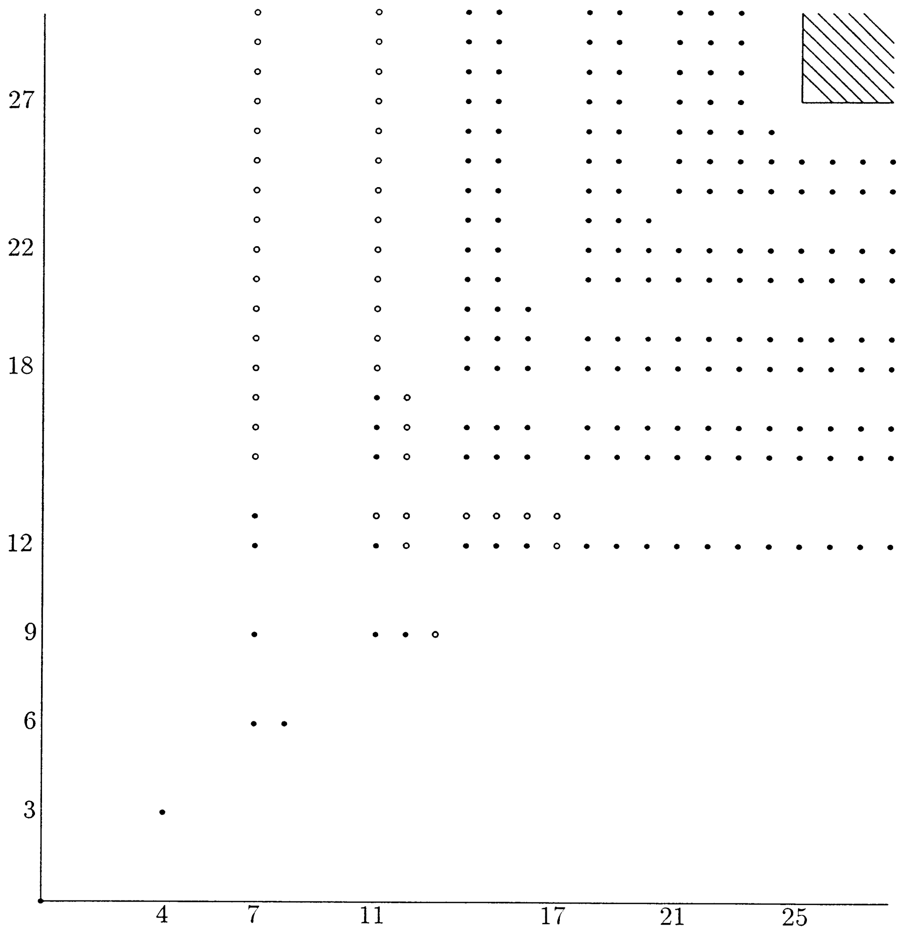

Fig. 3. A good semigroup which is not the semigroup of a ring.

Corollary 2.18. *Let S be a semigroup. Then* $d(\mathbb{N}^d \backslash C(S)) \geq 2d(S \backslash C(S))$.

Proof. By Proposition 2.17, $d(\mathbb{N}^d \backslash S) \geq d(K \backslash C(S))$. Hence $d(\mathbb{N}^d \backslash C(S)) = d(\mathbb{N}^d \backslash S) + d(S \backslash C(S)) \geq d(K \backslash C(S)) + d(S \backslash C(S))$. Since $K \supseteq S$, the conclusion follows. $\square$

3. Some classes of (Semilocal) rings well described by their semigroups of values

If R is a semilocal ring with maximal ideals $m_1, \dots, m_r$, we have seen in the Preliminaries that $v(R) = \prod_{i=1}^{r} v(R_{m_i})$, so we can reduce problems to the local case. Since we have assumed that $\bar{R}$ is a product of DVR's, we can prove something more, as a consequence of the following result, that we could not find in the literature:

Proposition 3.1. *Let A be a Noetherian ring such that its integral closure $\bar{A}$ is a finite product of local domains* (*not necessarily of dimension one*). *Then A is a semilocal ring isomorphic to the product of all its localizations at maximal ideals.*

Proof. Let $\bar{A}=A_1\times\cdots\times A_d$. Then the maximal ideals in $\bar{A}$ are $n_1\times A_2\times\cdots\times A_d,\dots,$ $A_1\times\cdots\times A_{d-1}\times n_d$, where n_i is the maximal ideal in A_i. The minimal primes in $\bar{A}$ are $0\times A_2\times\cdots\times A_d,\dots,A_1\times\cdots\times A_{d-1}\times 0$. $\operatorname{Spec}(\bar{A})$ is the disjoint union of $\operatorname{Spec}(A_i)$, $i=1,\dots,d$. Every maximal ideal in A is the contraction of a maximal ideal in $\bar{A}$, thus A is semilocal with maximal ideals $m_1,\dots,m_r$, say. Each minimal prime in A lies in exactly one maximal ideal (since they are contracted from $\bar{A}$). Let $\{P_i;\ i\in I_k\}$ be the minimal prime ideals of A which lie in m_k, and let $Q_k=\bigcap_{i\in I_k}P_i$. The natural map $A\to A/Q_1\times\cdots\times A/Q_r$ is injective since $\bigcap_1^r Q_i=0$ (A is reduced, since $\bar{A}$ is reduced). Since for any j and for any $k\neq j$, m_j is a prime ideal that does not contain P_i, for any $i\in I_k$, we have $Q_k\not\subseteq\bigcup_{j\neq k}m_j$ and $\bigcap_{j\neq k}Q_j\not\subseteq m_k$. Hence, if we pick $a\in Q_k\setminus\bigcup_{j\neq k}m_j$ and $b\in\bigcap_{j\neq k}Q_j\setminus m_k$, $a+b\in Q_k+Q_j$ but does not belong to any maximal ideal, so the Q_j's are comaximal. Thus, the map is surjective, so $A\simeq A/Q_1\times\cdots\times A/Q_r=(A/Q_1)_{m_1}\times\cdots\times(A/Q_r)_{m_r}$. Now $(Q_1)_{m_1}=(\bigcap_{i\in I_1}P_i)_{m_1}$, and the intersection can be extended to all minimal primes since $(P_k)_{m_1}=A_{m_1}$ if $k\notin I_1$. Thus $Q_{1_{m_1}}=0_{m_1}$. $\square$

We now return to our setting of one-dimensional rings.

Corollary 3.2. *If $(R,m_1,\dots,m_r)$ is a semilocal ring with $\bar{R}=V_1\times\cdots\times V_d$, where V_i is a DVR, $i=1,\dots,d$, then $R\simeq R_{m_1}\times\cdots\times R_{m_r}$.*

We can use the corollary above to reduce many questions on semilocal rings to the local case.

First we need to fix some notation. Notice that if (R,m) is a local ring and $S=v(R)$, then S is a (good) local semigroup and $v(m)=S\setminus\{\mathbf{0}\}$ is a good relative ideal of S. If S is any (good) local semigroup, we will denote by M the relative ideal $S\setminus\{\mathbf{0}\}$ (and call it the maximal ideal of S).

If $(R,m_1,\dots,m_r)$ is a semilocal ring with $\bar{R}=\prod_{i=1}^d V_i$ (a product of DVR's), by Corollary 3.2, $R\simeq R_{m_1}\times\cdots\times R_{m_r}$, so the Jacobson radical of R is $m=\bigcap_{i=1}^r m_i=\prod_{i=1}^r m_iR_{m_i}$. Considering the values, we get that $S=v(R)$ is a good semigroup, $S\subseteq\mathbb{N}^d$. Of course $r\le d$ and, assuming that the maximal ideals $n_{i_1},\dots,n_{i_l}$ of $\bar{R}$ are those lying over the maximal ideal m_i of R, we have that $v(m_i)=\{\boldsymbol{\alpha}\in S\mid\alpha_{i_1}>0,\dots,\alpha_{i_l}>0\}$ and $M=v(m)=\{\boldsymbol{\alpha}\in S\mid\alpha_i>0\text{ for }i=1,\dots,d\}$ are good relative ideals of S. Moreover, $S=v(R_{m_1})\times\cdots\times v(R_{m_r})=S_1\times\cdots\times S_r$, where for $i=1,\dots,r$, $S_i=v(R_{m_i})$ is a local semigroup with maximal ideal $M_i=v(m_iR_{m_i})$. It is easily checked that $M=M_1\times\cdots\times M_r$.

If S is any (good) semigroup, $S\subseteq\mathbb{N}^d$, we know by Theorem 2.5 that S is a product of local semigroups, $S=S_{A_1}\times\cdots\times S_{A_r}$, for a suitable partition $A_1,\dots,A_r$ of $\{1,\dots,d\}$. Set $M(A_i)=\{\boldsymbol{\alpha}\in S\mid\alpha_h>0\text{ for }h\in A_i\}$. We have that $M(A_i)$ is a good relative ideal of S and, recalling that $S_{A_i}=\pi_{A_i}(S)$ (cf. Theorem 2.5), the maximal ideal of the local

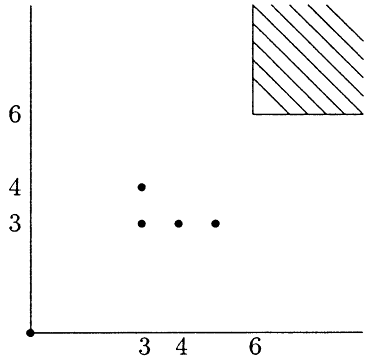

Fig. 4. $v(R) - v(m) = S - M$.

semigroup S_{A_i} is $\pi_{A_i}(M(A_i)) = M_{A_i}$. We define the *Jacobson radical* of S to be the relative ideal $\mathrm{rad}(S) = M = M(A_1) \cap \cdots \cap M(A_r) = M_{A_1} \times \cdots \times M_{A_r}$.

The object of this section is to recall some fundamental notions for rings and compare them with the analogous notions for semigroups, in the local and semilocal case.

3.1. Type of rings and semigroups

3.1.1. The local case

It is well known that the *type* (or *CM-type*) of a local one-dimensional ring (R, m) equals $l_R(R:m/R)$, see e.g. [10, Proposition 2.16]. Suppose now S is a (good) local semigroup with maximal ideal M. When $S - M$ is a good relative ideal of S, we define $\mathrm{type}(S) = d((S-M)\backslash S)$. This definition extends that given for numerical semigroups (cf. [10]). Unfortunately, $S - M$ is not necessarily a good relative ideal, even if S is the value semigroup of a ring, as the following example shows.

Example 3.3. The semigroup S of Example 2.10 is the semigroup of values of the following ring: $R = k[[(t^7, u^6), (t^6, u^7), (t^9, u^{11}), (t^{10}, u^{10}), (t^{11}, u^9), (t^{11}, u^{10}), (t^{12}, u^{12}), (t^{13}, -u^{13}), (t^{20}, u^{12}), (t^{16}, u^{20}), (t^{12}, u^{20})]] = k[[x_1, \ldots, x_{11}]]/(x_5 - x_6, x_2^2 - x_7, x_7 - x_{11}, x_1x_2 - x_8, x_2x_4 - x_{10}, x_1x_8 - x_9) \cap (x_4 - x_6, x_1^2 - x_7, x_7 - x_9, x_1x_2 + x_8, x_2x_8 + x_{10}, x_{10} - x_{11})$, where k is a field. In this example there is no possibility to define $d_{S-M}(\mathbf{0}, \boldsymbol{\alpha})$ if $\boldsymbol{\alpha} \in C(S)$, in a way which is independent of the chain from $\mathbf{0}$ to $\boldsymbol{\alpha}$ (cf. Fig. 4). On the other hand the type of R (of course well defined) can be computed as $d(v(R:m)\backslash v(R))$, cf. Proposition 2.11(iii). It turns out that $v(R:m)$, which is a good relative ideal of $v(R)$ since it is the set of values of a fractional ideal of R, is the subset of $v(R) - v(m)$ depicted in Fig. 5.

We will in the next two sections consider two classes of rings R for which it is possible to define the type of $S = v(R)$. Since $v(R:m) \subseteq v(R) - v(m)$, notice that, when $\mathrm{type}(S)$ is well defined, we have $\mathrm{type}(R) \le \mathrm{type}(S)$.

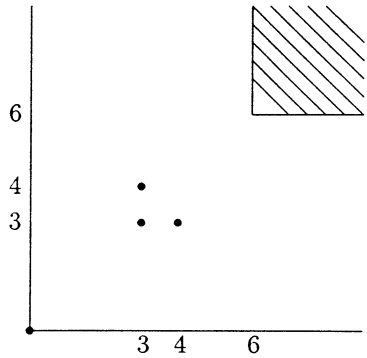

Fig. 5. $v(R:m)$.

3.1.2. The semilocal case

Let $(R, m_1, \ldots, m_r)$ be a semilocal ring with Jacobson radical $m = \bigcap_{i=1}^r m_i$. We define the type of R to be $l_R(R:m/R)$. Since $m = \Pi m_i R_{m_i}$, we have $R:m = \Pi R_{m_i} : \Pi m_i R_{m_i} = \Pi(R_{m_i} : m_i R_{m_i})$, hence we get $\text{type}(R) = \sum_{i=1}^r \text{type}(R_{m_i})$. In a similar way, if S is a good semigroup, with Jacobson radical M, when $S - M$ is a good relative ideal of S, we define $\text{type}(S) = d(S - M \backslash S)$. If $S = S_1 \times \cdots \times S_r$ is the representation of S in its local components (cf. Theorem 2.5) then, by Proposition 2.3, type(S) is well defined if and only if type(S_i) is well defined for all $i = 1, \ldots, r$, and in this case $\text{type}(S) = \sum_{i=1}^r \text{type}(S_i)$.

3.2. Almost symmetric semigroups and almost Gorenstein rings

A Gorenstein ring can be characterized by means of its semigroup of values, cf. [7, Theorem 4.8]. We will in this section investigate a larger class of rings, the almost Gorenstein rings. We start with local rings.

3.2.1. The local case

In this section (R, m) will always be a local ring and S a good local semigroup with maximal ideal $M = S \backslash \{\mathbf{0}\}$ and canonical ideal $K = K(S)$. We denote by ω a canonical ideal of R, such that $R \subseteq \omega \subseteq \bar{R}$.

Lemma 3.4. *For any semigroup S, we have $S - M \subseteq K \cup \Delta(\gamma)$.*

Proof. Let $\boldsymbol{\alpha} \in S - M$, $\boldsymbol{\alpha} \notin \Delta(\gamma)$. If $\boldsymbol{\alpha} \notin K$, there would be a $\boldsymbol{\beta} \in \Delta^S(\gamma - \boldsymbol{\alpha})$, $\boldsymbol{\beta} \neq \mathbf{0}$, so that $\boldsymbol{\alpha} + \boldsymbol{\beta} \in \Delta^S(\gamma)$, a contradiction. $\square$

A ring R is called *almost Gorenstein* if it fulfils the following equivalent conditions:

(1) $l_R(\bar{R}/R) = l_R(R/(R:\bar{R})) + \text{type}(R) - 1$.

(2) $\text{type}(R) = l_R(\omega/R) + 1$.

(3) $m\omega = m$.
(4) $\omega \subseteq m : m$.
(cf. [5, Definition-Proposition 20]). Gorenstein rings are exactly the almost Gorenstein rings of type 1. We call almost Gorenstein rings of type 2 *Kunz rings*. Gorenstein rings (Kunz rings, resp.) are characterized by the equality $l_R(\bar{R}/R) = l_R(R/(R:\bar{R}))$ ($l_R(\bar{R}/R) = l_R(R/(R:\bar{R})) + 1$, resp., cf. [5]).

Definitions. The semigroup S is called *symmetric* whenever $\boldsymbol{\alpha} \in S$ if and only if $\Delta^S(\gamma - \boldsymbol{\alpha}) = \emptyset$ (cf. [7, Definition 4.4(2)]). Since $S \subseteq K$ this is equivalent to $K = S$. We call S *almost symmetric* if $M = K + M$.

Notice that any symmetric semigroup is almost symmetric.

Lemma 3.5. *For a semigroup S the following conditions are equivalent*:
(i) *S is almost symmetric.*
(ii) $S - M = K \cup \Delta(\gamma)$.
(iii) type(S) *is well defined and* type$(S) = d(K \backslash S) + 1$.

Proof. (i) $\Leftrightarrow$ (ii): We have $K + M = M$ if and only if $K + M \subseteq M$ which is true if and only if $K + M \subseteq S$, i.e., if and only if $K \subseteq S - M$. Since $\Delta(\gamma) \subseteq S - M$, it follows from Lemma 3.4 that S is almost symmetric if and only if $S - M = K \cup \Delta(\gamma)$.

(ii) $\Leftrightarrow$ (iii): Since, for each $\boldsymbol{\alpha} \in \mathbb{N}^d$, $\boldsymbol{\alpha} \geq \gamma$, we have $\boldsymbol{\alpha} \in K \cup \Delta(\gamma)$ and since K is a good relative ideal of S, also $K \cup \Delta(\gamma)$ is. Hence, if $S - M = K \cup \Delta(\gamma)$, type$(S)$ is well defined. When type(S) is well defined, since $S - M \subseteq K \cup \Delta(\gamma)$ (cf. Lemma 3.4) we have type$(S) = d(S - M \backslash S) \leq d((K \cup \Delta(\gamma)) \backslash K) + d(K \backslash S) = 1 + d(K \backslash S)$ (where the last equality holds since, for any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in (K \cup \Delta(\gamma)) \backslash K$, $\boldsymbol{\alpha}, \boldsymbol{\beta} < \gamma + (1, \ldots, 1)$, $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are not comparable). Hence type$(S) = 1 + d(K \backslash S)$ if and only if $S - M = K \cup \Delta(\gamma)$. □

Proposition 3.6. *A semigroup S is symmetric if and only if S is almost symmetric and* type$(S) = 1$.

Proof. We already noticed that, if S is symmetric, then S is almost symmetric. Moreover, if S is almost symmetric, by Lemma 3.5(iii), $K = S$ (i.e., S is symmetric) is equivalent to type$(S) = 1$. □

If R is an almost Gorenstein ring, then $v(m) - v(m) = v(\omega + \bar{R}(\gamma))$, where $\bar{R}(\gamma) = \{r \in \bar{R} \mid v(r) \geq \gamma\}$, since $v(\omega + \bar{R}(\gamma)) \subseteq v(m : m) \subseteq v(m) - v(m) \subseteq K \cup \Delta(\gamma) = v(\omega + \bar{R}(\gamma))$. Since $\omega + \bar{R}(\gamma)$ is a fractional R-ideal, the type of S is defined if R is almost Gorenstein.

Proposition 3.7. *The ring R is almost Gorenstein if and only if $S = v(R)$ is almost symmetric and* type$(S) =$ type(R).

Proof. For every ring R, such that type(S) is well defined, we have $l_R(\omega/R) = d(K \setminus S) \geq \text{type}(S) - 1 \geq \text{type}(R) - 1$ (cf. the proof of Lemma 3.5 for the first inequality). By definition R is almost Gorenstein if $l_R(\omega/R) = \text{type}(R) - 1$, thus if and only if both the inequalities are equalities, thus if and only if S is almost symmetric and type(S) = type(R). □

Proposition 3.7 can be made more precise for type$(S) = 1$ or 2. The first part of the next corollary is the result from [7] mentioned above.

Corollary 3.8. (i) *The ring R is Gorenstein if and only if $S = v(R)$ is symmetric.*
(ii) *The ring R is Kunz if and only if $S = v(R)$ is almost symmetric and* type$(S) = 2$.

Proof. (i) It is well known that R is Gorenstein if and only if $\omega = R$, i.e., if and only if $K = S$.

(ii) If R is Kunz, then by Proposition 3.7, S is almost symmetric and type$(S) = 2$. Conversely, suppose S is almost symmetric and type$(S) = 2$. We have type$(R) \leq$ type(S). If type$(R) = 1$, then R is Gorenstein and type$(S) = 1$ (cf. (i) and Proposition 3.6). So type(R) = type$(S) = 2$ and, by Proposition 3.7, R is Kunz. □

Remark 3.9. Notice that we can have type$(R) = 2$ and $S = v(R)$ almost symmetric with type$(S) = 3$ (cf. [4, Example II.1.19]).

3.2.2. The semilocal case

In this section, $(R, m_1, \ldots, m_r)$ will be a semilocal ring with Jacobson radical $m = \bigcap_{i=1}^r m_i$, and S will be a good semigroup, $S = S_1 \times \cdots \times S_r$ the representation of S in its local components (cf. Theorem 2.5), $M = M_1 \times \cdots \times M_r$ will be the Jacobson radical of S, and $K(S)$ the canonical ideal. The ring R is defined to be *Gorenstein* if and only if R_{m_i} is Gorenstein, for all i. We define R to be *almost Gorenstein* if and only if R_{m_i} is almost Gorenstein, for all i. We call R *Kunz* if and only if R_{m_i} is Kunz, for all i.

We can give for S the same definition of symmetric and almost symmetric as in the local case, where now M is not the maximal ideal, but the Jacobson radical of S. We have:

Lemma 3.10. *$S = S_1 \times \cdots \times S_r$ is symmetric (almost symmetric, resp.) if and only if S_i is symmetric (almost symmetric, resp.) for all i.*

Proof. For the symmetric property apply Proposition 2.15. S is almost symmetric if and only if $K(S) + M \subseteq M$, i.e., if and only if $K(S_i) + M_i \subseteq M_i$ for all i, i.e. if and only if S_i is almost symmetric for all i. □

Reducing to the local case, analogous results to those in Section 3.2.1 can be stated also in the semilocal case.

Proposition 3.11. *$S = S_1 \times \cdots \times S_r$ is symmetric if and only if S is almost symmetric and* $\text{type}(S) = r$.

Proof. We only have to use Lemma 3.10, Proposition 3.6, and the equality $\text{type}(S) = \sum_{i=1}^{r} \text{type}(S_i)$. $\square$

Proposition 3.7 remains unchanged:

Proposition 3.12. *The ring R is almost Gorenstein if and only if $S = v(R)$ is almost symmetric and* $\text{type}(S) = \text{type}(R)$.

Proof. Here we use the definition that R is almost Gorenstein if all localizations are, then Proposition 3.7, Lemma 3.10, and the equalities $\text{type}(R) = \sum \text{type}(R_{m_i})$, $\text{type}(S) = \sum \text{type}(S_i)$. $\square$

Corollary 3.8 is changed to:

Proposition 3.13. (i) *The ring R is Gorenstein if and only if $S = v(R)$ is symmetric.*
(ii) *The ring R is Kunz if and only if $S = v(R)$ is almost symmetric and* $\text{type}(S_i) = 2$ *for each local component S_i of S.*

Proof. (i) Use Corollary 3.8(i) and Lemma 3.10.
(ii) Use Corollary 3.8(ii) and Proposition 3.12. $\square$

Alternatively, we could define directly semilocal almost Gorenstein rings, as rings satisfying the following equivalent conditions:
(1) $l_R(\bar{R}/R) = l_R(R/(R:\bar{R})) + \text{type}(R) - r$.
(2) $\text{type}(R) = l_R(\omega/R) + r$.
(3) $m\omega = m$.
(4) $\omega \subseteq m:m$.
Here R is semilocal with r maximal ideals and Jacobson radical m, and ω is a canonical ideal, $R \subseteq \omega \subseteq \bar{R}$. That the conditions are equivalent follows as in [5, Definition-Proposition 20], using now $l_R(R/m) = r$. Then exactly the same results as in the local case of Section 3.2.1 could be proved in the semilocal case, replacing 1 with r (also Lemma 3.5, replacing $\Delta(\gamma)$ with $\mathbb{N}^d(\gamma) = \{\boldsymbol{\alpha} \in \mathbb{N}^d \mid \boldsymbol{\alpha} \geq \gamma\}$).

3.3. Maximal embedding dimension and Arf rings and semigroups

3.3.1. The local case

In this section we assume rings and semigroups to be local, with the same notation as in Section 3.2.1. We denote the multiplicity of the ring R by $e(R)$ and the embedding dimension of R, i.e., $l_R(m/m^2)$ by $\text{edim}(R)$. It is well known that $\text{edim}(R) \leq e(R)$ (cf. [1, Theorem 1]) and R is said to be of *maximal embedding dimension* if $\text{edim}(R) = e(R)$. Recall that R is of maximal embedding dimension if and only if the maximal ideal

m is a stable ideal (cf. [11, Corollary 1.10]), where an ideal in R is called *stable* if $z(I:I)=I$ for some $z\in I$ (then, in our hypotheses, z is necessarily of minimal value in I). We define a semigroup ideal E of S to be *stable* if $\boldsymbol{\alpha}+(E-E)=E$ for some $\boldsymbol{\alpha}\in E$ (then $\boldsymbol{\alpha}$ is necessarily the minimal value in E). Note that if the good relative ideal E is stable, then $E-E=E-\boldsymbol{\alpha}$ is a good relative ideal of S. Hence, if $S\neq\mathbb{N}^d$, $M-S=M-M$ and type(S) is well defined if M is stable. In particular, the type of $S=v(R)$ is well defined if $v(m)$ is stable.

Proposition 3.14. *The following are equivalent*:
(i) *The ideal I of R is stable.*
(ii) *$v(I:I)=v(I)-v(I)$ and $v(I)$ is stable.*

Proof. For any ideal I and any element $z\in I$ we have $v(z(I:I))=v(z)+v(I:I)\subseteq v(z)+(v(I)-v(I))\subseteq v(I)$. The ideal I is stable if and only if $v(z(I:I))=v(I)$ for some $z\in I$, hence if and only if $v(I:I)=v(I)-v(I)$ and $v(z)+(v(I)-v(I))=v(I)$. □

Corollary 3.15. *Let $S=v(R)$. The following are equivalent*:
(i) *R is of maximal embedding dimension.*
(ii) *$v(m)$ is stable, and* type$(R)=$type(S).

Proof. We have $v(m:m)=v(m)-v(m)$ if and only if type$(R)=$type(S). □

Corollary 3.16. *Let $S=v(R)$. The following are equivalent*:
(i) *R is almost Gorenstein of maximal embedding dimension.*
(ii) *S is almost symmetric, $v(m)$ is stable, and* type$(R)=$type(S).

Notice that $e(R)=1$ if and only if R is a DVR. Otherwise if $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d)$ is the minimal positive value in $S=v(R)$, then $e(R)=\alpha_1+\cdots+\alpha_d$ (cf. [14, Theorem 1]). Moreover, since R is local, for any $i\geq 1$, we have $\alpha_i\geq 1$ and so $d\leq e(R)$. In the same way we can define the *multiplicity of a local semigroup S* as $e(S)=d_{\mathbb{N}^d}(\mathbf{0},\boldsymbol{\alpha})=\alpha_1+\cdots+\alpha_d$ if $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_d)$ is the minimal positive value in S.

Proposition 3.17. (i) *If R is not a DVR, we have* type$(R)\leq e(R)-1$ *with equality if and only if m is stable.*
(ii) *Suppose* type(S) *is well defined. If $S\neq\mathbb{N}$, then* type$(S)\leq e(S)-1$ *with equality if and only if M is stable.*

Proof. (i) Let z be an element of minimal value in m. Since R is not a DVR, we have type$(R)=l_R(m:m/R)=l_R(z(m:m)/zR)\leq l_R(m/zR)=l_R(R/zR)-1=e(R)-1$ with equality if and only if m is stable.

(ii) Since $S\neq\mathbb{N}$, we have type$(S)=d((S-M)\backslash S)=d((M-M)\backslash S)$. Moreover, $M-M\subseteq M-\boldsymbol{\alpha}$, where $\boldsymbol{\alpha}$ is the minimal value in M. We have type$(S)=d(M-M\backslash S)=d(\boldsymbol{\alpha}+(M-M)\backslash\boldsymbol{\alpha}+S)\leq d(M\backslash\boldsymbol{\alpha}+S)=d(S\backslash\boldsymbol{\alpha}+S)-1$. Now, if $\boldsymbol{\beta}$ is sufficiently large, we have $d(S\backslash\boldsymbol{\alpha}+S)=d_S(\mathbf{0},\boldsymbol{\beta})-d_{\boldsymbol{\alpha}+S}(\mathbf{0},\boldsymbol{\beta})=d_S(\mathbf{0},\boldsymbol{\beta})-d_S(\mathbf{0},\boldsymbol{\beta}-\boldsymbol{\alpha})=d_S(\boldsymbol{\beta}-$

$\boldsymbol{\alpha},\boldsymbol{\beta})=d_{\mathbb{N}^d}(\mathbf{0},\boldsymbol{\alpha})=e(S)$. Hence, $\operatorname{type}(S)\leq e(S)-1$ and the equality holds if and only if M is stable. □

Arf rings were introduced in [2] under the name *canonical* for a special class of analytically irreducible rings. In [11] there is a general definition; a ring is called Arf if every regular integrally closed ideal is stable (an ideal is called *regular* if it contains a nonzerodivisor). If $\boldsymbol{\alpha}\in\mathbb{Z}^d$, we denote in the sequel the ideal $\{r\in R \mid v(r)\geq\boldsymbol{\alpha}\}$ of R by $R(\boldsymbol{\alpha})$. Moreover we recall that $\boldsymbol{\delta}=\boldsymbol{\gamma}+(1,\ldots,1)$, where $\boldsymbol{\gamma}$ is the Frobenius vector of $S=v(R)$.

Lemma 3.18. *A regular ideal in R is integrally closed if and only if it is of the form $R(\boldsymbol{\alpha})$ for some $\boldsymbol{\alpha}\in\mathbb{Z}^d$. In particular, R is Arf if $R(\boldsymbol{\alpha})$ is stable for any $\boldsymbol{\alpha}\in\mathbb{Z}^d$.*

Proof. Let I be a regular ideal and let $x\in I$ be an element of minimal value. It is shown in [11, Remark (a), p. 659] that $z\in R$ is integral over the ideal xR if and only if $z/x\in\bar{R}$, i.e., if and only if $v(z)\geq v(x)$. Thus the integral closure of xR is $R(v(x))$. Since $xR\subseteq I\subseteq R(v(x))$, the claim follows. □

The semigroup S is called Arf if the semigroup ideal $S(\boldsymbol{\alpha})=\{\boldsymbol{\beta}\in S \mid \boldsymbol{\beta}\geq\boldsymbol{\alpha}\}$ is stable for any $\boldsymbol{\alpha}\in\mathbb{Z}^d$.

Proposition 3.19. *The following are equivalent*:

(1) *R is Arf.*

(2) *$S=v(R)$ is Arf and $v(R(\boldsymbol{\alpha}):R(\boldsymbol{\alpha}))=S(\boldsymbol{\alpha})-S(\boldsymbol{\alpha})$ for any $\boldsymbol{\alpha}\in\mathbb{Z}^d$.*

Proof. This follows directly from Proposition 3.14. □

First we show that we need only check finitely many ideals for stability.

Lemma 3.20. *If $R(\boldsymbol{\alpha})$ is stable for each $\boldsymbol{\alpha}\leq\boldsymbol{\delta}$, then R is Arf.*

Proof. Suppose that $R(\boldsymbol{\beta})$ is stable for each $\boldsymbol{\beta}\leq\boldsymbol{\delta}=(\delta_1,\ldots,\delta_d)$ and suppose that $\boldsymbol{\alpha}\not\leq\boldsymbol{\delta}$, say $\alpha_1>\delta_1$. Without loss of generality, we can assume that $\boldsymbol{\alpha}\in S$. Let $\boldsymbol{\alpha}'=(\delta_1,\alpha_2,\ldots,\alpha_d)$. It follows from Proposition 2.1, that $(k,\alpha_2,\ldots,\alpha_d)\in S$ for all $k\geq\delta_1$. This gives $v(R(\boldsymbol{\alpha}'))-v(R(\boldsymbol{\alpha}'))=v(R(\boldsymbol{\alpha}))-v(R(\boldsymbol{\alpha}))$. We also get $R(\boldsymbol{\alpha}'):R(\boldsymbol{\alpha}')\subseteq R(\boldsymbol{\alpha}):R(\boldsymbol{\alpha})$, so $v(R(\boldsymbol{\alpha}'):R(\boldsymbol{\alpha}'))\subseteq v(R(\boldsymbol{\alpha}):R(\boldsymbol{\alpha}))$. By induction on the number of coordinates α_i of $\boldsymbol{\alpha}$ for which $\alpha_i>\delta_i$ we can assume that $R(\boldsymbol{\alpha}')$ is stable. Thus, we have $v(R(\boldsymbol{\alpha}'):R(\boldsymbol{\alpha}'))=v(R(\boldsymbol{\alpha}'))-v(R(\boldsymbol{\alpha}'))$ by Proposition 3.14. Now $v(R(\boldsymbol{\alpha}'):R(\boldsymbol{\alpha}'))\subseteq v(R(\boldsymbol{\alpha}):R(\boldsymbol{\alpha}))\subseteq v(R(\boldsymbol{\alpha}))-v(R(\boldsymbol{\alpha}))=v(R(\boldsymbol{\alpha}'))-v(R(\boldsymbol{\alpha}'))$. This gives $v(R(\boldsymbol{\alpha}):R(\boldsymbol{\alpha}))=v(R(\boldsymbol{\alpha}))-v(R(\boldsymbol{\alpha}))$. Furthermore $v(R(\boldsymbol{\alpha}'))-v(R(\boldsymbol{\alpha}'))=v(R(\boldsymbol{\alpha}'))-\boldsymbol{\alpha}'$, which gives $v(R(\boldsymbol{\alpha}))-v(R(\boldsymbol{\alpha}))=v(R(\boldsymbol{\alpha}))-\boldsymbol{\alpha}$, since $v(R(\boldsymbol{\alpha}))-\boldsymbol{\alpha}=v(R(\boldsymbol{\alpha}'))-\boldsymbol{\alpha}'$, that is $v(R(\boldsymbol{\alpha}))$ is stable so, by Proposition 3.14, $R(\boldsymbol{\alpha})$ is stable. □

Proposition 3.21. *The following conditions are equivalent*:
(i) *R is almost Gorenstein and Arf.*
(ii) *$S=v(R)$ is Arf and almost symmetric and* type$(R)=$type(S).

Proof. (i)$\Rightarrow$(ii): This follows from Propositions 3.7 and 3.19.

(ii)$\Rightarrow$(i): Proposition 3.7 gives that R is almost Gorenstein. If we can show that, for each $\boldsymbol{\alpha}\le\boldsymbol{\delta},\boldsymbol{\alpha}\in S$, we have $v(R(\boldsymbol{\alpha}):R(\boldsymbol{\alpha}))=S(\boldsymbol{\alpha})-S(\boldsymbol{\alpha})$, then we can conclude, by Proposition 3.14, that $R(\boldsymbol{\alpha})$ is stable and so that R is Arf by Lemma 3.20. Let $\boldsymbol{\alpha}\le\boldsymbol{\delta}$ and take a saturated chain from 0 to $\boldsymbol{\delta}$ through $\boldsymbol{\alpha}$, say $\mathbf{0}=\boldsymbol{\alpha}_0<\boldsymbol{\alpha}_1<\cdots<\boldsymbol{\alpha}_i=\boldsymbol{\alpha}<\cdots<\boldsymbol{\alpha}_n=\boldsymbol{\delta}$, so $n=d(S\backslash C(S))$. We can suppose that $n>0$, otherwise $S=\mathbb{N}$, R is a DVR and the statement trivially holds. Then $S\subset S(\boldsymbol{\alpha}_1)-S(\boldsymbol{\alpha}_1)\subset\cdots\subset S(\boldsymbol{\alpha})-S(\boldsymbol{\alpha})\subset\cdots\subset S(\boldsymbol{\delta})-S(\boldsymbol{\delta})$ $=\mathbb{N}^d$. The inclusions are strict, since $\boldsymbol{\delta}-\boldsymbol{\alpha}_{i+1}+\mathbb{N}^d$ is contained in $S(\boldsymbol{\alpha}_{i+1})-S(\boldsymbol{\alpha}_{i+1})$, but not contained in $S(\boldsymbol{\alpha}_i)-S(\boldsymbol{\alpha}_i)$. We have $d(\mathbb{N}^d\backslash S)=\sum_{i=1}^n d((S(\boldsymbol{\alpha}_i)-S(\boldsymbol{\alpha}_i))\backslash(S(\boldsymbol{\alpha}_{i-1})-S(\boldsymbol{\alpha}_{i-1})))$ by Proposition 2.11(i). Since R is almost Gorenstein, we have (cf. Proposition 3.7 and the definition of almost Gorenstein) $d(\mathbb{N}^d\backslash S)=l_R(\bar{R}/R)=l_R(R/(R:\bar{R}))+\text{type}(R)-1=d(S\backslash C(S))+\text{type}(S)-1=n+\text{type}(S)-1$. Now $d((S(\boldsymbol{\alpha}_1)-S(\boldsymbol{\alpha}_1))\backslash(S(\boldsymbol{\alpha}_0)-S(\boldsymbol{\alpha}_0)))=d((M-M)\backslash S)=d((S-M)\backslash S)=\text{type}(S)$, so $d((S(\boldsymbol{\alpha}_i)-S(\boldsymbol{\alpha}_i))\backslash(S(\boldsymbol{\alpha}_{i-1})-S(\boldsymbol{\alpha}_{i-1})))=1$ if $i>1$. Since type$(S)=$type(R), we have $M-M=v(m:m)$. Furthermore $l_R(\bar{R}/R)=\sum_{i=1}^n l_R((R(\boldsymbol{\alpha}_i):R(\boldsymbol{\alpha}_i))/(R(\boldsymbol{\alpha}_{i-1}):R(\boldsymbol{\alpha}_{i-1})))=\text{type}(R)+\sum_{i=2}^n l_R((R(\boldsymbol{\alpha}_i):R(\boldsymbol{\alpha}_i))/(R(\boldsymbol{\alpha}_{i-1}):R(\boldsymbol{\alpha}_{i-1})))$. Since the inclusions $(R(\boldsymbol{\alpha}_i):R(\boldsymbol{\alpha}_i))\subset(R(\boldsymbol{\alpha}_{i-1}):R(\boldsymbol{\alpha}_{i-1}))$ are strict (we have that $\bar{R}(\boldsymbol{\delta}-\boldsymbol{\alpha}_{i+1})$ is contained in $R(\boldsymbol{\alpha}_{i+1}):R(\boldsymbol{\alpha}_{i+1})$ but not in $R(\boldsymbol{\alpha}_i):R(\boldsymbol{\alpha}_i)$), and since $v(R(\boldsymbol{\alpha}_i):R(\boldsymbol{\alpha}_i))\subseteq(S(\boldsymbol{\alpha}_i)-S(\boldsymbol{\alpha}_i))$, we get equality also for each $i\ge 2$. $\square$

It is shown in [11, Proposition-Definition 3.1] that among the Arf rings between R and $\bar{R}$ there is a smallest R', called the *Arf closure* of R. We will next give an "algorithm" to find the Arf closure of a ring of our class. We need a lemma.

Lemma 3.22. *The ideal I of R is stable if and only if $x^{-1}I$ is a ring, where x is any element of minimal value in I.*

Proof. Suppose that I is stable. Then $x^{-1}I=I:I$, which is a ring. Suppose that $x^{-1}I$ is a ring. We have to show that if $y\in I$, then $y/x\in I:I$. Let $z\in I$. Then $z\cdot y/x=x\cdot z/x\cdot y/x=x\cdot v/x=v$ for some $v\in I$ since $x^{-1}I$ is a ring. Hence the claim. $\square$

Proposition 3.23. *Let $R(\boldsymbol{\alpha})$ be a nonstable ideal of R with $\boldsymbol{\alpha}\le\boldsymbol{\delta}$. (If such an ideal does not exist, then $R=R'$.) Let U be the smallest ring in $\bar{R}$ containing $x^{-1}R(\boldsymbol{\alpha})$ and let $R_1=R+xU$, where $x\in R$ and $v(x)=\boldsymbol{\alpha}$. Repeat the construction on R_1 if R_1 is not Arf. After a finite number of steps we reach the Arf closure R'.*

Proof. In R' every ideal is stable so, by Lemma 3.22, $x^{-1}R'(\boldsymbol{\alpha})$ is a ring for each $\boldsymbol{\alpha}\in v(R'),v(x)=\boldsymbol{\alpha}$. In particular, if $\boldsymbol{\alpha}\in v(R)$ and $x\in R$, then $x^{-1}R'(\boldsymbol{\alpha})\supseteq U$, so $R'(\boldsymbol{\alpha})\supseteq xU$. Since $x^{-1}R(\boldsymbol{\alpha})$ is strictly contained in U, so also $R(\boldsymbol{\alpha})$ is strictly contained in R_1.

Thus $R \subset R_1 \subseteq R'$, and hence $R' = R'_1$. That the process is finite follows from the fact that R' is finite over R. $\square$

Remark 3.24. Observe that, if $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_d)$ is the minimal positive value in $v(R)$, then $\boldsymbol{\alpha}$ is also the minimal value in $v(R')$.

Example 3.25. Let $R = k[[(t^3, u^3), (t^4, u^5), (t^5, u^4)]]$, k a field. Consider the ring $R((3,3))$. Then we reach the Arf closure in one step, since $R_1 = k + k(t^3, u^3) + (t^4, u^4)k[[t]] \times k[[u]]$, which is Arf.

Example 3.26. Let $R = k[[t^6, t^8, t^{10} + t^{11}]]$, where k is a field of characteristic $\neq 2$. Then $v(R) = \{0, 6, 8, 10, 12, 14\} \cup (16 + \mathbb{N})$, which is an Arf semigroup. However R is not Arf. Let, with the notation of Proposition 3.23, $\alpha = 6$ and $x = t^6$. Then $U = k[[t^2, t^5]]$, and thus $R_1 = k[[t^6, t^8, t^{10}, t^{11}, t^{13}, t^{15}]]$, which is Arf, and so the Arf closure of R. Note that $v(R_1) = \{0, 6, 8\} \cup (10 + \mathbb{N})$.

For future reference we also need:

Lemma 3.27. *Let $r \in R$ be an element with $v(r) \leq \boldsymbol{\delta}$, and let $C = R : \bar{R}$ be the conductor. Then the ideal $I = rR + C$ is stable.*

Proof. We have to show that $I^2 = rI$. Now $I^2 = r^2R + rC + C^2 = r^2R + rC = rI$, since $rC = R(v(r) + \boldsymbol{\delta}) \supseteq C^2 = R(2\boldsymbol{\delta})$. $\square$

We now give a class of simple examples.

Example 3.28. Let $\boldsymbol{\alpha} \in \mathbb{N}^d$, $\boldsymbol{\alpha} \geq (1, \dots, 1)$ and $\boldsymbol{\delta} = n\boldsymbol{\alpha}$ for some $n \geq 1$. Let R be a ring such that $S = v(R) = \{k\boldsymbol{\alpha} \mid 0 \leq k \leq n-1\} \cup (\boldsymbol{\delta} + \mathbb{N}^d)$. We claim that R is Arf of type $\alpha_1 + \cdots + \alpha_d - 1$. For each $\boldsymbol{\beta} \in \mathbb{Z}^d$, $\boldsymbol{\beta} \leq \boldsymbol{\delta}$, we have $R(\boldsymbol{\beta}) = R(k\boldsymbol{\alpha})$ for some $k \leq n$. Let $x \in R$ be an element of value $\boldsymbol{\alpha}$. We have $R(k\boldsymbol{\alpha}) = x^kR + \bar{R}(\boldsymbol{\delta}) = x^kR + C$, since these ideals have the same value set and one is included in the other. So, by Lemma 3.27, these ideals are stable and, by Lemma 3.20, R is Arf. Moreover, since we have $v(m:m) = v(R(\boldsymbol{\alpha}) : R(\boldsymbol{\alpha})) = v(x^{-1}R(\boldsymbol{\alpha})) = S(\boldsymbol{\alpha}) - \boldsymbol{\alpha}$, then $\text{type}(R) = l_R((R:m)/R) = l_R((m:m)/R) = d((S(\boldsymbol{\alpha}) - \boldsymbol{\alpha}) \backslash S) = d_{S(\boldsymbol{\alpha}) - \boldsymbol{\alpha}}(\mathbf{0}, n\boldsymbol{\alpha}) - d_S(\mathbf{0}, n\boldsymbol{\alpha}) = n - 1 + \alpha_1 + \cdots + \alpha_d - n = \alpha_1 + \cdots + \alpha_d - 1$. We have $l_R(\bar{R}/R) = n(\alpha_1 + \cdots + \alpha_d - 1)$ and $l_R(R/C) = n$, hence a simple calculation gives that R is almost Gorenstein if and only if $n = 1$, or $d = 1$ and $\alpha_1 = 2$, or $d = 2$ and $\alpha_1 = \alpha_2 = 1$. In case $d = 2$ it is not hard to see that $m = R(\boldsymbol{\alpha})$ is minimally generated by $(t_1^{\alpha_1}, t_2^{\alpha_2}), (t_1^{(n+1)\alpha_1}, t_2^{n\alpha_2+1}), (t_1^{(n+1)\alpha_1}, t_2^{n\alpha_2+2}), \dots, (t_1^{(n+1)\alpha_1}, t_2^{(n+1)\alpha_2-1})$, $(t_1^{(n+1)\alpha_1-1}, t_2^{(n+1)\alpha_2}), (t_1^{(n+1)\alpha_1-2}, t_2^{(n+1)\alpha_2}), \dots, (t_1^{n\alpha_1}, t_2^{(n+1)\alpha_2})$, where $\bar{R}$ is the product of the DVRs V_1 and V_2 with uniformizing parameters t_1 and t_2, respectively.

Seminormal analytically irreducible rings are normal. We will now show that if we have $d \geq 2$ minimal primes, then seminormal rings are almost Gorenstein. This generalizes [3, Corollary 5.2].

Proposition 3.29. *If R is seminormal, then R is almost Gorenstein and Arf. If, in addition, R has $d \geq 2$ minimal primes, then* $\mathrm{type}(R) = d - 1$, *in particular, if $d = 2$, then R is Gorenstein and, if $d = 3$, then R is Kunz.*

Proof. As in [3, Section 5], it follows that $S = v(R) = \{\mathbf{0}\} \cup \{\boldsymbol{\alpha} \mid \boldsymbol{\alpha} \geq (1, \ldots, 1)\}$. Thus we have a ring as in Example 3.28, with $\boldsymbol{\alpha} = \boldsymbol{\delta} = (1, \ldots, 1)$ and $n = 1$, and the claim follows from what we have proved there. □

3.3.2. The semilocal case

As in the previous section it is easy to generalize the results to semilocal rings. With the same notation and assumptions as in Section 3.2.2, if R is semilocal with Jacobson radical m, we define the *multiplicity* of R as $e(R) = \sum_{i=1}^{r} e(R_{m_i})$ and say that R is of *maximal embedding dimension* if $l_R(m/m^2) = e(R)$. We define the *multiplicity* of S as $e(S) = \sum_{i=1}^{r} e(S_i)$, where S_i are the local components of S. We use the same definition as above of *Arf rings* and *Arf semigroups* in the semilocal case. It is well known that R is Arf if and only if R_m is Arf for every maximal ideal m in R. For future reference we also notice:

Proposition 3.30. *$S = S_1 \times \cdots \times S_r$ is Arf if and only if each local component S_i of S is Arf.*

Proposition 3.14 goes through with the same proof in the semilocal case. We can use Corollary 3.2 to show that Corollaries 3.15 and 3.16 are true in the semilocal case. Proposition 3.17 is changed to:

Proposition 3.31. (i) *If $v(R) \neq \mathbb{N}^d$, we have* $\mathrm{type}(R) \leq e(R) - r$ *with equality if and only if m is stable.*

(ii) *Suppose* $\mathrm{type}(S)$ *is well defined and $S \neq \mathbb{N}^d$. Then* $\mathrm{type}(S) \leq e(S) - r$ *with equality if and only if M is stable.*

Propositions 3.19 and 3.20 go through unchanged in the semilocal case.

Proposition 3.21 can be proved by reducing to the local components of R and S. Concerning the Arf closure R' of a semilocal ring R, recall that, for each maximal ideal m_i of R there is exactly one maximal ideal m_i' of R' over m_i and $(R_{m_i})' = (R')_{m_i'}$ (cf. [11, Theorem 3.4(i) and Corollary 3.3]). So, in our hypotheses, $R' = (R_{m_1})' \times \cdots \times (R_{m_r})'$ and the local results of Section 3.3.1 can be used component by component. Finally R is seminormal if and only if R_{m_i} is seminormal for all i. Thus the first part of the statement in Proposition 3.29 holds in the semilocal case. The second part of the statement can be used to calculate $\mathrm{type}(R) = \sum_{i=1}^{r} \mathrm{type}(R_{m_i})$.

4. Rings of multiplicity at most 3

Our aim in this section is to classify the local rings R with low multiplicity $e(R)$ in terms of their semigroup of values $v(R) = S$. It will turn out that in this situation all good semigroups are semigroups of rings. In [3] rings R with two minimal primes that are maximal with fixed conductor in $\bar{R}$ (in the sense that every overring in $\bar{R}$ to R has a larger conductor in $\bar{R}$ than R has) are classified. We show that the rings of multiplicity at most three, that are maximal with fixed conductor, are exactly the Gorenstein and Kunz rings. Since $\text{type}(S) \leq e(S) - 1 \leq 2$ if $e(R) = e(S) \leq 3$ (cf. Proposition 3.17(ii)), it follows from Corollary 3.8 that R is Gorenstein (Kunz, resp.) if and only if S is symmetric (almost symmetric, resp.).

Since $d \leq e(R)$ it is easily seen that $e(R) = 1$ if and only if $R = \bar{R}$ is a DVR. Suppose now $e(R) = 2$. Then the ring R has $d = 1$ or 2 minimal primes. Moreover, if $d = 1$, then $2 \in S$ (and so $2n \in S$ for each $n \geq 0$), the Frobenius number γ is odd and $S = \{0, 2, 4, \ldots, \gamma - 1\} \cup (\gamma + 1 + \mathbb{N})$. On the other hand, if $d = 2$, then $(1,1) \in S$ (and so $(n,n) \in S$, for each $n \geq 0$). It follows easily from Proposition 2.1 that $S = \{(0,0), (1,1), \ldots, (\gamma,\gamma)\} \cup \{(\alpha_1,\alpha_2) \mid (\alpha_1,\alpha_2) \geq (\gamma + 1, \gamma + 1)\}$, with Frobenius vector $\boldsymbol{\gamma}(S) = (\gamma,\gamma)$.

We denote by $\Sigma_e(\boldsymbol{\gamma})$ the set of good semigroups S with $e(S) = e$ and with $\boldsymbol{\gamma}(S) = \boldsymbol{\gamma}$. Moreover we call $\boldsymbol{\gamma}$ an *admissible Frobenius vector* for the multiplicity e if $\Sigma_e(\boldsymbol{\gamma})$ is not empty. Of course if R is a ring, then $v(R) \in \Sigma_{e(R)}(\boldsymbol{\gamma})$ for some admissible Frobenius vector $\boldsymbol{\gamma}$. With this terminology, we can state the above observations for rings of multiplicity two in the following way:

Proposition 4.1. (i) *The admissible Frobenius vectors for $e = 2$ are any odd natural number ≥ 1 $(d = 1)$ and any (γ,γ), $\gamma \in \mathbb{N}$ $(d = 2)$.*

(ii) *For a fixed admissible Frobenius vector $\boldsymbol{\gamma}$ for $e = 2$, the set $\Sigma_2(\boldsymbol{\gamma})$ has a unique element.*

Remark 4.2. (a) It is easy to construct examples of rings for each admissible Frobenius vector for $e = 2$. The ring $R = k[[t^2, t^{\gamma+2}]] = k[[x,y]]/(y^2 - x^{\gamma+2})$ satisfies $v(R) \in \Sigma_2(\gamma)$, $\gamma \in \mathbb{N}$ and $k[[(t,u),(0,u^{\gamma+1})]] = k[[x,y]]/(y) \cap (x^{\gamma+1} - y) = k[[x,y]]/(y^2 - yx^{\gamma+1})$ satisfies $v(R) \in \Sigma_2((\gamma,\gamma))$.

(b) Notice that, even if we consider rings with a fixed integral closure, the unique element S of $\Sigma_2(\boldsymbol{\gamma})$ (where $\boldsymbol{\gamma}$ is a fixed Frobenius vector for $e = 2$) is the semigroup of values of several rings. For example, if $\gamma = 3$, and so $S = \{0, 2\} \cup (4 + \mathbb{N})$, the two rings $R_1 = k[[x^2, x^5]]$ and $R_2 = k[[x^2 + x^3, x^5]]$ have S as semigroup of values.

The rings of multiplicity two are very special, in fact:

Corollary 4.3. *The following are equivalent for a ring R*:

(i) *$e(R) = 2$.*

(ii) *R is Gorenstein of maximal embedding dimension.*

(iii) *R is Gorenstein and Arf.*

Proof. (i) ⇒ (iii): Suppose that $e(R) = 2$. Then by Proposition 4.1 if $d = 1$, $S = \{0, 2, 4, \ldots, \gamma - 1\} \cup (\gamma + 1 + \mathbb{N})$ (for some odd $\gamma \geq 1$) or, if $d = 2$, $S = \{(0,0), (1,1), \ldots, (\gamma,\gamma)\} \cup$

$(\mathbb{N}^2+(\gamma+1,\gamma+1))$ (for some $\gamma \in \mathbb{N}$). In both cases S is a symmetric semigroup because for any $\boldsymbol{\alpha} \in \mathbb{N}^d$, we have $\boldsymbol{\alpha} \in S$ if and only if $\Delta^S(\boldsymbol{\gamma} - \boldsymbol{\alpha}) = \emptyset$. So by Corollary 3.8(i), R is Gorenstein. Moreover in both cases S is Arf and so, by Proposition 3.21, R is Gorenstein and Arf.

(iii) $\Rightarrow$ (ii): Immediate.

(ii) $\Rightarrow$ (i): Since R is Gorenstein, type$(R) = 1$ and, by Proposition 3.17(i), since R is of maximal embedding dimension, we have $1 = e(R) - 1$, hence $e(R) = 2$. □

Suppose now $e(R) = 3$. Since $d \le e(R)$, the ring R has $d = 1, 2$, or 3 minimal primes. Moreover:

Proposition 4.4. *The admissible Frobenius vectors for* $e = 3$ *are*:

(1) *Any natural number* γ, $\gamma \not\equiv 0 \pmod 3$, $\gamma \ge 2$.

($2a_k$) $(\gamma_1, 2\gamma_1 + 2k + 1)$, $k \ge 0$ *or* $(2\gamma_2 + 2k + 1, \gamma_2)$, $k \ge 0$, *for any* $\gamma_i \in \mathbb{N}$.

(2b) $(\gamma_1, 2\gamma_1)$ *or* $(2\gamma_2, \gamma_2)$, *for any* $\gamma_i \in \mathbb{N}$, $\gamma_i \ge 1$.

(3_k) $(\gamma_1, \gamma_1 + k, \gamma_1 + k), (\gamma_2 + k, \gamma_2, \gamma_2 + k), (\gamma_3 + k, \gamma_3 + k, \gamma_3)$, $k \ge 0$, *for any* $\gamma_i \in \mathbb{N}$.

Proof. If $d = 1$, i.e., if $S \subseteq \mathbb{N}$, the proof is easy.

If $d = 2$, i.e., if $S \subseteq \mathbb{N}^2$, since $e(S) = 3$, the minimal value in $S \backslash \mathbf{0}$ is (1,2) or (2,1). By symmetry we need only consider the first case. Since $(1, 2) \in S$ then $(2, 4), \dots, (n, 2n) \in S$ for each $n \ge 0$. Using Proposition 2.1 it follows easily that the possibilities for $\gamma(S)$ are those stated in $2a_k$ and 2b.

If $d = 3$, i.e., if $S \subseteq \mathbb{N}^3$, since $e(S) = 3$, the minimal value in $S \backslash \mathbf{0}$ is (1,1,1) and so $(n, n, n) \in S$ for each $n \ge 0$. Using Proposition 2.1 it follows that $\gamma(S)$ is necessarily of one of the forms stated. Notice that if $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is an admissible Frobenius vector for $e = 3$ and $\gamma_1 = \max\{\gamma_i\}$, then $\gamma_2 = \gamma_1$ or $\gamma_3 = \gamma_1$, because otherwise $(\gamma_1, \gamma_1, \gamma_1) \in \Delta^S(\boldsymbol{\gamma})$, in contradiction to $\Delta^S(\boldsymbol{\gamma}) = \emptyset$, (cf. Lemma 2.4(ii)). □

Proposition 4.5. *For a fixed admissible Frobenius vector* $\boldsymbol{\gamma}$ *for* $e = 3$, *the set* $\Sigma_3(\boldsymbol{\gamma})$ *has a minimal element* S (*with respect to inclusion*). *Moreover* S *is minimal if and only if* S *is Arf.*

Proof. For reasons of symmetry we need only consider the first example in each case of Proposition 4.4.

In case 1 in Proposition 4.4, S obviously contains $\{3n \mid n \in \mathbb{N}\} \cup \{n \mid n > \gamma\}$. It is easily checked that this set is a semigroup which is Arf. If S is not minimal, then S contains $\gamma - 1$ (if $\gamma \equiv 2 \pmod 3$) or $\gamma - 2$ (if $\gamma \equiv 1 \pmod 3$). In both cases $S(\gamma - 2) = \{\alpha \in S \mid \alpha \ge \gamma - 2\}$ is a nonstable relative ideal, so S is not Arf.

In case $2a_k$, S must contain $\{(n, 2n) \mid n \in \mathbb{N}\} \cup ((\gamma_1 + 1, 2\gamma_1 + 2k + 2) + \mathbb{N}^2) \bigcup_{n=1}^{k} \{(x, 2\gamma_1 + 2n) \mid x \ge \gamma_1 + 1\}$. If $k = 0$, S must contain just the first two sets in the union above (the same convention is used several times in the sequel: when $k = 0$,

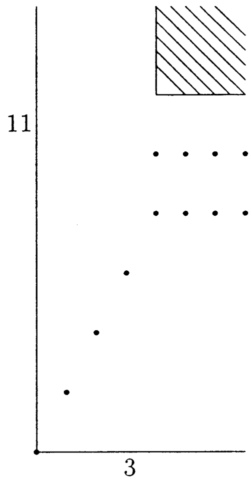

Fig. 6. The minimal $S \in \Sigma_3(3,11)$.

$\bigcup_{n=1}^{k} E_n$ is supposed to be empty). This set constitutes a semigroup which is Arf. If S is not minimal, then S must contain $\{(\gamma_1, 2\gamma_1 + 2k + 1), (\gamma_1, 2\gamma_1 + 2k)\}$. Then $S((\gamma_1, 2\gamma_1 + 2k))$ is a nonstable relative ideal, so S is not Arf (an example with $k = 2$ is depicted in Fig. 6).

In case 2b, S must contain $\{(n, 2n) \mid n \in \mathbb{N}\} \cup ((\gamma_1 + 1, 2\gamma_1 + 1) + \mathbb{N}^2)$. This set constitutes an Arf semigroup. If S is not minimal, then S must contain $\{(x, 2\gamma_1 - 1) \mid x \geq \gamma_1\}$, and then $S((\gamma_1, 2\gamma_1 - 1))$ is a nonstable relative ideal, so S is not Arf.

In cases 3_k the minimal semigroup is $\{(n,n,n) \mid n \in \mathbb{N}\} \bigcup_{n=1}^{k} \{(x, \gamma_1 + n, \gamma_1 + n) \mid x \geq \gamma_1 + 1\} \cup ((\gamma_1 + 1, \gamma_1 + k + 1, \gamma_1 + k + 1) + \mathbb{N}^3)$, which is Arf. If S is not minimal, then S must contain $\{(x, \gamma_1 + k, \gamma_1 + k), (\gamma_1, x + k, \gamma_1 + k), (\gamma_1, \gamma_1 + k, x + k) \mid x \geq \gamma_1\}$, and then $S(\gamma)$ is nonstable. □

Theorem 4.6. (i) *For a fixed admissible Frobenius vector γ for $e = 3$, the set $\Sigma_3(\gamma)$ is finite and linearly ordered* (*with respect to inclusion*).

(ii) (*a*) *If $\sum_{i=1}^{d} \gamma_i + d$ is even, then S is maximal in $\Sigma_3(\gamma)$ if and only if $d(\mathbb{N}^d \setminus S) = d(S \setminus C(S))$.*

(b) *If $\sum_{i=1}^{d} \gamma_i + d$ is odd, then S is maximal in $\Sigma_3(\gamma)$ if and only if $d(\mathbb{N}^d \setminus S) = d(S \setminus C(S)) + 1$.*

Proof. By symmetry reasons we need only consider the first example in each case of Proposition 4.4.

(i) Fix an admissible Frobenius vector γ for $e = 3$ and let $\boldsymbol{\alpha}$ be the minimal value in $S \setminus \mathbf{0}$. Notice that $A = \{n\boldsymbol{\alpha} \mid n \in \mathbb{N}\} \subseteq S$. Since $\Delta^S(\gamma) = \emptyset$, also $\Delta^S(\gamma - n\boldsymbol{\alpha}) = \emptyset$ for all $n \in \mathbb{N}$. Set $V = \bigcup_{n \in \mathbb{N}} \Delta(\gamma - n\boldsymbol{\alpha})$. Depending on the fixed admissible Frobenius vector γ

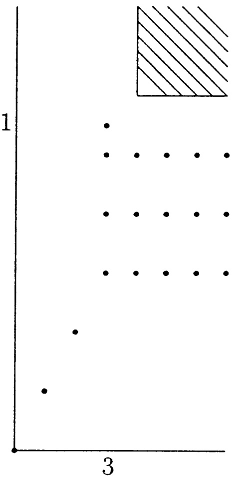

Fig. 7. $S \cup L_3$ (cf. proof of Theorem 4.6).

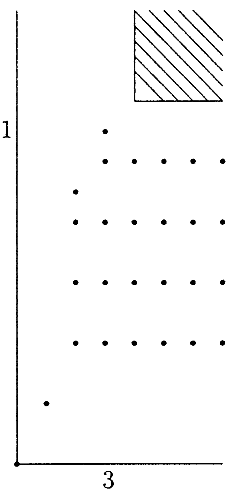

Fig. 8. The maximal $T \in \Sigma_3(3,11), T = S \cup L_3 \cup L_2$.

we define the set L_i in the following way: (for the case $2a_k$, with $k=2$, cf. Figs. 7 and 8)

(1a) If $\gamma \in \mathbb{N}$, $\gamma \equiv 1 \pmod 3$, set $L_i = \{3i - 2\gamma - 2\}$.

(1b) If $\gamma \in \mathbb{N}$, $\gamma \equiv 2 \pmod 3$, set $L_i = \{3i - 2\gamma - 1\}$.

($2a_k$) If $\boldsymbol{\gamma} = (\gamma_1, 2\gamma_1 + 2k + 1)$, set $L_i = \{(i, 2i+2k+1)\} \bigcup_{n=1}^{k} \{(i, 2i+2n)\} \cup \{(x, 2i) \mid x > i\}$.

(2b) If $\boldsymbol{\gamma} = (\gamma_1, 2\gamma_1)$, set $L_i = \{(x, 2i-1) \mid x \geq i\}$.

(3_k) If $\boldsymbol{\gamma} = (\gamma_1, \gamma_1 + k, \gamma_1 + k)$, set $L_i = \{(i, i+k, i+k+x) \mid x \geq 0\} \cup \{(i, i+k+x, i+k) \mid x \geq 0\} \cup \{(x, i, i) \mid x > i\} \bigcup_{n=1}^{k-1} \{(i, i+n, i+n)\}$.

It is not difficult to see that, if γ is an admissible Frobenius vector for $e=3$ and if $S\in\Sigma_3(\gamma)$, then

(1) $L_{i+1}=L_i+\boldsymbol{\alpha}$.

(2) $L_i\subseteq S$ implies that $L_{i+1}\subseteq S$.

(3) $L_i\cap S\neq\emptyset$ if and only if $L_i\subseteq S$.

(4) $\mathbb{N}^d\subseteq A\cup V\cup(\bigcup_{i\geq 0}L_i)$.

The first and the fourth property are easily checked, the second follows from the first, and the third follows applying Proposition 2.1. If γ is a fixed admissible Frobenius vector for $e=3$ and if $S,T\in\Sigma_3(\gamma)$, then the minimal value in $S\backslash\mathbf{0}$ and in $T\backslash\mathbf{0}$ is the same. Supposing that i (j, resp.) is the smallest index such that $L_i\subseteq S$ ($L_j\subseteq T$, resp.), we have that $S\supseteq T$ if and only if $i\leq j$. It follows that $\Sigma_3(\gamma)$ is finite and linearly ordered.

We now turn to (ii). For $d=1$ it is well known, cf. [4, Theorem II.1.14 and Proposition II.1.12], that

(a) If $\sum\gamma_i+d=\gamma+1$ is even (i.e., if γ is odd) then S is maximal in $\Sigma(\gamma)$ if and only if $\mathrm{Card}(\mathbb{N}\backslash S)=\mathrm{Card}(S\backslash C(S))$, where $\Sigma(\gamma)$ is the set of *all* semigroups T with $\gamma(T)=\gamma$.

(b) If $\sum\gamma_i+d=\gamma+1$ is odd (i.e., if γ is even) then S is maximal in $\Sigma(\gamma)$ if and only if $\mathrm{Card}(\mathbb{N}\backslash S)=\mathrm{Card}(S\backslash C(S))+1$.

Moreover it is easy to see that, if S is maximal in $\Sigma_3(\gamma)$, then S is maximal in $\Sigma(\gamma)$.

Now consider case 3_k. Any $S\in\Sigma_3(\gamma)$ has the same conductor C. We have $d(\mathbb{N}^d\backslash C)=\sum\gamma_i+d=3\gamma_1+2k+3$. To compute $d(S\backslash C)$, notice that, if S is minimal in $\Sigma_3(\gamma)$, then $d(S\backslash C)=\gamma_1+k+1$. Moreover, if $S_1=S\cup L_{\gamma_1}$, then $d(S_1\backslash C)=d(S\backslash C)+1$, if $S_2=S\cup L_{\gamma_1}\cup L_{\gamma_1-1}$, then $d(S_2\backslash C)=d(S\backslash C)+2$ and so on. It follows that, if $\sum\gamma_i+d=3\gamma_1+2k+3$ is even (i.e., if γ_1 is odd), the maximal element in $\Sigma_3(\gamma)$ is $T=S\bigcup_{i\geq(\gamma_1+1)/2}L_i$ (where S is the minimal semigroup), since $d(T\backslash C)=(3\gamma_1+2k+3)/2$ and, for any semigroup U, by Corollary 2.18 $d(\mathbb{N}^d\backslash C(U))\geq 2d(U\backslash C(U))$. If, on the other hand, $\sum\gamma_i+d$ is odd (i.e., if γ_1 is even) the maximal element in $\Sigma_3(\gamma)$ is $T=S\bigcup_{i\geq(\gamma_1+2)/2}L_i$. Actually $d(T\backslash C)=(3\gamma_1+2k+2)/2$ and the claim follows since $d(\mathbb{N}^d\backslash C)=3\gamma_1+2k+3$.

With similar computations in the other cases the theorem follows. □

Remark 4.7. Notice that the proof of the proposition gives an explicit description of all semigroups in $\Sigma_3(\gamma)$.

Arf studies algebroid analytically irreducible curves in [2]. It is shown in [2] that if R is Arf, then $v(R)$ is an Arf semigroup, but that the converse does not hold (cf. also Proposition 3.19 and Example 3.26). In the case of small multiplicity the situation is better.

Proposition 4.8. *Let R be a ring with $v(R)$ Arf and $e(R)\leq 3$. Then R is an Arf ring.*

Proof. If $e(R)=1$, R is a DVR, if $e(R)=2$, by Corollary 4.3 R is Arf. In both cases the proposition trivially holds. If $e(R)=3$, also $e(v(R))=3$ and we have a complete classification of Arf semigroups of multiplicity 3. In each case it is easy to see that all ideals $R(\boldsymbol{\alpha})$ with $\boldsymbol{\alpha} \le \boldsymbol{\gamma} + (1,\dots,1)$ are of the form $xR + (R:\bar{R})$ and thus stable by Lemma 3.27. $\square$

Corollary 4.9. *Let R be a ring with $e(R)=3$, let $S=v(R)$, and let $\boldsymbol{\gamma}=\boldsymbol{\gamma}(S)$. If $\sum \gamma_i + d$ is even (odd, resp.), then R is Gorenstein (Kunz, resp.) if and only if S is maximal in $\Sigma_3(\boldsymbol{\gamma})$.*

Proof. We have $l_R(\bar{R}/R) = d(\mathbb{N}^d \setminus S)$ and $l_R(R/(R:\bar{R})) = d(S \setminus C(S))$. Since R is Gorenstein (Kunz, resp.) if and only if $l_R(\bar{R}/R) = l_R(R/(R:\bar{R}))$ ($l_R(\bar{R}/R) = l_R(R/(R:\bar{R})) + 1$, resp.), cf. Section 3.2.1, by Theorem 4.6(ii), the proof is complete. $\square$

Some rings of multiplicity 3 are very special:

Corollary 4.10. *The following are equivalent for a ring R:*
(i) *R is Kunz of maximal embedding dimension.*
(ii) *$e(R)=3$, $\sum \gamma_i + d$ is odd and $S=v(R)$ is maximal in $\Sigma_3(\boldsymbol{\gamma})$.*

Proof. (i)$\Rightarrow$(ii): Since R is Kunz, type$(R)=2$ and, by Proposition 3.17, since R is of maximal embedding dimension, we have $2 = e(R) - 1$, hence $e(R)=3$. Corollary 4.9 gives the remaining claims.

(ii)$\Rightarrow$(i): By Theorem 4.6(ii)(b), $d(\mathbb{N}^d \setminus S) = d(S \setminus C(S)) + 1$, this means that R is Kunz. Moreover type$(R) = 2 = e(R) - 1$ and so, by Proposition 3.17, R is of maximal embedding dimension. $\square$

Corollary 4.11. *The following are equivalent for a ring R:*
(i) *R is Kunz and Arf.*
(ii) *$e(R)=3$, $\sum \gamma_i + d$ is odd and S is maximal and minimal in $\Sigma_3(\boldsymbol{\gamma})$.*
(iii) *S is one of the following:*
(1) $S = \langle 3,5,7 \rangle$.
($2a_k$) $S = \{(0,0)\} \bigcup_{n=1}^{k} \{(x,2n) \mid x \ge 1\} \cup ((1,2k+2) + \mathbb{N}^2)$ *or* $S = \{(0,0)\} \bigcup_{n=1}^{k} \{(2n,x) \mid x \ge 1\} \cup ((2k+2,1) + \mathbb{N}^2)\}$.
(2b) $S = \{(0,0),(1,2)\} \cup ((2,3) + \mathbb{N}^2)$ *or* $S = \{(0,0),(2,1)\} \cup ((3,2) + \mathbb{N}^2)$.
(3_k) $S = \{(0,0,0)\} \bigcup_{n=1}^{k} \{(x,n,n) \mid x \ge 1\} \cup ((1,k+1,k+1) + \mathbb{N}^3)$ *or* $S = \{(0,0,0)\} \bigcup_{n=1}^{k} \{(n,x,n) \mid x \ge 1\} \cup ((k+1,1,k+1) + \mathbb{N}^3)$ *or* $S = \{(0,0,0)\} \bigcup_{n=1}^{k} \{(n,n,x) \mid x \ge 1\} \cup ((k+1,k+1,1) + \mathbb{N}^3)$.

Proof. (i)$\Rightarrow$(ii): Since, if R is Arf, then R is of maximal embedding dimension, we have by Corollary 4.10 that $e(R)=3$, $\sum \gamma_i + d$ is odd and S is maximal in $\Sigma_3(\boldsymbol{\gamma})$. Moreover, since R is Arf, then S is Arf (cf. Proposition 3.19) and by Proposition 4.5 this is equivalent to S minimal in $\Sigma_3(\boldsymbol{\gamma})$.

(ii)$\Leftrightarrow$(iii): From the description of the semigroups of $\Sigma_3(\boldsymbol{\gamma})$ given in the proof of Theorem 4.6, it is not difficult to see that the only ones which are at the same time maximal and minimal are those listed in (iii).

(ii)$\Rightarrow$(i): By Corollary 4.10, R is Kunz. By Proposition 4.5, S is Arf, and by Theorem 4.6 we have $d(\mathbb{N}^d\backslash S)=d(S\backslash C(S))+1$. It follows that S is almost symmetric and type$(S)=$type$(R)=2$. By Proposition 3.21 R is Kunz and Arf. $\square$

Example 4.12. For any admissible Frobenius vector γ for $e=3$ and for any semigroup S with $\sum \gamma_i + d$ odd and maximal in $\Sigma_3(\gamma)$, we give an explicit example of a ring R such that $v(R)=S$. By Corollary 4.10 such a ring is Kunz of maximal embedding dimension.

For $d=1$ we can trivially consider the ring $R=k[[t^S]]$. This is the ring $k[[t^3, t^{(\gamma+6)/2}, t^{\gamma+3}]]=k[[x,y,z]]/(xz-y^2, x^{(\gamma+2)/2}-z^2, x^{(\gamma+4)/2}-yz)$.

For $d=2$, in correspondence with cases 2a$_k$, and 2b of Theorem 4.6, we can consider:
(2a$_k$) The subring $k[[(t,u^2),(t^{(\gamma_1+2)/2},u^{\gamma_1+2k+3}),(t^{\gamma_1+2},u^{2\gamma_1+2k+3})]]$ of $k[[t]]\times k[[u]]$, where γ_1 is even, $\gamma_1 \geq 0$. This is the ring

$$k[[x,y,z]]/(x^{(\gamma_1+2)/2}-y, x^{\gamma_1+2}-z)\cap(x^{\gamma_1+2k+3}-y^2, x^{\gamma_1/2}y-z).$$

(2b) The subring $k[[(t,u^2),(t^{(\gamma_1+3)/2},u^{\gamma_1+2}),(t^{\gamma_1+2},u^{2\gamma_1+2})]]$ of $k[[t]]\times k[[u]]$, where γ_1 is odd, $\gamma_1 \geq 1$. This is the ring

$$k[[x,y,z]]/(x^{(\gamma_1+3)/2}-y, x^{\gamma_1+2}-z)\cap(x^{\gamma_1+2}-y^2, x^{\gamma_1+1}-z).$$

If $d=3$ we have:
(3$_k$) The subring of $k[[t]]\times k[[u]]\times k[[v]]$

$$k[[(t,u,v),(t^{(\gamma_1+2)/2}, -u^{k+(\gamma_1+2)/2}, v^{k+(\gamma_1+2)/2}),(t^{\gamma_1+1}, u^{\gamma_1+k+2}, v^{\gamma_1+k+2})]].$$

This is the ring

$$k[[x,y,z]]/(x^{(\gamma_1+2)/2}-y, x^{\gamma_1+1}-z)\cap(x^{k+(\gamma_1+2)/2}+y, x^{k+\gamma_1+2}-z)\cap (x^{k+(\gamma_1+2)/2}-y, x^{k+\gamma_1+2}-z).$$

Remark 4.13. In the same way as we have done above for the maximal semigroups, it is possible to give examples, for *each* possible semigroup S of multiplicity $e(S)=3$, of a ring R such that $v(R)=S$. Thus, in case $e(S)\leq 3$, the good semigroups coincide with semigroups of rings (for $e(S)=2$ cf. Remark 4.2(a)).

We conclude this section with an example concerning intersection numbers.

Example 4.14. Let $A\subseteq\{1,\dots,d\}$, let $P_1,\dots,P_d$ be the minimal primes of R and let $P_A=\bigcap_{i\in A}P_i$. If A and B are disjoint subsets of $\{1,\dots,d\}$, the *intersection number* of the branches in A with those in B is defined to be $I_{A,B}=l_R(R/(P_A+P_B))$. Now let $d=3$, $A=\{1\},B=\{2\},C=\{3\}$. Then Garcia shows (cf. [9]) that $\mathbb{I}:=I_{\{1,2\},\{3\}}-I_{\{1\},\{3\}}-I_{\{2\},\{3\}}=l_R(R/(R:\bar{R}))-l_R(\bar{R}/R)\leq 0$ if all branches are nonsingular. We can make this a bit more precise. That all branches are nonsingular is equivalent to $e(R)=3$. For $e=3$ we have a complete classification (case 3$_k$) of possible semigroups, cf. Theorem 4.6. For a given γ, $\mathbb{I}$ varies between 0 and $-\min\{\gamma_i\}-1$ and is minimal if and only if R is Arf, while ▯ is maximal if and only if R is Gorenstein ($\gamma_1+\gamma_2+\gamma_3$ odd, ▯ $=0$) or Kunz ($\gamma_1+\gamma_2+\gamma_3$ even, ▯ $=-1$).

5. The multiplicity forest of a ring

Recall that, if I is an ideal of R, the blowing up R^I of I is $\bigcup_{n>0}(I^n : I^n)$. We have $I^n : I^n \subseteq I^{n+1} : I^{n+1}$ for each n, and $R^I = I^{n_0} : I^{n_0}$ for some n_0 since R is Noetherian. Recall that we can associate to R, as in [11, p. 666], a sequence of semilocal rings $R=R_0 \subseteq R_1 \subseteq \cdots$ where R_{i+1} is obtained from R_i by blowing up $\mathrm{rad}(R_i)$, the Jacobson radical of R_i. We call this sequence the *Lipman sequence*. Since, in our hypotheses, $\bar{R}$ is a finitely generated R-module, this sequence stabilizes for some n and $R_h=\bar{R}$, for $h \geq n$. Recall also that, given a maximal ideal n_j of $\bar{R}$ the *branch sequence* of R along n_j is the sequence of rings $(R_i)_{n_j \cap R_i}$ (cf. [11, p. 669]) and the *multiplicity sequence* of R along n_j is given by the multiplicities of these rings (cf. [11, p. 669]). If R is an Arf ring, all the overrings R_i of the Lipman sequence are also Arf (cf. [11, Corollary 2.5]). Moreover, for any ring R, the Arf closure R' has the same multiplicity sequence as R, cf. [11, Corollary 3.7].

Example 5.1. If R is the subring $k[[(t,u^2),(t,u^7),(t^2,u^7)]]$ of $k[[t]] \times k[[u]]$ (cf. Example $2a_k$ of Section 4 with $\gamma_1=0$ and $k=2$), we get the Lipman sequence $R_1=k[[t]] \times k[[u^2,u^5]]$, $R_2=k[[t]] \times k[[u^2,u^3]]$, $R_3=R_4=\cdots=\bar{R}=k[[t]] \times k[[u]]$.

To a local ring R with $\bar{R}=V_1 \times \cdots \times V_d$, a rooted tree, the *blowing up tree* of R, is associated in the following way: The nodes are all local rings occurring in all branch sequences. The root (at level 0) is R, and on level 1 there are the localizations (at its maximal ideals) of $R_1=R^{\mathrm{rad}(R)}$, and so on. If U is a local ring in the tree and $\bar{U}=V_{i_1} \times \cdots \times V_{i_k}$, then U has k minimal primes $q_1,\ldots,q_k$. We define the *fine multiplicity* of U to be $\mathbf{e}(U)=(e_1(U),\ldots,e_d(U))$, where $e_j(U)=0$ if $j \notin \{i_1,\ldots,i_k\}$ and $e_{i_j}(U)=e(U/q_j)$, $j=1,\ldots,k$. (Thus the usual multiplicity of U is $\sum_{i=1}^{d} e_i(U)$.) If we replace the local rings in the tree with their fine multiplicities, we get the *multiplicity tree* of R. If the ring R is semilocal, we define the *blowing up forest* and the *multiplicity forest* of R to be the disjoint union of the corresponding trees of all localizations at maximal ideals of R.

In the example above we get the following blowing up tree and multiplicity tree (Fig. 9):

Remark 5.2. Notice that, since we have assumed that $\bar{R}$ is a product of DVR's, by Corollary 3.2, each ring R_i of the Lipman sequence associated to R is the direct product of its localizations at maximal ideals, i.e., the direct product of the local rings which appear at level i in the blowing up forest. Notice also that the multiplicity sequences of Lipman can be read off in our multiplicity forest moving upwards, summing the coordinates of the vectors. In the example above, the multiplicity sequences along the two branches are respectively $3,1,1,\ldots$ and $3,2,2,1,1,\ldots$.

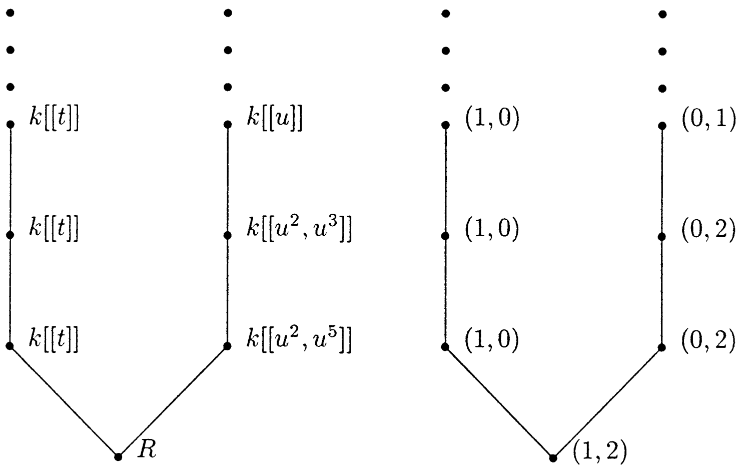

Fig. 9. The blowing up and multiplicity trees of Example 5.1.

Proposition 5.3. *If R' is the Arf closure of R, then the multiplicity forests of R and R' are the same.*

Proof. If m is a maximal ideal of R, then R' has exactly one maximal ideal m' over m (cf. [11, Theorem 3.4(i)]) and the fine multiplicity of R is the same as the fine multiplicity of R', by Remark 3.24, since Arf closure commutes with localization [11, Corollary 3.3]. We can iterate this argument since, by [11, Theorem 3.5], the Arf closure commutes with blowing up. Thus, if R_1 ($(R')_1$, resp.) is the first overring in the Lipman sequence of R (of R', resp.) we have $(R')_1 = (R_1)'$. $\square$

Our aim is to descend along the Lipman sequence starting from a product of DVR's $R^0 = V_1 \times \cdots \times V_d$, in the Arf case, giving a construction that permits to obtain any Arf R ring such that $\bar{R} = V_1 \times \cdots \times V_d$. A similar construction in the analytically irreducible case is given in [6, Corollary 4.5]. Since the multiplicity forests of R and R' are the same (cf. Proposition 5.3), any multiplicity forest of a ring of our class is achieved.

Notice that if aR is a minimal reduction of I (i.e., in our class of rings, if $a \in R$ is an element of minimal value), then the multiplicity of I equals $e(I) = \sum_i v_i(a)$.

In the following we will use the following notation: $\bar{R} = V_1 \times \cdots \times V_d$, where, for $h = 1, \ldots, d$, $V_h = (V_h, N_h)$ is a DVR and $V_h/N_h \simeq k$, and that R has r maximal ideals $n_1, \ldots, n_r$ (of course $r \le d$). We know, cf. Corollary 3.2, that $R \simeq R_{n_1} \times \cdots \times R_{n_r}$. Let $U = {}_1U \times \cdots \times {}_lU$. With this notation, consider now the following:

Construction A. Let $A_1, \ldots, A_l$ be a partition of $\{1, \ldots, r\}$. Let, for $k = 1, \ldots, l$, $R_{A_k} = \prod_{i \in A_k} R_{n_i}$ (so that $R = R_{A_1} \times \cdots \times R_{A_l}$). For every $j = 1, \ldots, l$ consider a principal ideal

$I_j = a_jR_{A_j}$ in R_{A_j}, generated by a nonzero divisor in the Jacobson radical of R_{A_j} such that $k \hookrightarrow R_{A_j}/I_j$, let ${}_jU$ be the pullback of $k \hookrightarrow R_{A_j}/I_j \leftarrow R_{A_j}$, and let $U = {}_1U \times \cdots \times {}_lU$.

With this notation we can prove that:

Proposition 5.4. (1) *For every* $j = 1,\dots,l$, ${}_jU$ *is local with maximal ideal* I_j, I_j *is stable, and* $I_j : I_j = R_{A_j}$.

(2) U *is semilocal with integral closure* $\bar{U} = \bar{R} = V_1 \times \cdots \times V_d$ *and with* l *maximal ideals* $m_1,\dots,m_l$, *where* $m_j = {}_1U \times \cdots \times {}_{j-1}U \times I_j \times {}_{j+1}U \times \cdots \times {}_lU$.

(3) *For* $j = 1,\dots,l$, $U_{m_j} = {}_jU$ *and the multiplicity of the ring* U_{m_j} *equals the multiplicity of the ideal* I_j.

(4) *The Jacobson radical of* U *is* $\mathrm{rad}(U) = I_1 \times \cdots \times I_l = U : R$.

(5) $\mathrm{rad}(U)$ *is a stable ideal of* U *and* $U^{\mathrm{rad}(U)} = \mathrm{rad}(U) : \mathrm{rad}(U) = R$.

(6) R *is Arf if and only if* U *is Arf.*

Proof. (1) I_j is a maximal ideal of ${}_jU$, because ${}_jU/I_j = k$ is a field. Moreover, since $I_j = a_jR_{A_j}$ is principal in R_{A_j}, we have $I_j : I_j = a_jR_{A_j} : a_jR_{A_j} = R_{A_j}$, so I_j is a stable ideal. To show that ${}_jU$ is local, notice that the extension ${}_jU \subseteq R_{A_j}$ is integral. Thus, if there is another maximal ideal n in ${}_jU$, then there is in R_{A_j} a prime ideal over n, and I_j is not contained in $\mathrm{rad}(R_{A_j})$, a contradiction to our assumptions.

(2) By construction U and R share the ideal $I_1 \times \cdots \times I_l$, so $\bar{U} = \bar{R}$. Since U is the product of the local rings $({}_jU, I_j)$, it has the following maximal ideals: $m_j = {}_1U \times \cdots \times {}_{j-1}U \times I_j \times {}_{j+1}U \times \cdots \times {}_lU$, $j = 1,\dots,l$.

(3) We have $U_{m_j} = ({}_1U \times \cdots \times {}_lU)_{{}_1U \times \cdots \times I_j \times \cdots \times {}_lU} \simeq ({}_jU)_{I_j} = {}_jU$. The multiplicity of $U_{m_j} = {}_jU$ equals the sum of the values $v_i (i \in A_j)$ of an element of minimal positive value in U_{m_j}. On the other hand, the multiplicity of the ideal I_j of R_{A_j} also equals this sum as is noticed above.

(4) It follows from the expression of the maximal ideals in (2) that $m_1 \cap \cdots \cap m_l = I_1 \times \cdots \times I_l$. Moreover, since ${}_jU : R_{A_j} = I_j (I_j$ is an ideal the two rings share and is the maximal ideal of ${}_jU)$ we have $U : R = ({}_1U \times \cdots \times {}_lU) : (R_{A_1} \times \cdots \times R_{A_l}) = ({}_1U : R_{A_1}) \times \cdots \times ({}_lU : R_{A_l}) = I_1 \times \cdots \times I_l$.

(5) Since we know that, by (1), for $j = 1,\dots,l$, I_j is stable (i.e., I_j is principal in $I_j : I_j = R_{A_j}$), we get that $\mathrm{rad}(U) = I_1 \times \cdots \times I_l$ is stable (i.e., $I_1 \times \cdots \times I_l$ is principal in $(I_1 \times \cdots \times I_l) : (I_1 \times \cdots \times I_l) = (I_1 : I_1) \times \cdots \times (I_l : I_l) = R_{A_1} \times \cdots \times R_{A_l} = R$). Thus we have also $U^{\mathrm{rad}(U)} = \mathrm{rad}(U) : \mathrm{rad}(U) = R$.

(6) If U is Arf then R is Arf since, by (5), it is the blowing up of $\mathrm{rad}(U)$, cf. [11, Corollary 2.5]. Conversely assume that R is Arf. Then each R_{A_j} is Arf. Since I_j is the maximal ideal of ${}_jU$, any integrally closed ideal of ${}_jU$ is of the form ${}_jU(\boldsymbol{\beta}) = I_j(\boldsymbol{\beta})$, for some $\boldsymbol{\beta} \in v({}_jU)$. It follows that $a_j^{-1}I_j(\boldsymbol{\beta}) \subseteq a_j^{-1}I_j = R_{A_j}$ is an ideal of R_{A_j}. Moreover $a_j^{-1}I_j(\boldsymbol{\beta}) = \{x/a_j \mid x \in I_j, v(x) \geq \boldsymbol{\beta}\} = \{r \in R_{A_j} \mid v(r) \geq \boldsymbol{\beta} - v(a_j)\} = R_{A_j}(\boldsymbol{\beta} - v(a_j))$. Since R_{A_j} is Arf then $R_{A_j}(\boldsymbol{\beta} - v(a_j)) = a_j^{-1}I_j(\boldsymbol{\beta})$ is stable and therefore also $I_j(\boldsymbol{\beta})$ is stable (since if $a_j^{-1}I_j(\boldsymbol{\beta})$ is principal in $a_j^{-1}I_j(\boldsymbol{\beta}) : a_j^{-1}I_j(\boldsymbol{\beta}) = I_j(\boldsymbol{\beta}) : I_j(\boldsymbol{\beta})$, also $I_j(\boldsymbol{\beta})$ is). Hence ${}_jU$ is Arf for any j, and then U is Arf. $\square$

Theorem 5.5. *Let $R^0 = V_1 \times \cdots \times V_d$, where, for $h = 1,\dots,d, V_h = (V_h, N_h)$ is a DVR and $V_h/N_h = k$.*
(i) *Let $R^0 \supseteq R^1 \supseteq \cdots \supseteq R^n$ be a sequence of subrings of R^0 such that, for $i = 1,\dots, n-1, R^{i+1}$ is obtained from R^i as in Construction A. More precisely, suppose that $A^1_1,\dots,A^1_{l_1}$ is a partition of $\{1,\dots,d\}$, that, for $j = 1,\dots,l_1$, I^1_j is a principal ideal of $R^0_{A^1_j} = \prod_{i\in A^1_j} V_i$ generated by an element a^1_j of value $v_h(a^1_j) = \alpha^1_{j,h}$ (for $h \in A^1_j$), and that $R^1 = {}_1R^1 \times \cdots \times {}_{l_1}R^1$ is the ring obtained from R^0 as in Construction A, (with respect to the partition and the principal ideals above). Suppose that $A^2_1,\dots,A^2_{l_2}$ is a partition of $\{1,\dots,d\}$ which is coarser than $A^1_1,\dots,A^1_{l_1}$ and that $R^2 = {}_1R^2 \times \cdots \times {}_{l_2}R^2$ is the ring obtained from R^1 as in Construction A, with respect to the principal ideals $I^2_1,\dots,I^2_{l_2}$, where I^2_j is generated by an element a^2_j of value $v_h(a^2_j) = \alpha^2_{j,h}$ for $h \in A^2_j$, and so on. Then $R = R^n$ is a semilocal Arf ring such that*:

(1) *$\bar{R} = V_1 \times \cdots \times V_d$ and $R_i = R^{n-i}$, for $i = 0,\dots,n$, where R_i is the i'th overring of R in the Lipman sequence.*

(2) *For every $j = 1,\dots,d$ consider the sequence $j_1, j_2,\dots$ such that $j \in A^1_{j_1}$, $j \in A^2_{j_2},\dots$, $j \in A^n_{j_n}$, then the sequence $e({}_{j_n}R^n), e({}_{j_{n-1}}R^{n-1}),\dots, e({}_{j_1}R^1), e(V_j) = 1,1,1,\dots$ equals the sequence $\sum_{h\in A^n_{j_n}} \alpha^n_{j_n,h},\dots,\sum_{h\in A^1_{j_1}} \alpha^1_{j_1,h}, 1,1,\dots$ and is exactly the multiplicity sequence of R along the maximal ideal $n_j = V_1 \times \cdots \times N_j \times \cdots \times V_d$ of $\bar{R}$.*
(ii) *Every semilocal Arf ring R with $\bar{R} = V_1 \times \cdots \times V_d$ can be obtained from $\bar{R}$ as in* (i).

Proof. Applying Proposition 5.4(2) n times, we see that R is semilocal with integral closure $\bar{R} = V_1 \times \cdots \times V_d$. Consider now the Lipman sequence of overrings associated to $R = R^n$. By Proposition 5.4(5), we get $(R^{i+1})^{\mathrm{rad}(R^{i+1})} = R^i$, and so $R_1 = R^{n-1},\dots, R_n = R^0$. To show that R is Arf, since $V_1 \times \cdots \times V_d$ is Arf, it is enough to apply Proposition 5.4(6). Let's prove (2): Given a maximal ideal $n_j = V_1 \times \cdots N_j \times \cdots \times V_d$ of $\bar{R}$, consider the sequence $j_1, j_2,\dots$ such that $j \in A^1_{j_1}, j \in A^2_{j_2},\dots, j \in A^n_{j_n}$, we get that $n_j \cap R_i = n_j \cap R^{n-i}$ is the maximal ideal of ${}_{j_{n-i}}R^{n-i}$. It follows that the multiplicity sequence of R along n_j equals $e({}_{j_n}R^n), e({}_{j_{n-1}}R^{n-1}),\dots, e({}_{j_1}R^1), e(V_j) = 1,1,\dots$. By Proposition 5.4(3), this is the sequence of multiplicities of the principal ideals $I^n_{j_n},\dots, I^1_{j_1}$, $N_j, N_j,\dots$, that is the sequence $\sum_{h\in A^n_{j_n}} \alpha^n_{j_n,h},\dots,\sum_{h\in A^1_{j_1}} \alpha^1_{j_1,h}, 1,1,\dots$.

(ii) Suppose that R is a semilocal Arf ring with $\bar{R} = V_1 \times \cdots \times V_d$. Consider the Lipman sequence associated to $R, R = R_0 \subseteq R_1 \subseteq \cdots$. We know, by Lipman, that any ring R_i is semilocal and Arf and that the sequence stops at $R_n = \bar{R} = V_1 \times \cdots \times V_d$. We want to show that R_i is obtained from R_{i+1} as in Construction A. Set $R_i = U$. Let $m_1,\dots,m_r$ be the maximal ideals of U. By Proposition 3.1, we know that $U = U_{m_1} \times \cdots \times U_{m_r}$ and, by [11, Corollary 2.5], for each j, U_{m_j} is a local Arf ring. Moreover, since $\mathrm{rad}(U) = m_1U_{m_1} \times \cdots \times m_rU_{m_r}$, we have $U^{\mathrm{rad}(U)} = \mathrm{rad}(U) : \mathrm{rad}(U) = (m_1U_{m_1} : m_1U_{m_1}) \times \cdots \times (m_rU_{m_r} : m_rU_{m_r}) = U_{m_1}^{m_1U_{m_1}} \times \cdots \times U_{m_r}^{m_rU_{m_r}}$. To complete the proof, it is enough to show that U_{m_j} is the pullback of the diagram $k \hookrightarrow U_{m_j}^{m_jU_{m_j}}/I_j \longleftarrow U_{m_j}^{m_jU_{m_j}} = (m_jU_{m_j} : m_jU_{m_j})$ where I_j is a principal ideal of $U_{m_j}^{m_jU_{m_j}}$ contained in the

Jacobson radical. In fact this is true, setting $I_j = m_j U_{m_j}$, which is principal in $U_{m_j}^{m_j U_j}$ by definition of blowing up and, since the extension is integral, $m_j U_{m_j}$ is contained in the Jacobson radical of $U_{m_j}^{m_j U_{m_j}}$. Notice that, since $k \simeq U_{m_j}/m_j U_{m_j}$, $k \hookrightarrow U_{m_j}^{m_j U_{m_j}}/I_j$. $\square$

Our aim now is to repeat for semigroups what we have done for rings and get results from comparing the two constructions.

Let S be a Arf subsemigroup of $\mathbb{N}^d$. We recall that all semigroups we consider are good semigroups (cf. Section 2 for the definition). Since the radical ideal $M = \mathrm{rad}(S)$ is stable, then $M - M = M - \boldsymbol{\alpha}$ (where $\boldsymbol{\alpha}$ is the minimum element in M) is also a (good) Arf semigroup. So, setting $S_{(0)} = S$ and $S_{(i+1)} = \mathrm{rad}(S_{(i)}) - \mathrm{rad}(S_{(i)})$, we have a chain of Arf semigroups $S = S_{(0)} \subseteq S_{(1)} \subseteq \cdots$ and, for some n, $S_{(n)} = \mathbb{N}^d$, so $S_{(m)} = \mathbb{N}^d$, for $m \geq n$. Let us call this the *Lipman sequence* associated to the Arf semigroup S. Since in the following we will use only the significant part of the Lipman sequence, i.e., $S = S_{(0)} \subseteq \cdots \subseteq S_{(n)} = \mathbb{N}^d$, where n is minimal, we will, with a small abuse of terminology, call this the Lipman sequence of S. As with rings, for any Arf semigroup we can construct the blowing up forest and the multiplicity forest. We need for further results a more precise description of these forests. Suppose $S = S_{(0)} \subseteq \mathbb{N}^d$. $S_{(0)}$ has a representation as a product of local semigroups, $S_{(0)} = S_{(0)}^1 \times \cdots \times S_{(0)}^{l^0}$. Of course the d components of $\mathbb{N}^d$ are partitioned in these local semigroups, so that $S_{(0)}^1 \subseteq \mathbb{N}^{|B_0^1|}, \ldots, S_{(0)}^{l^0} \subseteq \mathbb{N}^{|B_0^{l^0}|}$, for a certain partition $B_0^1, \ldots, B_0^{l^0}$ of $\{1, \ldots, d\}$. Blowing up the radical ideal of $S_{(0)}$ we get $S_{(1)}$, that also is a product of local semigroups, $S_{(1)} = S_{(1)}^1 \times \cdots \times S_{(1)}^{l^1}$, where $S_{(1)}^1 \subseteq \mathbb{N}^{|B_1^1|}, \ldots, S_{(1)}^{l^1} \subseteq \mathbb{N}^{|B_1^{l^1}|}$, for a partition $B_1^1, \ldots, B_1^{l^1}$ of $\{1, \ldots, d\}$ that is a refinement of the previous one. We have in this way the blowing up forest of S.

Suppose that the minimal positive element of the local semigroup $S_{(i)}^j$ is $\boldsymbol{\alpha}_i^j$ and denote by $(\boldsymbol{\alpha}_i^j)_h$ its hth component. We get the multiplicity forest of the Arf semigroup S replacing the local semigroup $S_{(i)}^j$ in the blowing up forest with the vector $\mathbf{e}_{(i)}^j = (e_{i,1}^j, \ldots, e_{i,d}^j) \in \mathbb{N}^d$, defined in the following way: $e_{i,h}^j = (\boldsymbol{\alpha}_i^j)_h$, if $h \in B_i^j$ and $e_{i,h}^j = 0$ otherwise. Suppose now that S is a semigroup and $S = S_1 \times \cdots \times S_r$ is the representation of S in its local components (cf. Theorem 2.5). Consider the following:

Construction B. Let $A_1, \ldots, A_l$ be a partition of $\{1, \ldots, r\}$. We have $S = S_{A_1} \times \cdots \times S_{A_l}$, where $S_{A_j} = \prod_{i \in A_j} S_i$. For any j, $j = 1, \ldots, l$, consider an element $\boldsymbol{\alpha}_j \in S_{A_j}$, with $(\boldsymbol{\alpha}_j)_h > 0$, for each $h \in A_j$, and the principal ideal $I_j = \boldsymbol{\alpha}_j + S_{A_j}$. Let ${}_jT = \mathbf{0} \cup (\boldsymbol{\alpha}_j + S_{A_j})$ and $T = {}_1T \times \cdots \times {}_lT$.

With this notation we can prove that:

Proposition 5.6. (1) *For each $j, j = 1, \ldots, l$, ${}_jT$ is a local semigroup with stable maximal ideal $I_j = \boldsymbol{\alpha}_j + S_{A_j}$, and $I_j - I_j = S_{A_j}$.*

(2) *T is a semigroup and $T = {}_1T \times \cdots \times {}_lT$ is its representation as a product of local semigroups.*

(3) $\mathrm{rad}(T) = I_1 \times \cdots \times I_l = T - S$.

(4) $\mathrm{rad}(T)$ *is a stable ideal of T and* $\mathrm{rad}(T) - \mathrm{rad}(T) = S$.

(5) *S is Arf if and only if T is Arf.*

Proof. (1) Since the principal ideal I_j of the semigroup S_{A_j} is a good relative ideal, ${}_jT = \mathbf{0} \cup I_j$ is a semigroup and it is local because $(\boldsymbol{\alpha}_j)_h > 0$, for each $h \in A_j$, with maximal ideal I_j. Moreover I_j is principal in $I_j - I_j = (\boldsymbol{\alpha}_j + S_{A_j}) - (\boldsymbol{\alpha}_j + S_{A_j}) = S_{A_j}$, and so I_j is stable.

(2) A product of semigroups is a semigroup (cf. Proposition 2.3).

(3) Since, for $j = 1, \ldots, l$, ${}_jT$ is local with maximal ideal I_j, we get $\mathrm{rad}(T) = I_1 \times \cdots \times I_l$. Since ${}_jT - S_{A_j} = I_j$, we get $T - S = ({}_1T \times \cdots \times {}_lT) - (S_{A_1} \times \cdots \times S_{A_l}) = ({}_1T - S_{A_1}) \times \cdots \times ({}_lT - S_{A_l}) = I_1 \times \cdots \times I_l$.

(4) Since I_j is stable in ${}_jT$, for each j (cf. 1)), we get that $\mathrm{rad}(T) = I_1 \times \cdots \times I_l$ is stable and applying again (1) $\mathrm{rad}(T) - \mathrm{rad}(T) = (I_1 \times \cdots \times I_l) - (I_1 \times \cdots \times I_l) = (I_1 - I_1) \times \cdots \times (I_l - I_l) = S_{A_1} \times \cdots \times S_{A_l} = S$.

(5) If $S = S_1 \times \cdots \times S_r$ is Arf, then each S_i is Arf, $i = 1, \ldots, r$, by Proposition 3.30. So also S_{A_j} is Arf, for $j = 1, \ldots, l$. It follows that also ${}_jT = \mathbf{0} \cup (\boldsymbol{\alpha}_j + S_{A_j})$ is Arf. Conversely, if T is Arf, then $\mathrm{rad}(T)$ is stable and $S = \mathrm{rad}(T) - \mathrm{rad}(T)$ is Arf. $\square$

Theorem 5.7. (i) *Consider a sequence of subsemigroups of $\mathbb{N}^d$*

$$S^{(0)} = \mathbb{N}^d \supseteq S^{(1)} \supseteq \cdots \supseteq S^{(n)},$$

where $S^{(i+1)}$ is obtained from $S^{(i)}$ as T from S in Construction B. More precisely, suppose that $A_1^1, \ldots, A_{l_1}^1$ is a partition of $\{1, \ldots, d\}$, that, for $j = 1, \ldots, l_1, \boldsymbol{\alpha}_j^{(1)} \in \mathbb{N}^{|A_j^1|}$ with $(\boldsymbol{\alpha}_j^{(1)})_h > 0$ for each $h \in A_j^1$, that ${}_jS^{(1)} = \mathbf{0} \cup (\boldsymbol{\alpha}_j^{(1)} + \mathbb{N}^{|A_j^1|})$, and that $S^{(1)} = \prod_{j=1}^{l_1} ({}_jS^{(1)})$. Suppose that $A_1^2, \ldots, A_{l_2}^2$ is a partition of $\{1, \ldots, d\}$ which is coarser than $A_1^1, \ldots, A_{l_1}^1$, and that $S^{(2)} = \prod_{j=1}^{l_2} ({}_jS^{(2)})$, where ${}_jS^{(2)} = \mathbf{0} \cup (\boldsymbol{\alpha}_j^{(2)} + {}_jS^{(1)})$ for some $\boldsymbol{\alpha}_j^{(2)} \in {}_jS^{(1)}$ with $(\boldsymbol{\alpha}_j^{(2)})_h > 0$ for each $h \in A_j^2$, and so on. Then, for $i = 0, \ldots, n$, $S^{(i)}$ is an Arf semigroup such that:

(1) *$S_{(i)} = S^{(n-i)}$ for $i = 0, \ldots, n$, where $S_{(i)}$ is the ith element of the Lipman sequence associated to the Arf semigroup $S^{(n)}$.*

(2) *The nodes $\mathbf{e}_{(i)}^j = (e_{i,1}^j, \ldots, e_{i,d}^j)$ of the multiplicity forest of $S^{(n)}$ are the vectors with the following components*: *for $0 \le i < n$, $e_{i,h}^j = (\boldsymbol{\alpha}_j^{(n-i)})_h$, if $h \in A_j^{n-i}$, and $e_{i,h}^j = 0$ otherwise*; *for $i \ge n, e_{i,h}^j = 1$ if $h = j$, and $e_{i,h}^j = 0$ otherwise.*

(ii) *Any Arf semigroup in $\mathbb{N}^d$ can be obtained as in* (i).

(iii) *Two Arf semigroups of $\mathbb{N}^d$ with the same multiplicity forest are equal.*

Proof. (i) By Proposition 5.6(5), since $\mathbb{N}^d$ is Arf, we get that $S^{(i)}$ is Arf for each i. By Proposition 5.6(4), we get $S_{(i)} = S^{(n-i)}$, so (1) is proved. (2) Since, for $i = 0, \ldots, n$, $S_{(i)} = S^{(n-i)}$, also the partition $B_i^1, \ldots, B_i^{l^i}$ of $\{1, \ldots, d\}$ induced by the semigroup $S_{(i)}$

coincides with the partition $A_1^{n-i},\dots,A_{l_{n-i}}^{n-i}$. It follows that the blowing up forest of $S^{(n)}$ is determined by the local semigroups ${}_jS^{(n-i)}=S^j_{(i)}$. Moreover, the minimum positive element of ${}_jS^{(n-i)}$ is by our construction $\boldsymbol{\alpha}_j^{(n-i)}$. So if $\mathbf{e}^j_{(i)}=(e^j_{i,1},\dots,e^j_{i,d})$ is the generic node in the multiplicity forest of $S^{(n)}$, we get, for $0\le i<n$, $e^j_{i,}=(\boldsymbol{\alpha}_j^{(n-i)})_h$ if $h\in B_i^j=A_j^{n-i}$, and $e^j_{i,h}=0$ otherwise, and, for $i\ge n$, the unit vectors $\mathbf{e}^j_{(i)}=(0,\dots,0,1,0,\dots,0)$, with 1 in the jth position.

(ii) Let S be an Arf semigroup in $\mathbb{N}^d$ and consider the associated Lipman sequence of semigroups $S=S_{(0)}\subseteq S_{(1)}\subseteq\cdots\subseteq S_{(n)}=\mathbb{N}^d$. We have to show that $S_{(i)}$ is obtained from $S_{(i+1)}$ as in Construction B. Set $T=S_{(i)}$ and suppose that $T=T_1\times\cdots\times T_r$ is its decomposition in local semigroups, with maximal ideals $M_1,\dots,M_r$. Since $\mathrm{rad}(T)=M_1\times\cdots\times M_r$, we have $S_{(i+1)}=\mathrm{rad}(T)-\mathrm{rad}(T)=(M_1-M_1)\times\cdots\times(M_r-M_r)$. For $j=1,\dots,r, M_j-M_j$ is a subsemigroup of $\mathbb{N}^{|A_j|}$ and $A_1,\dots,A_r$ is a partition of $\{1,\dots,d\}$. We have to show that, for $j=1,\dots,r, T_j=\mathbf{0}\cup I_j$, for some principal ideal I_j of M_j-M_j, generated by an element $\boldsymbol{\alpha}_j$ with $(\boldsymbol{\alpha}_j)_h>0$, for each $h\in A_j$. Setting $I_j=M_j$, since T_j is Arf, M_j is principal in M_j-M_j and its generator $\boldsymbol{\alpha}_j$, that is the minimum positive value of M_j has only positive coordinates, i.e., $(\boldsymbol{\alpha}_j)_h>0$, for each $h\in A_j$, because T_j is local. So the proof is complete.

(iii) By (ii) any Arf semigroup is obtained applying Construction B a finite number of times. So an Arf semigroup is uniquely determined by the choice of the partitions $A_1^i,\dots,A_{l_i}^i$ of $\{1,\dots,d\}$ and of the vectors $\boldsymbol{\alpha}_1^i,\dots,\boldsymbol{\alpha}_{l_i}^i$, i.e., it is uniquely determined by its multiplicity forest. $\square$

Corollary 5.8. *Any Arf semigroup is the semigroup of values of a semilocal Arf ring.*

Proof. Let S be an Arf semigroup and let $S=S_{(0)}\subseteq\cdots\subseteq S_{(n)}=\mathbb{N}^d$ be its Lipman sequence. Denote the blowing up forest of S with the notation introduced above. Consider now the ring $R^0=k[[t_1]]\times\cdots\times k[[t_d]]$, where $t_1,\dots,t_d$ are indeterminates over a field k with $|k|\ge d$. Our aim is to apply Construction A for rings, using the numerical information given by the multiplicity forest of S, in order to get a ring $R=R^n$, with $v(R)=S$. More precisely, suppose that $S_{(n-1)}=S^1_{(n-1)}\times\cdots\times S^{l^{n-1}}_{(n-1)}$ is the decomposition in local semigroups of $S_{(n-1)}$ and that $S^1_{(n-1)}\subseteq\mathbb{N}^{|B^1_{n-1}|},\dots,S^{l^{n-1}}_{(n-1)}\subseteq\mathbb{N}^{|B^{l^{n-1}}_{n-1}|}$, where $B^1_{n-1},\dots,B^{l^{n-1}}_{n-1}$ is a partition of $\{1,\dots,d\}$. Choose, for $j=1,\dots,l^{n-1}$, a principal ideal I^1_j of $R^0_{B^j_{n-1}}=\prod_{i\in B^j_{n-1}}k[[t_i]]$ generated by an element a^1_j of value $v_h(a^1_j)=e^j_{n-1,h}$, for $h\in B^j_{n-1}$, where $e^j_{n-1,h}$ is the hth component of the vector $\mathbf{e}^j_{(n-1)}$ in the multiplicity forest of S. Applying Construction A we get a ring $R^1={}_1R^1\times\cdots\times{}_{l_1}R^1$ $(l_1=l^{n-1})$. Now consider the partition $B^1_{n-2},\dots,B^{l^{n-2}}_{n-2}$ of $\{1,\dots,d\}$ determined by the semigroup $S_{(n-2)}$. This partition is coarser than the previous one and we can continue with Construction A, taking into account the vectors $\mathbf{e}^j_{(n-2)}$ of the multiplicity forest of S. In this way, after n steps, we get a semilocal Arf ring R^n, that, by Theorem 5.5, has the same multiplicity forest as S. Since, by Theorem 5.7(iii), there is only one Arf semigroup with a fixed multiplicity forest, we have $v(R^n)=S$. $\square$

According to Theorem 5.7, an Arf semigroup is completely described by its multiplicity forest. If $d=1$, the multiplicity forest of S is just a sequence of numbers $e_{(0)},e_{(1)},\dots,e_{(n)}=1,e_{(n+1)}=1,\dots$ and S is exactly the set $\{0,e_{(0)},e_{(0)}+e_{(1)},e_{(0)}+e_{(1)}+e_{(2)},\dots\}$. This generalizes, for $d\geq 1$, to:

Proposition 5.9. *Suppose that* $\mathbf{e}^1_{(i)},\dots,\mathbf{e}^{l^i}_{(i)}$ *are the nodes at level* i *in the multiplicity forest* $\mathbf{F}$ *of the semigroup* S. *Then* $S=\mathbf{0}\bigcup_{\mathbf{F}'}\{\sum_{\mathbf{e}^j_{(i)}\in\mathbf{F}'}\mathbf{e}^j_{(i)}\}$, *where* $\mathbf{0}\in\mathbb{N}^d$ *and* $\mathbf{F}'$ *ranges over all finite subforests of* $\mathbf{F}$ *rooted in a nonempty subset of* $\mathbf{e}^1_{(0)},\dots,\mathbf{e}^{l^0}_{(0)}$.

Proof. Let us argue by induction on n, where n is the length of the Lipman sequence associated to S, i.e., the smallest integer n such that $S_{(n)}=\mathbb{N}^d$. If $n=0, S=S_{(0)}=\mathbb{N}^d$, the multiplicity forest is the disjoint union of d lines with the unit vectors $(1,0,\dots,0),\dots,(0,\dots,0,1)$ as nodes, respectively, and in this case the statement trivially holds. For the inductive step, let $S=S_{(0)}\subseteq\cdots\subseteq S_{(n)}=\mathbb{N}^d$ be the Lipman sequence of S. We know, by Theorem 5.7, that $S_{(0)}=S^{(n)}$ is obtained from $S_{(1)}=S^{(n-1)}$ as in construction B. So $S_{(1)}=S_{(1)A_1}\times\cdots\times S_{(1)A_l}$ and $S_{(0)}=\prod_{j=1}^{l}(\mathbf{0}\cup(\boldsymbol{\alpha}_j+S_{(1)A_j}))$, where $A_1,\dots,A_l$ is a partition of $\{1,\dots,d\}$. Since for $j=1,\dots,l$ any $S_{(1)A_j}$ is Arf and has a Lipman sequence of length $\leq n-1$, we have by the inductive hypothesis $S_{(1)A_j}=\mathbf{0}\cup(\bigcup_{\mathbf{F}'}\sum_{\mathbf{e}^j_{(i)}\in\mathbf{F}''}\mathbf{e}^j_{(i)})$, where $\mathbf{F}''$ ranges over all finite subforests of $\mathbf{F}$ rooted in a nonempty subset of $\{\mathbf{e}^h_{(1)}|h\in A_j\}$. Moreover, by Theorem 5.7(i)(2), the multiplicity forest of S at level 0 is defined by the vectors $\boldsymbol{\alpha}_j$. More precisely, $\mathbf{e}^j_{(0)}=(e^j_{0,1},\dots,e^j_{0,d})$, where $e^j_{0,h}=(\boldsymbol{\alpha}_j)_h$, if $h\in A_j$ and $e^j_{0,h}=0$ otherwise. To conclude, observe that, since $A_1,\dots,A_l$ is a partition of $\{1,\dots,d\}$, a generic element of $S=S_{(0)}$ is of the form $\sum_{j=1}^{l}\mathbf{x}_j$, where $\mathbf{x}_j\in\mathbb{N}^d$ is such that $\pi_{A_j}(\mathbf{x}_j)\in(\mathbf{0}\cup(\boldsymbol{\alpha}_j+S_{(1)A_j}))$ and $(\mathbf{x}_j)_h=0$ if $h\notin A_j$. $\square$

Proposition 5.10. *The following are equivalent*:

(1) *R is Arf.*

(2) *$S=v(R)$ is Arf and the multiplicity forests of R and S are the same.*

Proof. (1)$\Rightarrow$(2): Suppose R is Arf. By Proposition 3.19, $S=v(R)$ is Arf. Let $R=R_0\subseteq R_1\subseteq\cdots$ and $S=S_{(0)}\subseteq S_{(1)}\subseteq\cdots$ be the Lipman sequences associated to R and S, respectively. We can prove by induction that, for $i\geq 0$, $v(R_i)=S_{(i)}$. The induction step is easily checked, because R_i is Arf, so $R_{i+1}=\mathrm{rad}(R_i):\mathrm{rad}(R_i)$ and, by Proposition 3.19, $v(R_{i+1})=v(\mathrm{rad}(R_i):\mathrm{rad}(R_i))=v(\mathrm{rad}(R_i))-v(\mathrm{rad}(R_i))=\mathrm{rad}(S_{(i)})-\mathrm{rad}(S_{(i)})=S_{(i+1)}$. Thus, in particular, the multiplicity forests of R and S coincide.

(2)$\Rightarrow$(1): Let $\mathbf{F}$ be the multiplicity forest of R and S and let R' be the Arf closure of R. By Proposition 5.3, the multiplicity forest of R' is also $\mathbf{F}$. Apply (1)$\Rightarrow$(2) to the Arf ring R'. We have that $v(R')$ is an Arf semigroup and its multiplicity forest is $\mathbf{F}$. By Theorem 5.7(iii), we get $S=v(R')$. Since $R\subseteq R'$ have the same semigroup of values and are both fractional ideals of R, we have, by Proposition 2.11(iii) that $R=R'$. $\square$

It is natural to ask which are the numerical conditions for a forest of vectors of $\mathbb{N}^d$ in order to be the multiplicity forest of an Arf semigroup. For $d=1$ it is immediate that a sequence of positive integers $e_{(0)}, e_{(1)}, \dots$ is the multiplicity sequence of a numerical Arf semigroup S if and only if

(1) there exists an $n \in \mathbb{N}$ such that, for $m \geq n$, $e_{(m)}=1$ and

(2) for each $i \geq 0$, $e_{(i)} = \sum_{s=1}^{r} e_{(i+s)}$, for some $r \geq 1$.

We generalize, this to any $d \geq 1$, and collect our results in:

Theorem 5.11. *Let* $\mathbf{F}$ *be a forest of vectors* $\{\mathbf{e}^j_{(i)} = (e^j_{i,1}, \dots, e^j_{i,d})\}$ *of* $\mathbb{N}^d$. *The following are equivalent*:

(1) $\mathbf{F}$ *is the multiplicity forest of an Arf semigroup.*

(2) $\mathbf{F}$ *is the multiplicity forest of a ring.*

(3) $\mathbf{F}$ *satisfies the three conditions* (a), (b) *and* (c) *below*

 (a) *there exists* $n \in \mathbb{N}$ *such that, for* $m \geq n$, $\mathbf{e}^j_{(m)} = (0, \dots, 0, 1, 0, \dots, 0)$ (*the nonzero coordinate in the* j*-th position*) *for any* $j = 1, \dots, d$.

 (b) $e^j_{i,h} = 0$ *if and only if* $\mathbf{e}^j_{(i)}$ *is not in the* h*-th branch of the forest* (*the* h*-th branch of the forest is the unique maximal path containing the* h*-th unit vectors*).

 (c) $e^j_{(i)} = \sum_{\mathbf{e} \in T | \mathbf{e}^j_{(i)}} \mathbf{e}$ *for some finite subtree* $\mathbf{T}$ *of* $\mathbf{F}$, *rooted in* $\mathbf{e}^j_{(i)}$.

Proof. The equivalence between (1) and (2) follows from Proposition 5.3, Proposition 5.10, and Corollary 5.8.

As for the equivalence between (1) and (3), if we assume that $\mathbf{F}$ is a multiplicity tree of an Arf semigroup S, properties (a) and (b) follow immediately from the definition. Moreover since any Arf semigroup can be obtained as in Theorem 5.7(i), we have that the non-zero components of $\mathbf{e}^j_{(i)}$ form a vector belonging to ${}_jS^{(n-i-1)}$ (cf. the statement of Theorem 5.7(i)). Applying Theorem 5.9 to this semigroup we get condition (c).

Conversely conditions (a), (b) and (c) are sufficient to construct, starting from $\mathbb{N}^d$ an Arf semigroup (cf. the statement of Theorem 5.7(i)). □

Example 5.12. Consider the forest (in fact tree) in $\mathbb{N}^3$ where the three branches continue upwards with unit vectors (Fig. 10). This is the multiplicity tree of an Arf semigroup, since the conditions of Theorem 5.11 are satisfied. If we replace $\mathbf{e}_{(0)} = (4,5,6)$ for example with (4,4,4), we have a tree that is not the multiplicity tree of an Arf semigroup, since condition (3) of Theorem 5.11 is not satisfied for the second component of (4,4,4).

Corollary 5.13. *Let* $\mathbf{F}$ *be the multiplicity forest of an Arf semigroup* S *and let* $\mathbf{F}_C$ *be the set of nodes of* $\mathbf{F}$ *which are not unit vectors. Then* $\mathbf{F}_C$ *is a finite subforest and* $S - \mathbb{N}^d = \sum_{\mathbf{e} \in \mathbf{F}_C} \mathbf{e} + \mathbb{N}^d$. *If* R *is an Arf ring with* $v(R) = S$ *and if* $\bar{R} = V_1 \times \cdots \times V_d$, *then* $R : \bar{R} = t_1^{\delta_1} V_1 \times \cdots \times t_d^{\delta_d} V_d$, *where for* $i = 1, \dots, d$, t_i *is the uniformizing parameter of* V_i *and* $(\delta_1, \dots, \delta_d) = \sum_{\mathbf{e} \in \mathbf{F}_C} \mathbf{e}$.

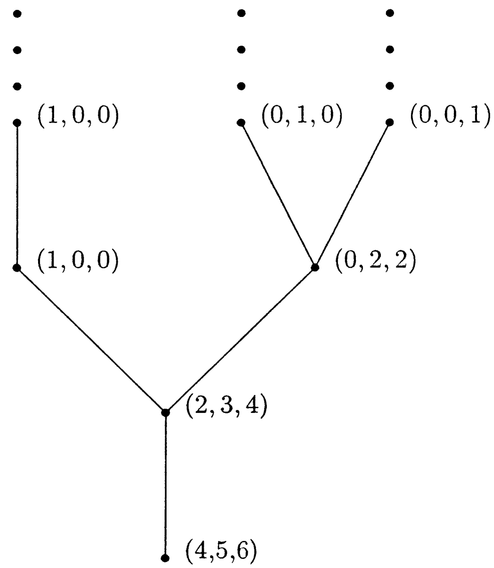

Fig. 10. A concrete example.

Proof. By Theorem 5.11 we have in particular that $\mathbf{e}^{j}_{(i)} \geq \mathbf{e}^{j_1}_{(i+1)}$ if $\mathbf{e}^{j_1}_{(i+1)}$ covers $\mathbf{e}^{j}_{(i)}$. It follows that $\mathbf{F}_C$ is a subforest of $\mathbf{F}$. The second part of the statement also follows from Theorem 5.11, since the vector $\sum_{\mathbf{e}\in\mathbf{F}_C} \mathbf{e}$ is characterized by being the smallest vector such that all larger vectors belong to S. Since $v(R)=S$, $R:\bar{R}$ is the principal ideal of $\bar{R}$ generated by an element of value $\gamma(S)+(1,\dots,1)$. Since $S-\mathbb{N}^d=\sum_{\mathbf{e}\in\mathbf{F}_C}\mathbf{e}+\mathbb{N}^d$, we have $\gamma(S)+(1,\dots,1)=\sum_{\mathbf{e}\in\mathbf{F}_C}\mathbf{e}=(\delta_1,\dots,\delta_d)$. $\square$

Example 5.14. Consider the tree of Fig. 10. As we noticed, it is the multiplicity tree of an Arf semigroup. By Corollary 5.8, it is the semigroup of values of an Arf ring. The proof of Corollary 5.8 indicates how to construct such a ring. Start with $R^0=k[[x]]\times k[[y]]\times k[[z]]$, where x, y, and z are indeterminates over a field k and $|k|\geq 3$, and let $a^1_1=x, a^1_2=(y^2,z^2)$. Then $R^1=k[[x]]\times((1,1)k+y^2k[[y]]\times z^2k[[z]])=U_1\times U_2$. Let $a^2=(x^2,y^3,z^4)$. Then $R^2=(1,1,1)k+(x^2,y^3,z^4)R_1=(1,1,1)k+(x^2k[[x]]\times((y^3,z^4)k+(y^5k[[y]]\times z^6k[[z]]))$. Now, if $a^3=(x^4,y^5,z^6)$, we get $R=R^3=(1,1,1)k+(x^4,y^5,z^6)R^2=(1,1,1)k+(x^4,y^5,z^6)k+(x^6k[[x]]\times((y^8,z^{10})k+(y^{10}k[[y]]\times z^{12}k[[z]])))$ that has the given multiplicity tree. The multiplicity sequences along the three branches are $(15,9,1,1,1,\dots),(15,9,4,1,1,\dots)$, and $(15,9,4,1,1,\dots)$. The conductor $R:\bar{R}=R:R^0$ equals $(x^6,y^{10},z^{12})R^0$ (cf. Corollary 5.13).

Example 5.15. As an example of Theorem 5.11, we give all possible multiplicity trees for local rings of multiplicity 3. It is easy to check that the result agrees with the Arf semigroups we have classified in Section 4, cf. Proposition 4.4. Fig. 11 shows the trees.

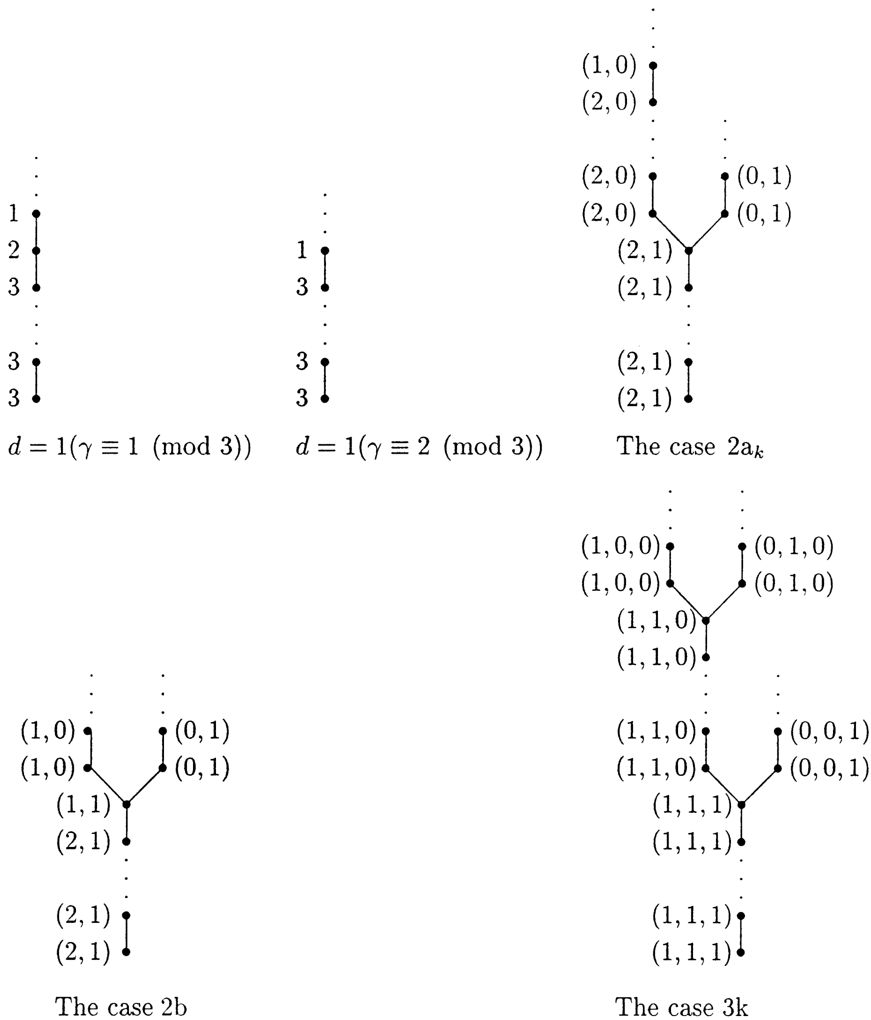

Fig. 11. The possible multiplicity trees for local rings R with $e(R)=3$. In the case $2a_k$ (3_k, resp.), $k \geq 0$ is the number of times (2,0) ((1,1,0), resp.) occur.

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