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Equivalent conditions for noncentral generalized Laplacianness and independence of matrix quadratic forms

Jianhua Hu [1]

School of Statistics and Management, Shanghai University of Finance and Economics, 777 Guoding Rd., Shanghai 200433, PR China

A R T I C L E I N F O

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Let Y be an $n \times p$ multivariate normal random matrix with general covariance Σ_Y and W be a symmetric matrix. In the present article, the property that a matrix quadratic form $Y'WY$ is distributed as a difference of two independent (noncentral) Wishart random matrices is called the (noncentral) generalized Laplacianness (GL). Then a set of algebraic results are obtained which will give the necessary and sufficient conditions for the (noncentral) GL of a matrix quadratic form. Further, two extensions of Cochran's theorem concerning the (noncentral) GL and independence of a family of matrix quadratic forms are developed.

1. Introduction

In the research of the distribution of quadratic forms, the problem that a quadratic form is distributed as a difference of two independent chi-squire random variables and its generalization have been investigated by many scholars. Usually, the equivalent algebraic conditions are expected to characterize the property that a quadratic form is distributed as a difference of two independent chi-squire random variables.

E-mail address: frank.jianhuahu@gmail.com

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Luther [9] established the equivalence between the distribution of a quadratic form as the unique difference of two stochastically independent chi-square distributions and the tripotency of its underlying matrix. Later, Baldessari [2] developed necessary and sufficient conditions under which a quadratic form, in normal random variables, is distributed as a given linear combination of independent chi-square random variables, generalizing Luther's result. Tan [22] extended Baldessari's results to a quadratic form in possibly singular normal random variables. Khatri [8] further extended Baldessari's result to the singular covariance matrix, to quadratic forms and to quadratic expressions.

Moreover, Tan [21] extended the problem from the univariate case to the multivariate case and obtained some extensions of Cochran's theorem concerning differences of independent noncentral Wishart random matrices, where the covariance of normal random matrix Y is the structure of Kronecker product $A \otimes \Sigma$. Wong and Wang [24] extended Tan's results to the case of a general covariance matrix, meaning that the collection of all np elements in Y has an arbitrary $np \times np$ covariance matrix. Masaro and Wong [11] derived a set of necessary and sufficient conditions for Laplace–Wishart distribution associated with matrix quadratic forms when Y follows a multivariate normal distribution with zero mean and Laplace–Wishart distribution has a diagonal covariance. Brief summaries of the related development are available in Anderson and Styan [1] and Hu [6].

Other scholars who also worked on chi-square difference and their generalizations, in distribution function approach, include Pearson et al. [15], Gurland [5], Shah [20], Robinson [19], Press [16] and Provost [17]. The similar research also appeared for gamma difference, see Mathai [13]. Cochran's theorem was proposed in Cochran [3]. A summary of the extensions of Cochran's theorem concerning chi-squareness or Wishartness and independence is given in Hu [6,7], and recently Masaro and Wong [12].

This article will extend Tan's results to the general covariance Σ_Y of Y as did in Wong and Wang [24]. The new results obtained in this article, based on Masaro and Wong's work [11] greatly improve Wong and Wang's works as well as extend Masaro and Wong's work.

In this article, Y denotes an $n \times p$ multivariate normal random matrix with general covariance Σ_Y and W denotes a symmetric matrix. The property that a matrix quadratic form $Y'WY$ is distributed as a difference of two independent (noncentral) Wishart random matrices is called the (noncentral) generalized Laplacianness (GL). The terminology is quoted from Mathai [13]. The organization of this article is as follows.

In Section 2, some necessary preliminaries are summarized. Conditions for the GL of a matrix quadratic form are established in Section 3 and a general extension of Cochran's theorem concerning the GL and independence of a family of matrix quadratic forms is developed in Section 4. The parallel results to, respectively, the noncentral GL of a matrix quadratic form and an extension of Cochran's theorem concerning the noncentral GL and independence of a family of matrix quadratic forms are established in Sections 5 and 6. The concluding remarks is briefly stated in Section 7. The related lemmas are presented in Appendix.

2. Preliminaries

In this paper, $\mathcal{M}_{n\times p}$ denotes the set of $n \times p$ matrices over the real set $\Re$. The trace inner product $\langle , \rangle$ equipped in $\mathcal{M}_{n\times p}$ is defined as $\langle A, B\rangle = \mathrm{tr}(AB')$ for all $A,\ B \in \mathcal{M}_{n\times p}$, where B' is the transpose of B. $\|.\|$ denotes the trace norm in $\mathcal{M}_{n\times p}$ and $|A|$ denotes the determinant of A. $\mathcal{S}_p$ denotes the set of symmetric matrices of order p over the real set $\Re$. $\mathcal{N}_p$ denotes the set of nonnegative definite matrices of order p over the real set $\Re$. I_m denotes the identity matrix of order m.

We use $\mathbf{e}_{ij}$ to denote the matrix whose ijth entry is 1 and all other entries 0 and E_{ij} the symmetric matrix of order p whose ijth entry and jith entry both are 1 and all other entries 0. Write $\mathcal{B}_p = \{E_{ij} : 1 \leqslant i \leqslant j \leqslant p\}$, called the basic base of the set $\mathcal{S}_p$.

For a nonnegative definite matrix Σ of order p, there exists an orthogonal matrix H such that $H'\Sigma H = \mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_p]$ with $\sigma_i \geqslant 0$. Write $\mathcal{H}_p = \{HE_{ij}H' :\ 1 \leqslant i \leqslant j \leqslant p,\ E_{ij} \in \mathcal{B}_p\}$, called the similar base, associated with Σ, of the set $\mathcal{S}_p$.

We use A^+ to denote the Moore–Penrose inverse of matrix A and $Sr(A)$ the spectral radius of square matrix A. A square matrix A is said to be tripotent if $A^3 = A$.

For an $n \times p$ matrix Y, Y is written into $Y = [\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n]'$, $\mathbf{y}_i \in \Re^p$, where $\Re^p$ is the p dimensional real space, and $\text{vec}(Y)$ denotes np dimensional vector $[\mathbf{y}_1', \mathbf{y}_2', \ldots, \mathbf{y}_n']'$. Here the vec operator transforms a matrix into a vector by stacking the rows of the matrix one underneath the other. The Kronecker product $A \otimes B$ of matrices A and B is defined to be $A \otimes B = [a_{ij}B]$. And $(A \otimes B)\text{vec}(C) = \text{vec}(ACB')$.

The commutation matrix K_{np} is defined by $K_{np}\text{vec}(Y') = \text{vec}(Y)$, $Y \in \mathcal{M}_{n\times p}$. And $\Sigma_{Y'} = K_{np}'\Sigma_Y K_{np}$, see Magnus and Neudecker [10].

The following lemma, see Rao [18, Chapter 1], is useful for our subsequent discussion.

Lemma 2.1. *For matrices A, B and C, $AB'B = CB'B$ is equivalent to $AB' = CB'$. Similarly, $B'BA = B'BC$ is equivalent to $BA = BC$.*

Define

$$G(\mathbf{s}, \tilde{\mathbf{s}}, \Sigma, W, \Sigma_Y) = \Sigma_Y(W \otimes \mathbf{s})\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \tilde{\mathbf{s}})\Sigma_Y, \quad \text{for } \mathbf{s}, \tilde{\mathbf{s}} \in \mathcal{S}_p,$$

and

$$\Gamma(\mathbf{s}, \tilde{\mathbf{s}}, \Sigma, W, L) = L(\mathbf{s} \otimes W)L'L(\Sigma^+ \otimes W)L'L(\tilde{\mathbf{s}} \otimes W)L', \quad \text{for } \mathbf{s}, \tilde{\mathbf{s}} \in \mathcal{S}_p.$$

Using the commutation matrix, the properties of the Kronecker product and Lemma 2.1, the following lemma is easily proved.

Lemma 2.2. *Let $\Sigma_Y \in \mathcal{N}_{np}$, $\Sigma \in \mathcal{N}_p$ and $W \in \mathcal{S}_n$. Then for $\mathbf{s}, \tilde{\mathbf{s}} \in \mathcal{S}_p$,*

$$\Sigma_Y[W \otimes (\mathbf{s}\Sigma\tilde{\mathbf{s}} + \tilde{\mathbf{s}}\Sigma\mathbf{s})]\Sigma_Y = G(\mathbf{s}, \tilde{\mathbf{s}}, \Sigma, W, \Sigma_Y) + G(\tilde{\mathbf{s}}, \mathbf{s}, \Sigma, W, \Sigma_Y)$$

is equivalent to

$$L[(\mathbf{s}\Sigma\tilde{\mathbf{s}} + \tilde{\mathbf{s}}\Sigma\mathbf{s}) \otimes W]L' = \Gamma(\mathbf{s}, \tilde{\mathbf{s}}, \Sigma, W, L) + \Gamma(\mathbf{s}, \tilde{\mathbf{s}}, \Sigma, W, L),$$

where $\Sigma_{Y'} = L'L$, $L = [L_1, L_2, \ldots, L_p]$, $q = \text{rank}(\Sigma_Y)$ and $L_i \in \mathcal{M}_{q\times n}$, $i = 1, 2, \ldots, p$.

When we decompose $\Sigma_{Y'}$ as $\Sigma_{Y'} = L'L$, $L = [L_1, L_2, \ldots, L_p]$ with $L_i \in \mathcal{M}_{q\times n}$ $(i = 1, 2, \ldots, p)$ and $r(\Sigma_{Y'}) \leqslant q \leqslant np$, we assume $q = np$ without loss of the generality in our discussion. If $q < np$, we just replace L' by $[L', \mathbf{0}] \in \mathcal{M}_{np\times np}$.

Suppose that $\sigma_1, \sigma_2, \ldots, \sigma_r$ are positive real numbers. Let $B_{ij} = (L_iWL_j' + L_jWL_i')/2\sqrt{\sigma_i\sigma_j}$, $i, j \leqslant r$. Assume that $L_iWL_i' \neq \mathbf{0}$, i.e. $B_{ii} \neq \mathbf{0}$, $(i \leqslant r)$ also without loss of generality.

For convenience, the following conditions (A1)–(A5) are called A-conditions.

(A1) $L[(\mathbf{t}\Lambda\tilde{\mathbf{t}} + \tilde{\mathbf{t}}\Lambda\mathbf{t}) \otimes W]L' = \Gamma(\mathbf{t}, \tilde{\mathbf{t}}, \Lambda, W, L) + \Gamma(\tilde{\mathbf{t}}, \mathbf{t}, \Lambda, W, L)$;
(A2) $L(\Lambda^+ \otimes W)L'L(\mathbf{t} \otimes W)L' = L(\mathbf{t} \otimes W)L'L(\Lambda^+ \otimes W)L'$;
(A3) $\{\mathbf{t} : L(\mathbf{t} \otimes W)L' = \mathbf{0}\} = \{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}\}$;
(A4) $\text{tr}(L(\Lambda^+ \otimes W)L'L(\mathbf{t} \otimes W)L') + \text{tr}(L(\mathbf{t} \otimes W)L') = 2m_1\text{tr}(\Lambda\mathbf{t})$; and
(A5) $\text{tr}(L(\Lambda^+ \otimes W)L'L(\mathbf{t} \otimes W)L') - \text{tr}(L(\mathbf{t} \otimes W)L') = 2m_2\text{tr}(\Lambda\mathbf{t})$.

The following conditions (C1)–(C6) are called C-conditions.

(C1) $L_iWL_j' + L_jWL_i' = \mathbf{0}$ for i or $j > r$;
(C2) $B_{ii}^3 = B_{ii}$, $\text{tr}(B_{ii}) = m_1 - m_2$, $\text{tr}(B_{ii}^2) = m_1 + m_2$;
(C3) $B_{ii}B_{jj} = \mathbf{0}$, $i \neq j$;
(C4) $4B_{ij}^2 = B_{ii}^2 + B_{jj}^2$, $i \neq j$;
(C5) $B_{ii}B_{ij} = B_{ij}B_{jj}$, $i \neq j$; and
(C6) $2(B_{ii} + B_{jj})(B_{ik}B_{jk} + B_{jk}B_{ik}) = B_{ij}$ for all distinct i, j, k.

If $A = X'X$, where $X \sim \mathcal{N}_{m\times p}(\boldsymbol{\mu}, I_m \otimes \Sigma)$ with $\Sigma \in \mathcal{N}_p$, then A is said to have the noncentral Wishart distribution with m degrees of freedom, covariance matrix Σ and noncentrality matrix $\boldsymbol{\lambda} = \boldsymbol{\mu}'\boldsymbol{\mu}$. Write $A \sim \mathcal{W}_p(m, \Sigma, \boldsymbol{\lambda})$. When $\boldsymbol{\mu} = \mathbf{0}$, A is said to have the Wishart distribution with m degrees of freedom and covariance matrix Σ, denoted by $A \sim \mathcal{W}_p(m, \Sigma)$. See Muirhead [14, Chapter 3].

For a symmetric matrix W, $Y'WY$ is called the matrix quadratic form in a normal random matrix Y. $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ means that $Y'WY$ is distributed as a difference of two independent noncentral Wishart random matrices (with a common covariance Σ), implying that $Y'WY$ has the noncentral generalized Laplacianness. Similarly, $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ means that $Y'WY$ is distributed as a difference of two independent Wishart random matrices (with a common covariance Σ), implying that $Y'WY$ has the generalized Laplacianness.

The moment generating function $M(\mathbf{s})$ of $Y'WY$ is defined as $M(\mathbf{s}) = E(e^{\langle \mathbf{s}, Y'WY\rangle})$, $\mathbf{s} \in \mathcal{S}_p$. The following lemma is due to Wong et al. [23].

Lemma 2.3. *Let $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ and W_i's be symmetric. Then the joint moment generating function $M(\mathbf{s})$ of $Y'W_iY$'s is given by*

$$M(\mathbf{s}) = |I_{np} - 2\Sigma^*|^{-1/2}\exp\left\{\langle \mathbf{s}, \boldsymbol{\lambda}\rangle + 2\left\langle \boldsymbol{\mu}^*, \Sigma_Y^{1/2}(I_{np} - 2\Sigma^*)^{-1}\Sigma_Y^{1/2}\boldsymbol{\mu}^*\right\rangle\right\},$$

where $\mathcal{S} = \mathcal{S}_p \times \mathcal{S}_p \times \cdots \times \mathcal{S}_p$ *(l times),* $\mathbf{s} = (\mathbf{s}_i) \in \mathcal{S}$, $\Sigma^* = \Sigma_Y^{1/2}[\sum_{i=1}^l (W_i \otimes \mathbf{s}_i)]\Sigma_Y^{1/2}$, $\boldsymbol{\mu}^* = \sum_{i=1}^l \mathrm{vec}(W_i\boldsymbol{\mu}\mathbf{s}_i)$, $\boldsymbol{\lambda}_i = \boldsymbol{\mu}'W_i\boldsymbol{\mu} \in \mathcal{S}_p$, $\boldsymbol{\lambda} = (\boldsymbol{\lambda}_i) \in \mathcal{S}$ *and* $Sr(\Sigma^*) < 1/2$.

Let $Y \sim \mathcal{N}_{m\times p}(\boldsymbol{\mu}, I_m \otimes \Sigma)$, then $M(\mathbf{s})$ of $Y'Y$, i.e. the moment generating function of the noncentral Wishart distribution $\mathcal{W}_p(m, \Sigma, \boldsymbol{\lambda})$, is equal to

$$M(\mathbf{s}) = |I_p - 2\Sigma_*|^{-m/2}\exp\{\langle \mathbf{s}, \boldsymbol{\lambda}\rangle + 2\langle \boldsymbol{\lambda}, \mathbf{s}\Sigma^{1/2}(I_p - 2\Sigma_*)^{-1}\Sigma^{1/2}\mathbf{s}\rangle\} \tag{2.1}$$

for all $\mathbf{s} \in \mathcal{S}_p$ such that $Sr(\Sigma_*) < 1/2$ with $\boldsymbol{\lambda} = \boldsymbol{\mu}'\boldsymbol{\mu}$, where $\Sigma_* = \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}$. And if $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$, the moment generating function $M(\mathbf{s})$ of $Y'WY$ can be expressed as

$$M(\mathbf{s}) = |I_p - 2\Sigma_*|^{-m_1/2}|I_p + 2\Sigma_*|^{-m_2/2}\exp\{\langle \mathbf{s}, \boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2\rangle + 2\Phi_1 + 2\Phi_2\} \tag{2.2}$$

for all $\mathbf{s} \in \mathcal{S}_p$ such that $Sr(\Sigma_*) < 1/2$, where $\Phi_1 = \langle \boldsymbol{\lambda}_1, \mathbf{s}\Sigma^{1/2}(I_p - 2\Sigma_*)^{-1}\Sigma^{1/2}\mathbf{s}\rangle$ and $\Phi_2 = \langle \boldsymbol{\lambda}_2, \mathbf{s}\Sigma^{1/2}(I_p + 2\Sigma_*)^{-1}\Sigma^{1/2}\mathbf{s}\rangle$.

We can extend (2.2) so that the case $m_1 = 0$ or $m_2 = 0$ or $\Sigma = \mathbf{0}$ is included.

The following result is useful for us to discuss the independence of random matrices, see Hu [6,7].

Lemma 2.4. *Let $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$, and W_i's be a family of symmetric matrices in $\mathcal{S}_n$. Then a family of matrix quadratic form $Y'W_iY$'s is independent if and only if for any distinct $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{t}, \tilde{\mathbf{t}}$ in the basic base $\mathcal{B}_p$,*

(1) $\Sigma_Y(W_i \otimes \mathbf{t})\Sigma_Y(W_j \otimes \tilde{\mathbf{t}})\Sigma_Y = \mathbf{0}$;
(2) $\Sigma_Y(W_i \otimes \mathbf{t})\Sigma_Y(W_j \otimes \tilde{\mathbf{t}})\mathrm{vec}(\boldsymbol{\mu}) = \mathbf{0}$; *and*
(3) $\mathrm{vec}(\boldsymbol{\mu})'(W_i \otimes \mathbf{t})\Sigma_Y(W_j \otimes \tilde{\mathbf{t}})\mathrm{vec}(\boldsymbol{\mu}) = 0$.

3. Algebraic conditions for the GL of a matrix quadratic form

In this section as well as next section, Y is an $n \times p$ multivariate normal random matrix with mean $\mathbf{0}$ and general covariance Σ_Y.

Our investigation begins with the following main theorem. We shall establish a class of sufficient and necessary algebraic conditions to characterize the GL of a matrix quadratic form, i.e. a matrix quadratic form $Y'WY$ being distributed as the difference of two independent Wishart random matrices. The two Wishart distribution have a common covariance Σ.

First let us consider the special case which the common covariance is a diagonal nonnegative definite matrix, written Λ.

Theorem 3.1. *Let $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ with $\Sigma_Y \in \mathcal{N}_{np}$ and W be symmetric. Then $Y'WY \sim \mathcal{W}_p(m_1, \Lambda) - \mathcal{W}_p(m_2, \Lambda)$ with $m_1, m_2 \in \{0, 1, 2, \ldots\}$ and $\Lambda = \mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_r, \mathbf{0}] \in \mathcal{N}_p$ if and only if there exist positive real numbers $\sigma_1, \sigma_2, \ldots, \sigma_r$ ($r \leqslant p$) such that, for any elements $\mathbf{t}, \tilde{\mathbf{t}}$ in the basic base $\mathcal{B}_p$,*

$$\Sigma_Y[W \otimes (\mathbf{t}\Lambda\tilde{\mathbf{t}} + \tilde{\mathbf{t}}\Lambda\mathbf{t})]\Sigma_Y = G(\mathbf{t}, \tilde{\mathbf{t}}, \Lambda, W, \Sigma_Y) + G(\tilde{\mathbf{t}}, \mathbf{t}, \Lambda, W, \Sigma_Y) \tag{3.1}$$

such that

$$\Sigma_Y(W \otimes \Lambda^+)\Sigma_Y(W \otimes \mathbf{t})\Sigma_Y = \Sigma_Y(W \otimes \mathbf{t})\Sigma_Y(W \otimes \Lambda^+)\Sigma_Y, \tag{3.2}$$

$$\{\mathbf{t} : \Sigma_Y(W \otimes \mathbf{t})\Sigma_Y = \mathbf{0}\} = \{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}\}, \tag{3.3}$$

$$\mathrm{tr}(\Sigma_Y(W \otimes \Lambda^+)\Sigma_Y(W \otimes \mathbf{t})) + \mathrm{tr}(\Sigma_Y(W \otimes \mathbf{t})) = 2m_1\mathrm{tr}(\Lambda\mathbf{t}) \tag{3.4}$$

and

$$\mathrm{tr}(\Sigma_Y(W \otimes \Lambda^+)\Sigma_Y(W \otimes \mathbf{t})) - \mathrm{tr}(\Sigma_Y(W \otimes \mathbf{t})) = 2m_2\mathrm{tr}(\Lambda\mathbf{t}). \tag{3.5}$$

Proof. Exactly as in the proof of Lemma 2.2, we easily derive the equivalence between A-conditions and (3.1)–(3.5). Then by Lemma 7.3, see Appendix, we only show that A-conditions are equivalent to C-conditions.

Suppose that C-conditions hold. We shall show that A-conditions hold.

Let $B = \sum_{i=1}^{r} B_{ii}$. Use $(ij, i'j')$ to represent combination $(\mathbf{t}, \tilde{\mathbf{t}})$ from the basic base $\mathcal{B}_p$. Then by (C1) we only consider these combinations $(ij, i'j')$, $1 \leqslant i \leqslant j \leqslant r, 1 \leqslant i' \leqslant j' \leqslant r$. Write $\Omega = \{(ij, i'j') : 1 \leqslant i \leqslant j \leqslant r, 1 \leqslant i' \leqslant j' \leqslant r\}$. Divide the index set Ω into the following seven index subsets:

$$D_1 = \{(ii, ii) : \ 1 \leqslant i \leqslant r\};$$

$$D_2 = \{(ij, ij) : \ 1 \leqslant i < j \leqslant r\};$$

$$D_3 = \{(ii, jj) : \ 1 \leqslant i, j \leqslant r;\ i \neq j\};$$

$$D_4 = \{(ii, ij) \cup (ij, ii) : \ 1 \leqslant i < j \leqslant r\};$$

$$D_5 = \{(ik, jk) : \ 1 \leqslant i, j < k \leqslant r;\ i, j \text{ distinct}\};$$

$$D_6 = \{(ii, i'j') \cup (i'j', ii) : \ 1 \leqslant i, i', j' \leqslant r;\ i, i', j' \text{ distinct}, i' < j'\}; \text{ and}$$

$$D_7 = \{(ij, i'j') : \ 1 \leqslant i < j \leqslant r, 1 \leqslant i' < j' \leqslant r;\ i, j, i', j' \text{ distinct}\}.$$

Note that by (C3), (C4) and Lemma 2.1,

$$B_{ij}B_{kk} = \mathbf{0} \text{ for distinct } i, j, k. \tag{3.6}$$

For $(ij, i'j') \in D_1$, (A1) reduces to $\sigma_i\sigma_j(B_{ii} + B_{jj}) = \sigma_i^2 B_{ii}BB_{ii}$, which follows from (C2) and (C3).

For $(ij, i'j') \in D_2$, (A1) reduces to $\sigma_i\sigma_j(B_{ii} + B_{jj}) = 4\sigma_i\sigma_j B_{ij}BB_{ij}$, which is derived from (C5) and (3.6).

For $(ij, i'j') \in D_3$, (A1) reduces to $\sigma_i\sigma_j(B_{ii}BB_{jj} + B_{jj}BB_{ii}) = \mathbf{0}$, which is obtained from (C3).

For $(ij, i'j') \in D_4$, (A1) reduces to $2\sqrt{\sigma_i\sigma_j}\sigma_i B_{ij} = 2\sqrt{\sigma_i\sigma_j}\sigma_i(B_{ii}BB_{ij} + B_{ij}BB_{ii})$, which follows from (C5), (C6) and (3.6).

For $(ij, i'j') \in D_5$, (A1) reduces to $2\sqrt{\sigma_i\sigma_j}\sigma_k B_{ij} = 4\sqrt{\sigma_i\sigma_j}\sigma_k(B_{ik}BB_{jk} + B_{jk}BB_{ik})$, which is gotten from (C5), (C6) and (3.6).

For $(ij, i'j') \in D_6 \cup D_7$, (A1) reduces to $4\sqrt{\sigma_i\sigma_j\sigma_{i'}\sigma_{j'}}(B_{ij}BB_{i'j'} + B_{i'j'}BB_{ij}) = \mathbf{0}$, which follows from (3.6). So (A1) holds.

For $(ij, i'j') \in \Omega$, (A2) or $BB_{ij} = B_{ij}B$ follows from (3.6) and (C5).

(C1) tells us the fact that $\{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}\} = \{\mathbf{t}_{ij} \in \mathcal{B}_p : i > r \text{ or } j > r\} \subseteq \{\mathbf{t} : L(\mathbf{t} \otimes W)L' = \mathbf{0}\}$. For $\mathbf{t}_{ij} \in \mathcal{B}_p, 1 \leqslant i \leqslant j \leqslant r, L(\mathbf{t}_{ij} \otimes W)L' = \sqrt{\sigma_i\sigma_j}B_{ij} \neq \mathbf{0}$ from (C2), (C4) and Lemma 2.1. So $\{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}\} = \{\mathbf{t} : L(\mathbf{t} \otimes W)L' = \mathbf{0}\} = \{\mathbf{t}_{ij} \in \mathcal{B}_p : i > r \text{ or } j > r\}$, which proves (A3).

Finally, for $E_{ii} \in \mathcal{B}_p$, $i = 1, 2, \ldots, r$, with simple calculation, (A4) and (A5) hold by (C2) and (C3). For B_{ij} $i, j > r$, it is a trivial thing. For B_{ij} $i \neq j, i, j \leqslant r$, the right side values of (A4) and (A5) always are zero. We only need to calculate the left side values (LSVs) of both (A4) and (A5). By (A.1) and (A.2) in Lemma 7.2, we have

$$\begin{aligned}\text{LSVs} &= \mathrm{tr}(L(\Lambda^+ \otimes W)L'L(B_{ij} \otimes W)L') \pm \mathrm{tr}(L(B_{ij} \otimes W)L') \\ &= 2\sqrt{\sigma_i\sigma_j}[\mathrm{tr}(BB_{ij}) \pm \mathrm{tr}(B_{ij})] = 2\sqrt{\sigma_i\sigma_j}\mathrm{tr}((B_{ii} + B_{jj} \pm I)B_{ij}) \\ &= \sqrt{\sigma_i\sigma_j}\mathrm{tr}[(\mathbf{e}_{ii} \otimes A_{ii} + \mathbf{e}_{jj} \otimes A_{jj} \pm I)(\mathbf{e}_{ij} \otimes A_{ij} + \mathbf{e}_{ji} \otimes A_{ji})] = 0,\end{aligned}$$

which proves (A4) and (A5). Hence A-conditions hold.

Conversely, suppose A-conditions hold. We must prove that C-conditions hold.

(C1) follows from (A3), i.e. $L(B_{ij} \otimes W)L' = \mathbf{0}$, for i or $j > 0$. Fixing $(1 \leqslant i < j \leqslant r)$ and taking $\mathbf{t} = \tilde{\mathbf{t}} = E_{ii}$ in (A1) and (A2), we have

$$B_{ii} = B_{ii}BB_{ii}, \quad B_{ii}B = BB_{ii} \tag{3.7}$$

and

$$\mathrm{tr}(B_{ii}) = m_1 - m_2, \quad \mathrm{tr}(BB_{ii}) = m_1 + m_2. \tag{3.8}$$

Taking $\mathbf{t} = E_{ii}$ and $\tilde{\mathbf{t}} = E_{jj}$ in (A1) gives

$$B_{ii}BB_{jj} + B_{jj}BB_{ii} = \mathbf{0}. \tag{3.9}$$

By (3.7) and (3.9), we get $||B_{ii}B_{jj} \pm B_{jj}B_{ii}||^2 = 0$, i.e. $B_{ii}B_{jj} = \mathbf{0}$, which proves (C3). Further, we have $B^3 = B$, then $B_{ii}^3 = B_{ii}$ and $\mathrm{tr}(B_{ii}^2) = \mathrm{tr}(B_{ii}B_{ii}BB_{ii}) = \mathrm{tr}(B_{ii}^3B) = \mathrm{tr}(B_{ii}B) = m_1 + m_2$, which, with (3.8), proves (C2).

Taking $\mathbf{t} = E_{ii}$ and $\tilde{\mathbf{t}} = B_{ij}$ in (A1) gives $B_{ij} = B_{ii}BB_{ij} + B_{ij}BB_{ii}$. Taking $\mathbf{t} = B_{ij}$ and $\tilde{\mathbf{t}} = E_{jj}$ in (A1) and (A2) gives

$$B_{ij} = B_{ij}BB_{jj} + B_{jj}BB_{ij}, \ \ BB_{ij} = B_{ij}B. \tag{3.10}$$

So $B_{ii}B_{ij} = B_{ii}B_{ij}BB_{jj}$ and $B_{ij}B_{jj} = B_{ii}BB_{ij}B_{jj}$, which proves (C5).

Taking $\mathbf{t} = \tilde{\mathbf{t}} = B_{ij}$ in (A1) gives

$$4B_{ij}BB_{ij} = B_{ii} + B_{jj}. \tag{3.11}$$

From (C3), (C5) and (3.10)–(3.11), we obtain $4B_{ij}^2 = B_{ii}^2 + B_{jj}^2$, which proves (C4).

From (3.10), (C3) and (C5), we obtain, for distinct i, j, k,

$$B_{ij}B_{kk} = \mathbf{0}. \tag{3.12}$$

Taking $\mathbf{t} = E_{ik}$ and $\tilde{\mathbf{t}} = E_{jk}$ for distinct i, j, k in (A1) gives

$$B_{ij} = 2B_{ik}BB_{jk} + 2B_{jk}BB_{ik}. \tag{3.13}$$

So from (3.13), (3.12) and (C5), we get $B_{ij} = 2(B_{ii} + B_{jj})(B_{ik}B_{jk} + 2B_{jk}B_{ik})$, which proves (C6). Thus the desired result is proved. □

In Theorem 3.1, condition (3.1) reveals the most important inherent property for the GL of a matrix quadratic form. Condition (3.2) tells us that matrix $\Sigma_Y^{1/2}(W \otimes \Lambda^+)\Sigma_Y^{1/2}$ is commutative with any matrix $\Sigma_Y^{1/2}(W \otimes \mathbf{t})\Sigma_Y^{1/2}$ for any $\mathbf{t}$ in the basic base $\mathcal{B}_p$. Condition (3.3) says that two different linear transformations $\Sigma_Y(W \otimes \mathbf{t})\Sigma_Y$ and $\Lambda\mathbf{t}\Lambda$ have the same kernel or null space. Conditions (3.4) and (3.5), respectively, determines the degrees m_1, m_2 of freedom of two Wishart random matrices.

Next we shall extend Theorem 3.1 to the general case which the common covariance is a general nonnegative definite matrix Σ. A set of the corresponding sufficient and necessary algebraic conditions is summarized in the following theorem.

Theorem 3.2. *Suppose that* $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ *with* $\Sigma_Y \in \mathcal{N}_{np}$ *and* W *is symmetric. Then* $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ *with* $m_1, m_2 \in \{0, 1, 2, \ldots\}$ *and* $\Sigma \in \mathcal{N}_p$ *if and only if there exists some* $\Sigma \in \mathcal{N}_p$ *such that, for any elements* $\mathbf{h}, \tilde{\mathbf{h}}$ *in the similar base* $\mathcal{H}_p$ *associated with* Σ,

$$\Sigma_Y[W \otimes (\mathbf{h}\Sigma\tilde{\mathbf{h}} + \tilde{\mathbf{h}}\Sigma\mathbf{h})]\Sigma_Y = G(\mathbf{h}, \tilde{\mathbf{h}}, \Sigma, W, \Sigma_Y) + G(\tilde{\mathbf{h}}, \mathbf{h}, \Sigma, W, \Sigma_Y) \tag{3.14}$$

such that

$$\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y, \tag{3.15}$$

$$\{\mathbf{h} : \Sigma_Y(W \otimes \mathbf{h})\Sigma_Y = \mathbf{0}\} = \{\mathbf{h} : \Sigma\mathbf{h}\Sigma = \mathbf{0}\}, \tag{3.16}$$

$$\mathrm{tr}(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})) + \mathrm{tr}(\Sigma_Y(W \otimes \mathbf{h})) = 2m_1\mathrm{tr}(\Sigma\mathbf{h}) \tag{3.17}$$

and

$$\mathrm{tr}(\Sigma_Y(W \otimes \Sigma^+)\Sigma_Y(W \otimes \mathbf{h})) - \mathrm{tr}(\Sigma_Y(W \otimes \mathbf{h})) = 2m_2\mathrm{tr}(\Sigma\mathbf{h}). \tag{3.18}$$

Proof. If $\Sigma \in \mathcal{N}_p$, there is an orthogonal matrix H of order p such that

$$H'\Sigma H = \mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_r, \mathbf{0}] \equiv \Lambda, \quad r = r(\Sigma), \quad \sigma_i > 0, \quad i = 1, 2, \ldots, r \tag{3.19}$$

and $YH \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_{YH})$ where $\Sigma_{YH} = (I \otimes H')\Sigma_Y(I \otimes H)$.

Assume that there exists $\Sigma \in \mathcal{N}_p$ such that (3.14)–(3.18) hold. Let $\mathbf{t} = H'\mathbf{h}H$, then function $\mathbf{t} = H'\mathbf{h}H$ is a $1-1$ map from the similar base $\mathcal{H}_p$ associated with Σ onto the basic base $\mathcal{B}_p$. By replacing $\mathbf{h}$, $\tilde{\mathbf{h}}$ and Σ, respectively, with $H\mathbf{t}H'$, $H\tilde{\mathbf{t}}H'$ and $H\Lambda H'$ in (3.14)–(3.18), with necessary Kronecker product operations, (3.14)–(3.18) are equivalent to the following equations, for $\mathbf{t}, \tilde{\mathbf{t}} \in \mathcal{B}_p$,

$$\Sigma_{YH}\left[W \otimes (\mathbf{t}\Lambda\tilde{\mathbf{t}} + \tilde{\mathbf{t}}\Lambda)\right]\Sigma_{YH} = G(\mathbf{t}, \tilde{\mathbf{t}}, \Lambda, W_i, \Sigma_{YH}) + G(\tilde{\mathbf{t}}, \mathbf{t}, \Lambda, W_i, \Sigma_{YH}),$$

$$\Sigma_{YH}(W \otimes \Lambda^+)\Sigma_{YH}(W \otimes \mathbf{t})\Sigma_{YH} = \Sigma_{YH}(W \otimes \mathbf{t})\Sigma_{YH}(W \otimes \Lambda^+)\Sigma_{YH},$$

$$\{\mathbf{t} : \Sigma_{YH}(W \otimes \mathbf{t})\Sigma_{YH} = \mathbf{0}\} = \{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}\},$$

$$\mathrm{tr}(\Sigma_{YH}(W \otimes \Lambda^+)\Sigma_{YH}(W \otimes \mathbf{t})) + \mathrm{tr}(\Sigma_{YH}(W \otimes \mathbf{t})) = 2m_1\mathrm{tr}(\Lambda\mathbf{t})$$

and

$$\mathrm{tr}(\Sigma_{YH}(W \otimes \Lambda^+)\Sigma_{YH}(W \otimes \mathbf{t})) - \mathrm{tr}(\Sigma_{YH}(W \otimes \mathbf{t})) = 2m_2\mathrm{tr}(\Lambda\mathbf{t}),$$

where Λ is determined by (3.19).

By Theorem 3.1, $H'Y'WYH \sim \mathcal{W}_p(m_1, \Lambda) - \mathcal{W}_p(m_2, \Lambda)$. Hence $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$ by Theorem 3.2.4 of Muirhead (1982). The converse can be shown by following the above steps backwards. □

Conditions (3.14)–(3.18) are same as the conditions (3.1)–(3.5) except for using the similar base $\mathcal{H}_p$ associated with Σ to replace the basic base $\mathcal{B}_p$ of $\mathcal{S}_p$.

Whenever $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$, then the degrees m_1, m_2 of freedom are determined by

$$\begin{aligned} m_1 &= [\mathrm{tr}(\Sigma_Y(W \otimes \Sigma^+))^2 + \mathrm{tr}(\Sigma_Y(W \otimes \Sigma^+))]/2\mathrm{r}(\Sigma) \text{ and} \\ m_2 &= [\mathrm{tr}(\Sigma_Y(W \otimes \Sigma^+))^2 - \mathrm{tr}(\Sigma_Y(W \otimes \Sigma^+))]/2\mathrm{r}(\Sigma), \end{aligned} \tag{3.20}$$

where $\mathrm{r}(A)$ denotes the rank of matrix A.

In Theorem 3.2, if $\mathbf{y}$ is an $n \times 1$ random normal vector with mean vector $\mathbf{0}$ and covariance C of order n, the conditions (3.14)–(3.18) reduce to the familiar algebraic conditions.

Corollary 3.3. *Let $y \sim \mathcal{N}_n(\mathbf{0}, C)$ with $C \in \mathcal{N}_n$ and W be a symmetric matrix of order n. Then $\mathbf{y}'W\mathbf{y} \sim \chi^2(m_1) - \chi^2(m_2)$, a difference of two independent chi-square random variables, with m_1, $m_2 \in \{0, 1, 2, \ldots\}$ if and only if*

(1) $CWC = CWCWCWC \neq \mathbf{0}$; *and*
(2) $\mathrm{tr}(CW)^2 + \mathrm{tr}(CW) = 2m_1$, $\mathrm{tr}(CW)^2 - \mathrm{tr}(CW) = 2m_2$.

If $C = I$ in Corollary 3.3, statement (1) of Corollary 3.3 reduces to the well-known tripotent condition, $W^3 = W$, which is the necessary and sufficient condition to a quadratic form $\mathbf{y}'W\mathbf{y}$ being distributed as a difference of two independent chi-squared random variables, see Luther [9] and Graybill [4].

4. Algebraic conditions for the GL and independence of a family of matrix quadratic forms

Based on Theorem 3.1 and (1) of Lemma 2.4, we shall establish the following result on the GL and independence of a family of matrix quadratic forms. In other words, we shall provide an extension of Cochran's theorem concerning the GL and independence of a family of matrix quadratic forms.

Similar to the discussion in Section 3, first let us consider the simplest case where the common covariance Λ of independent Wishart random matrices is diagonal.

Theorem 4.1. *Suppose that $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ with $\Sigma_Y \in \mathcal{N}_{np}$ and W_i's are symmetric matrices of order n. Then $Y'W_1Y, Y'W_2Y, \ldots, Y'W_lY$ are independent and, for any $i \in \{1, 2, \ldots, l\}$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Lambda) - \mathcal{W}_p(m_{2i}, \Lambda)$ with $m_{1i}, m_{2i} \in \{0, 1, 2, \ldots\}$ and $\Lambda = \mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_r, \mathbf{0}] \in \mathcal{N}_p$ if and only if there exist positive real numbers $\sigma_1, \sigma_2, \ldots, \sigma_r$ $(r \leqslant p)$ such that, for any distinct $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{t}, \tilde{\mathbf{t}}$ in the basic base $\mathcal{B}_p$,*

(a) $\Sigma_Y[W_i \otimes (\mathbf{t}\Lambda\tilde{\mathbf{t}} + \tilde{\mathbf{t}}\Lambda\mathbf{t})]\Sigma_Y = G(\mathbf{t}, \tilde{\mathbf{t}}, \Lambda, W_i, \Sigma_Y) + G(\tilde{\mathbf{t}}, \mathbf{t}, \Lambda, W_i, \Sigma_Y)$;
(b) $\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_i \otimes \mathbf{t})\Sigma_Y = \Sigma_Y(W_i \otimes \mathbf{t})\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y$;
(c) $\{\mathbf{t} : \Sigma_Y(W_i \otimes \mathbf{t})\Sigma_Y = \mathbf{0}\} = \{\mathbf{t} : \Lambda\mathbf{t}\Lambda = \mathbf{0}\}$;
(d) $\mathrm{tr}(\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_i \otimes \mathbf{t})) + \mathrm{tr}(\Sigma_Y(W_i \otimes \mathbf{t})) = 2m_{1i}\mathrm{tr}(\Lambda\mathbf{t})$,
$\mathrm{tr}(\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_i \otimes \mathbf{t})) - \mathrm{tr}(\Sigma_Y(W_i \otimes \mathbf{t})) = 2m_{2i}\mathrm{tr}(\Lambda\mathbf{t})$; *and*
(e) $\Sigma_Y(W_i \otimes \Lambda^+)\Sigma_Y(W_j \otimes \Lambda^+)\Sigma_Y = \mathbf{0}$.

Proof. Let $\{Y'W_iY\}_{i=1}^{l}$ be an independent family of random matrices and $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Lambda) - \mathcal{W}_p(m_{2i}, \Lambda)$ with $m_{1i}, m_{2i} \in \{0, 1, 2, \ldots\}$ and $\Lambda = diag[\sigma_1, \sigma_2, \ldots, \sigma_r, \mathbf{0}] \in \mathcal{N}_p$, $i = 1, 2, \ldots, l$. Then (a)–(e) follow from Theorem 3.1 and (1) of Lemma 2.4.

Conversely, suppose that there exist positive real numbers $\sigma_1, \sigma_2, \ldots, \sigma_r$ $(r \leqslant p)$ such that (a)–(e) hold. For each i, from Theorem 3.1, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Lambda) - \mathcal{W}_p(m_{2i}, \Lambda)$ with $m_{1i}, m_{2i} \in \{0, 1, 2, \ldots\}$ and $\Lambda = \mathrm{diag}[\sigma_1, \sigma_2, \ldots, \sigma_r, \mathbf{0}] \in \mathcal{N}_p$. To prove the independence of a family of matrix quadratic forms, by Lemma 2.4, it suffices to show (a) of Lemma 2.4, or equivalently,

$$\Sigma_Y(W_i \otimes \mathbf{s}_i)\Sigma_Y(W_j \otimes \mathbf{s}_j)\Sigma_Y = \mathbf{0}, \quad \text{for } \mathbf{s}_i, \mathbf{s}_j \in \mathcal{S}_p, \tag{4.1}$$

from conditions (a)–(e).

Exactly as in the proof of Lemma 2.2, (4.1) is equivalent to

$$L(\mathbf{s}_i \otimes W_i)L'L(\mathbf{s}_j \otimes W_j)L' = \mathbf{0}, \quad \text{where } L'L = \Sigma_{Y'},\ \mathbf{s}_i, \mathbf{s}_j \in \mathcal{S}_p \tag{4.2}$$

and condition (e) amounts to

$$L(\Lambda^+ \otimes W_i)L'L(\Lambda^+ \otimes W_j)L' = \mathbf{0}. \tag{4.3}$$

Then we only need to obtain (4.2) from statements (a)–(e).

For matrix $\mathbf{s}_i$ in set $\mathcal{S}_p$, $\mathbf{s}_i$ can be written as

$$\mathbf{s}_i = \begin{bmatrix} \mathbf{a} & * \\ * & * \end{bmatrix}_{p\times p} \quad \text{where } \mathbf{a} \in \mathcal{S}_r.$$

Write

$$\mathbf{s}_i^* = \begin{bmatrix} \mathbf{a} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}_{p\times p} \quad \text{where } \mathbf{a} \in \mathcal{S}_r.$$

By (c), for any $\mathbf{s}_i, \mathbf{s}_j \in \mathcal{S}_p$,

$$L(\mathbf{s}_i \otimes W_i)L'L(\mathbf{s}_j \otimes W_j)L' = L(\mathbf{s}_i^* \otimes W_i)L'L(\mathbf{s}_j^* \otimes W_j)L'. \tag{4.4}$$

Since, by (a) and (b),

$$\begin{aligned} 2L(\mathbf{s}_i^* \otimes W_i)L' &= L[(\Lambda^+ \Lambda \mathbf{s}_i^* + \mathbf{s}_i^* \Lambda \Lambda^+) \otimes W_i]L' \\ &= [L(\Lambda^+ \otimes W_i)L']^2 L(\mathbf{s}_i^* \otimes W_i)L' + L(\mathbf{s}_i^* \otimes W_i)L'[L(\Lambda^+ \otimes W_i)L']^2 \\ &= 2L(\mathbf{s}_i^* \otimes W_i)L'[L(\Lambda^+ \otimes W_i)L']^2, \end{aligned} \tag{4.5}$$

similarly,

$$L(\mathbf{s}_j^* \otimes W_j)L' = [L(\Lambda^+ \otimes W_j)L']^2 L(\mathbf{s}_j^* \otimes W_j)L', \tag{4.6}$$

we obtain (4.2) from (4.3)–(4.6). So, we have completed the proof. □

In Theorem 4.1, condition (e) tells us that one equation can be used to reveal the independence of a set of matrix quadratic forms if these matrix quadratic forms have the GL.

Next, we will consider the general case where the common covariance Σ of these Wishart random matrices is nonnegative definite. Exactly as in the proof of Theorem 3.2, we can easily derive the following theorem, an extension of Cochran's theorem concerning the GL and independence of a set of matrix quadratic forms, from Theorem 4.1 and (1) of Lemma 2.4 with an appropriate modification by replacing $\mathcal{H}_p$ with $\mathcal{B}_p$. See Hu [6, Chapter 4].

Theorem 4.2. *Suppose that $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ with $\Sigma_Y \in \mathcal{N}_{np}$ and W_i's are symmetric matrix of order n. Then $Y'W_1Y, Y'W_2Y, \ldots, Y'W_lY$ are independent and, for any $i \in \{1, 2, \ldots, l\}$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma) - \mathcal{W}_p(m_{2i}, \Sigma)$ with $m_{1i}, m_{2i} \in \{0, 1, 2, \ldots\}$ and $\Sigma \in \mathcal{N}_p$ if and only if there exists some $\Sigma \in \mathcal{N}_p$ such that, for any distinct $i, j \in \{1, 2, \ldots, l\}$ and any $\mathbf{h}, \tilde{\mathbf{h}}$ in the similar base $\mathcal{H}_p$ associated with Σ,*

(a) $\Sigma_Y[W_i \otimes (\mathbf{h}\Sigma\tilde{\mathbf{h}} + \tilde{\mathbf{h}}\Sigma\mathbf{h})]\Sigma_Y = G(\mathbf{h}, \tilde{\mathbf{h}}, \Sigma, W_i, \Sigma_Y) + G(\tilde{\mathbf{h}}, \mathbf{h}, \Sigma, W_i, \Sigma_Y)$;
(b) $\Sigma_Y(W_i \otimes \Sigma^+)\Sigma(W_i \otimes \mathbf{h})\Sigma_Y = \Sigma_Y(W_i \otimes \mathbf{h})\Sigma_Y(W_i \otimes \mathbf{\Sigma}^+)\Sigma_Y$;
(c) $\{\mathbf{h} : \Sigma_Y(W_i \otimes \mathbf{h})\Sigma_Y = \mathbf{0}\} = \{\mathbf{h} : \Sigma\mathbf{h}\Sigma = \mathbf{0}\}$;
(d) $\mathrm{tr}(\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_i \otimes \mathbf{h})) + \mathrm{tr}(\Sigma_Y(W_i \otimes \mathbf{h})) = 2m_{1i}\mathrm{tr}(\Sigma\mathbf{h})$,
$\mathrm{tr}(\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_i \otimes \mathbf{h})) - \mathrm{tr}(\Sigma_Y(W_i \otimes \mathbf{h})) = 2m_{2i}\mathrm{tr}(\Sigma\mathbf{h})$; *and*
(e) $\Sigma_Y(W_i \otimes \Sigma^+)\Sigma_Y(W_j \otimes \Sigma^+)\Sigma_Y = \mathbf{0}$.

In Theorem 4.2, if we replace covariance Σ_Y of Y with the sum of special Kronecker products, we have the following corollary, an application of Theorem 4.2 on a special case. See Hu [6, Chapter 4].

Corollary 4.3. *Let $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ with $\Sigma_Y = \sum_{a=1}^{r} A_a \otimes E_{aa}$, $r \leqslant p$, $A_a \in \mathcal{N}_n$, and W_i's are symmetric matrices of order n. Then $Y'W_1Y, Y'W_2Y, \ldots, Y'W_lY$ are independent and, for any $i \in \{1, 2, \ldots, l\}$, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma) - \mathcal{W}_p(m_{2i}, \Sigma)$ with $m_{1i}, m_{2i} \in \{0, 1, 2, \ldots\}$ and $\Sigma = \sum_{b=1}^{r} \sigma_b E_{bb}$ if and only if there exist positive real numbers $\sigma_1, \sigma_2, \ldots, \sigma_r$ $(r \leqslant p)$ such that, for all $a, b, c \in \{1, 2, \ldots, r\}$ and any distinct $i, j \in \{1, 2, \ldots, l\}$,*

(1) $A_aW_iA_cW_iA_cW_iA_b = \sigma_c^2 A_aW_iA_b \neq \mathbf{0}$;
(2) $\sigma_b A_aW_iA_aW_iA_b = \sigma_a A_aW_iA_bW_iA_b$;
(3) $A_aW_iA_aW_jA_a = \mathbf{0}$;
(4) $\mathrm{tr}(A_aW_i)^2/\sigma_a^2 + \mathrm{tr}(A_aW_i)/\sigma_a^2 = 2m_{1i}$; *and*
(5) $\mathrm{tr}(A_aW_i)^2/\sigma_a^2 - \mathrm{tr}(A_aW_i)/\sigma_a^2 = 2m_{2i}$.

5. Conditions for the noncentral GL of a matrix quadratic form

In this section and next section, Y is an $n \times p$ multivariate normal random matrix with nonzero mean $\boldsymbol{\mu}$ and general covariance Σ_Y.

We shall use the moment generating function $M(\mathbf{s})$ of $Y'WY$ to extend Theorem 3.2 to the case of Y having nonzero mean. The following theorem summarizes a set of sufficient and necessary algebraic conditions for the noncentral GL of a matrix quadratic form.

Theorem 5.1. *Suppose $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ with $\Sigma_Y \in \mathcal{N}_{np}$ and W is a symmetric matrix. Then $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ with $m_1, m_2 \in \{0, 1, 2, \ldots\}$, $\Sigma \in \mathcal{N}_p$ and $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathcal{M}_{p\times p}$ if and only if there exists some $\Sigma \in \mathcal{N}_p$ such that, in addition to (3.14)–(3.18), for any $\mathbf{s}$ in a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathcal{S}_p$ and $k = 1, 2, \ldots,$*

$$\operatorname{tr}((\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k-1}) = \operatorname{tr}(\Delta(W \otimes \mathbf{s})[\Sigma_Y(W \otimes \mathbf{s})]^{2k-1}) \tag{5.1}$$

and

$$\operatorname{tr}((\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k}) = \operatorname{tr}(\Delta(W \otimes \mathbf{s})[\Sigma_Y(W \otimes \mathbf{s})]^{2k}) \tag{5.2}$$

with

$$\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W\boldsymbol{\mu} \tag{5.3}$$

where $\Delta = \operatorname{vec}(\boldsymbol{\mu})\operatorname{vec}(\boldsymbol{\mu})'$.

Proof. By Lemma 2.3, the moment generating function $M(\mathbf{s})$ of $Y'WY$ is expressed as

$$M(\mathbf{s}) = \left|I - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}\right|^{-1/2} \exp\{\langle \mathbf{s}, \boldsymbol{\mu}'W\boldsymbol{\mu}\rangle + 2\Phi_0\}, \tag{5.4}$$

where $\Phi_0 = \langle \Delta,\ (W \otimes \mathbf{s})\Sigma_Y^{1/2}[I - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}]^{-1}\Sigma_Y^{1/2}(W \otimes \mathbf{s})\rangle$ and the spectral radius of square matrix $\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}$ is less than 1/2. So, $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ is equivalent to $Y'WY = D_1 - D_2$, where D_1, D_2 are independent and $D_1 \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1)$, $D_2 \sim \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$. By (2.1), $M_1(\mathbf{s})$ of D_1 and $M_2(\mathbf{s})$ of $-D_2$ are given, respectively, by

$$M_1(\mathbf{s}) = |I - 2\Sigma_*|^{-m_1/2}\exp\{\langle \mathbf{s}, \boldsymbol{\lambda}_1\rangle + 2\Phi_1\} \tag{5.5}$$

and

$$M_2(\mathbf{s}) = |I + 2\Sigma_*|^{-m_2/2}\exp\{\langle -\mathbf{s}, \boldsymbol{\lambda}_2\rangle + 2\Phi_2\}, \tag{5.6}$$

where $\Sigma_* = \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}$ for $\mathbf{s} \in \mathcal{S}_p$ such that $Sr(\Sigma_*) < 1/2$ and Φ_i's are defined in (2.2).

The independence of D_1 and D_2 and (5.4)–(5.6) imply that there exists a neighborhood $\mathcal{N}_0$ of $\mathbf{0}$ in $\mathcal{S}_p$ such that $M(\mathbf{s}) = M_1(\mathbf{s})M_2(\mathbf{s})$ for $\mathbf{s} \in \mathcal{N}_0$. Using (5.4)–(5.6) and comparing the same items in both sides of $M(\mathbf{s}) = M_1(\mathbf{s})M_2(\mathbf{s})$, we obtain the following conditions:

(i) $|I - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}|^{-1/2} = |I - 2\Sigma_*|^{-m_1/2}|I + 2\Sigma_*|^{-m_2/2}$;
(ii) for any $\mathbf{s} \in \mathcal{N}_0$, $\Phi_0 = \Phi_1 + \Phi_2$; and
(iii) $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W\boldsymbol{\mu}$,

which proves (5.3) as required.

By Lemma 7.1, the condition (i) is equivalent to $(Y - \boldsymbol{\mu})'W(Y - \boldsymbol{\mu}) \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$. Thus (3.14)–(3.18) follow from Theorem 3.2.

For any symmetric matrix $\mathbf{s} \in \mathcal{N}_0$, we have

$$\Phi_1 = \operatorname{tr}\left(\boldsymbol{\lambda}_1[\mathbf{s}\Sigma\mathbf{s} + 2\mathbf{s}(\Sigma\mathbf{s})^2 + 2^2\mathbf{s}(\Sigma\mathbf{s})^3 + \cdots]\right), \tag{5.7}$$

$$\Phi_2 = \operatorname{tr}\left(\boldsymbol{\lambda}_2[\mathbf{s}\Sigma\mathbf{s} - 2\mathbf{s}(\Sigma\mathbf{s})^2 + 2^2\mathbf{s}(\Sigma\mathbf{s})^3 - \cdots]\right) \tag{5.8}$$

and

$$\Phi_0 = \operatorname{tr}\left(\Delta[(W \otimes \mathbf{s})\Upsilon + 2(W \otimes \mathbf{s})\Upsilon^2 + 2^2(W \otimes \mathbf{s})\Upsilon^3 + \cdots]\right), \tag{5.9}$$

where $\Upsilon = \Sigma_Y(W \otimes \mathbf{s})$. Putting (5.7)–(5.9) into the equation $\Phi_0 = \Phi_1 + \Phi_2$ and then comparing its both sides, with the arbitrariness of $\mathbf{s}$ close to $\mathbf{0}$, we obtain (5.1) and (5.2). Thus we have completed the proof of the desired result. □

The nonzero mean of Y results in conditions (5.1) and (5.2). In fact, we have obtained the following relation between $Y'WY$ and $(Y-\boldsymbol{\mu})'W(Y-\boldsymbol{\mu})$ in the proof of Theorem 5.1.

Corollary 5.2. *Let $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ with $\Sigma_Y \in \mathcal{N}_{np}$ and W be symmetric. Then $Y'WY \sim \mathcal{W}_p(m_1, \Sigma, \boldsymbol{\lambda}_1) - \mathcal{W}_p(m_2, \Sigma, \boldsymbol{\lambda}_2)$ with $m_1, m_2 \in \{0, 1, 2, \ldots\}$, $\Sigma \in \mathcal{N}_p$ and $\boldsymbol{\lambda}_1, \boldsymbol{\lambda}_2 \in \mathcal{M}_{p\times p}$ if and only if there exists some $\Sigma \in \mathcal{N}_p$ such that,*

(a) $(Y-\boldsymbol{\mu})'W(Y-\boldsymbol{\mu}) \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$; *and*
(b) *for any* $\mathbf{s}$ *in a neighborhood* $\mathcal{N}_0$ *of* $\mathbf{0} \in \mathcal{S}_p$ *and* $k = 1, 2, \ldots,$

$$\operatorname{tr}\left((\boldsymbol{\lambda}_1 + \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k-1}\right) = \operatorname{tr}\left(\Delta(W\otimes\mathbf{s})[\Sigma_Y(W\otimes\mathbf{s})]^{2k-1}\right),$$
$$\operatorname{tr}\left((\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2)\mathbf{s}(\Sigma\mathbf{s})^{2k}\right) = \operatorname{tr}\left(\Delta(W\otimes\mathbf{s})[\Sigma_Y(W\otimes\mathbf{s})]^{2k}\right)$$

with $\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 = \boldsymbol{\mu}'W\boldsymbol{\mu}$.

6. Conditions for the noncentral GL and independence of a family of matrix quadratic forms

In this section, we shall use the moment generating function $M(\mathbf{s})$ of $Y'WY$ to extend Theorem 4.2 to the nonzero mean case of Y.

Based on Theorem 5.1, we obtain the following extension of Cochran's theorem concerning the noncentral GL and independence of a family of matrix quadratic forms by putting Theorem 4.2, Theorem 5.1 and Lemma 2.4 together with an appropriate modification.

Theorem 6.1. *Let $Y \sim \mathcal{N}_{n\times p}(\boldsymbol{\mu}, \Sigma_Y)$ with $\Sigma_Y \in \mathcal{N}_{np}$ and W_i's be symmetric matrices of order n. Then $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma, \boldsymbol{\lambda}_{1i}) - \mathcal{W}_p(m_{2i}, \Sigma, \boldsymbol{\lambda}_{2i})$ with $m_{1i}, m_{2i} \in \{0, 1, 2, \ldots\}$, $\Sigma \in \mathcal{N}_p$ and $\boldsymbol{\lambda}_{1i}, \boldsymbol{\lambda}_{2i} \in \mathcal{M}_{p\times p}$ if and only if there exists some $\Sigma \in \mathcal{N}_p$ such that, in addition to conditions* (a)–(e) *of Theorem 4.2, the following statements* (f) *and* (g) *also hold.*

(f) *For any distinct* $i, j \in \{1, 2, \ldots, l\}$ *and* $\mathbf{t}, \tilde{\mathbf{t}} \in \mathcal{B}_p$,

$$\Sigma_Y(W_i\otimes\mathbf{t})\Sigma_Y(W_j\otimes\tilde{\mathbf{t}})\operatorname{vec}(\boldsymbol{\mu}) = \mathbf{0} \text{ and}$$
$$\operatorname{vec}(\boldsymbol{\mu})'(W_i\otimes\mathbf{t})\Sigma_Y(W_j\otimes\tilde{\mathbf{t}})\operatorname{vec}(\boldsymbol{\mu}) = 0; \text{ and}$$

(**g**) *for any* $\mathbf{s}$ *in a neighborhood* $\mathcal{N}_0$ *of* $\mathbf{0}$ *in* $\mathcal{S}_p$, $i = 1, 2, \ldots, l$ *and* $k = 1, 2, \ldots,$

$$\operatorname{tr}\left((\boldsymbol{\lambda}_{1i} + \boldsymbol{\lambda}_{2i})\mathbf{s}(\Sigma\mathbf{s})^{2k-1}\right) = \operatorname{tr}\left(\Delta(W_i\otimes\mathbf{s})[\Sigma_Y(W_i\otimes\mathbf{s})]^{2k-1}\right) \text{ and}$$
$$\operatorname{tr}\left((\boldsymbol{\lambda}_{1i} - \boldsymbol{\lambda}_{2i})\mathbf{s}(\Sigma\mathbf{s})^{2k}\right) = \operatorname{tr}\left(\Delta(W_i\otimes\mathbf{s})[\Sigma_Y(W_i\otimes\mathbf{s})]^{2k}\right)$$

with $\boldsymbol{\lambda}_{1i} - \boldsymbol{\lambda}_{2i} = \boldsymbol{\mu}'W_i\boldsymbol{\mu}$.

Finally, let us look at a special case $\Sigma_Y = A\otimes\Sigma$ of Theorem 6.1 investigated by Tan [21].

Corollary 6.2. *In Theorem 6.1, suppose that $\Sigma_Y = A\otimes\Sigma$ for some $A \in \mathcal{N}_n$ and $\Sigma \in \mathcal{N}_p$. Then $Y'W_1Y$, $Y'W_2Y, \ldots, Y'W_lY$ are independent and, for each i, $Y'W_iY \sim \mathcal{W}_p(m_{1i}, \Sigma, \boldsymbol{\lambda}_{1i}) - \mathcal{W}_p(m_{2i}, \Sigma, \boldsymbol{\lambda}_{2i})$ with $m_{1i}, m_{2i} \in \{0, 1, 2, \ldots\}$, $\Sigma \in \mathcal{N}_p$ and $\boldsymbol{\lambda}_{1i}, \boldsymbol{\lambda}_{2i} \in \mathcal{M}_{p\times p}$ if and only if for any distinct $i, j \in \{1, 2, \ldots, l\}$,*

(1) $AW_iAW_iAW_iA = AW_iA \neq \mathbf{0}$;
(2) $\operatorname{tr}(AW_i)^2 + \operatorname{tr}(AW_i) = 2m_{1i}$, $\operatorname{tr}(AW_i)^2 - \operatorname{tr}(AW_i) = 2m_{2i}$;
(3) $\boldsymbol{\lambda}_{1i} + \boldsymbol{\lambda}_{2i} = \boldsymbol{\mu}'W_iAW_i\boldsymbol{\mu} = \boldsymbol{\mu}'W_iAW_iAW_iAW_i\boldsymbol{\mu}$, $\boldsymbol{\lambda}_{1i} - \boldsymbol{\lambda}_{2i} = \boldsymbol{\mu}'W_i\boldsymbol{\mu} = \boldsymbol{\mu}'W_iAW_iAW_i\boldsymbol{\mu}$;
(4) $AW_iAW_jA = \mathbf{0}$;
(5) $AW_iAW_j\boldsymbol{\mu} = \mathbf{0}$; *and*
(6) $\boldsymbol{\mu}'W_iAW_j\boldsymbol{\mu} = \mathbf{0}$.

7. Concluding remarks

In this article, we obtain a set of the sufficient and necessary conditions under which a matrix quadratic form $Y'WY$ with a symmetric W and a general Σ_Y is distributed as a difference of two independent (noncentral) Wishart random matrices, namely, having the (noncentral) GL. Based on the results, we then establish some general extensions of Cochran's theorem concerning the (noncentral) GL and independence of a set of matrix quadratic forms.

It should be noted that it is challenging research for us to obtain a set of sufficient and necessary conditions under which a matrix quadratic form $Y'WY$ with symmetric matrix W and a general covariance Σ_Y of Y is distributed as a linear combination of independent Wishart random matrices.

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Appendix

Lemma 7.1. *Let $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ and $\Sigma \in \mathcal{N}_p$. Then the following statements are equivalent.*

(a) $Y'WY \sim \mathcal{W}_p(m_1, \Sigma) - \mathcal{W}_p(m_2, \Sigma)$;
(b) *for any* $\mathbf{s} \in \mathcal{S}_p$,

$$\left|I_{np} - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}\right| = \left|I_p - 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}\right|^{m_1}\left|I_p + 2\Sigma^{1/2}\mathbf{s}\Sigma^{1/2}\right|^{m_2};$$

(c) $\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}$ *and* $\mathrm{diag}[I_{m_1} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, -I_{m_2} \otimes \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}, \mathbf{0}]$ *have the same characteristic polynomial for all* $\mathbf{s} \in \mathcal{S}_p$; *and*
(d) *for any positive integer k and any* $\mathbf{s} \in \mathcal{S}_p$,

$$\mathrm{tr}(\Sigma_Y(W \otimes \mathbf{s}))^k = \left(m_1 + (-1)^k m_2\right)\mathrm{tr}(\Sigma\mathbf{s})^k.$$

Proof. Let $\Sigma_* = \Sigma^{1/2}\mathbf{s}\Sigma^{1/2}$. From Lemma 2.3 and (2.2) with $\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_2 = 0$, (a) is equivalent to (a′) for any $\mathbf{s} \in \mathcal{S}_p$ such that $Sr(\Sigma_*) < 1/2$ and $Sr(\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}) < 1/2$,

$$\left|I_{np} - 2\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}\right|^{-1/2} = |I_p - 2\Sigma_*|^{-m_1/2}|I_p + 2\Sigma_*|^{-m_2/2}.$$

It's obvious that (a′) follows from (b). And (b) is obtained from (a′) by analytic continuation. Thus (a) and (b) are equivalent.

Replace $\mathbf{s}$ with $\mathbf{s}/2\lambda$ (nonzero $\lambda \in \Re$) in (b) and multiplying both sides of (b) by λ^{np}. Then (b) amounts to

$$\left|\lambda I_{np} - \Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}\right| = |\lambda I_p - \Sigma_*|^{m_1}|\lambda I_p + \Sigma_*|^{m_2}|\lambda I_p - \mathbf{0}|^{(n-m_1-m_2)}.$$

So (c) is equivalent to (b). (c) amounts to that matrix $\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}$ and diagonal matrix $\mathrm{diag}[I_{m_1} \otimes \Sigma_*, -I_{m_2} \otimes \Sigma_*, \mathbf{0}]$ in $\mathcal{S}_{np}$ have the same spectrum $\{\lambda_j\}_{j=1}^{np}$. Equivalently, for any positive integer k and any $\mathbf{s} \in \mathcal{S}_p$, we have

$$\mathrm{tr}\left(\Sigma_Y^{1/2}(W \otimes \mathbf{s})\Sigma_Y^{1/2}\right)^k = \mathrm{tr}\left(\mathrm{diag}[I_{m_1} \otimes \Sigma_*, -I_{m_2} \otimes \Sigma_*, \mathbf{0}]\right)^k,$$

which proves the equivalence between (c) and (d) via appropriate Kronecker operations. So the proof is complete. □

Lemma 7.2. *Assume that C-conditions holds. Then there exists an orthogonal matrix H, not depending on i, such that*

$$B_{ii} = H(\mathbf{e}_{ii} \otimes A_{ii})H', \tag{A.1}$$

where $A_{ii} = \mathrm{diag}[U_{ii}, V_{ii}, \mathbf{0}] \in \mathcal{M}_{n\times n}$ *and* $U_{ii} = I_{m_1}, V_{ii} = -I_{m_2}$ *and*

$$B_{ij} = H(\mathbf{e}_{ij} \otimes A_{ij} + \mathbf{e}_{ji} \otimes A_{ji})H'/2, \tag{A.2}$$

where $A_{ij} = diag[U_{ij}, V_{ij}, \mathbf{0}] \in \mathcal{M}_{n\times n}$, $U_{ij} \in \mathcal{M}_{m_1\times m_1}, V_{ij} \in \mathcal{M}_{m_2\times m_2}$ *and* $A'_{ij} = A_{ji}, U_{ij}U'_{ij} = I_{m_1}$, $V_{ij}V'_{ij} = I_{m_2}$.

Proof. It follows from condition (C2) that

$$B_{ii}^{2k+1} = B_{ii},\ \mathrm{tr}\left(B_{ii}^{2k+1}\right) = m_1 - m_2,\quad \mathrm{tr}\left(B_{ii}^{2k}\right) = m_1 + m_2,\ k = 1, 2, \ldots \tag{A.3}$$

By (A.3) and (C3) we may choose an orthogonal matrix H, not depending on i, such that (A.1) holds. Thus using (C3), (C4) and (A.3) it is easily shown that for $i \neq j$, $||B_{ij} - 4B_{ij}^3||^2 = 0$ and so

$$B_{ij} = 4B_{ij}^3, \quad i \neq j. \tag{A.4}$$

Combining (C4) with (A.4) we obtain, for $i \neq j$, $(B_{ii}^2 + B_{jj}^2)B_{ij} = 4B_{ij}^2 B_{ij} = B_{ij}$. The symmetry of B_{ij} then yields

$$B_{ij} = (B_{ii}^2 + B_{jj}^2)B_{ij}(B_{ii}^2 + B_{jj}^2), \quad i \neq j. \tag{A.5}$$

For $i \neq j$, using $\mathbf{e}_{ii} \otimes A_{ii}$, $\mathbf{e}_{jj} \otimes A_{jj}$ and $H'B_{ij}H$ to replace B_{ii}, B_{jj} and B_{ij}, we get (A.2) from (A.5), which completes the proof. □

The following lemma is due to Masaro and Wong [11] and its proof is also found in Hu [6], Appendix.

Lemma 7.3. *Let* $Y \sim \mathcal{N}_{n\times p}(\mathbf{0}, \Sigma_Y)$ *with* $\Sigma_Y \in \mathcal{N}_{np}$ *and W be a symmetric matrix of order n. Then* $Y'WY \sim \mathcal{W}_p(m_1, \Lambda) - \mathcal{W}_p(m_2, \Lambda)$ *with* $m_1, m_2 \in \{0, 1, \ldots\}$ *and* $\Lambda = diag[\sigma_1, \ldots, \sigma_r, \mathbf{0}] \in \mathcal{N}_p$ *if and only if there exist positive real numbers* $\sigma_1, \sigma_2, \ldots, \sigma_r (r \leqslant p)$ *such that C-conditions hold.*

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