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Existence of approximate Hermitian–Einstein structures on semistable principal bundles

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Abstract

Let E_G be a principal G-bundle over a compact connected Kähler manifold, where G is a connected reductive linear algebraic group defined over $\mathbb{C}$. We show that E_G is semistable if and only if it admits approximate Hermitian–Einstein structures.

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1. Introduction

A holomorphic vector bundle E over a compact connected Kähler manifold (X, ω) is said to admit approximate Hermitian–Einstein metrics if for every $\varepsilon > 0$, there is a Hermitian metric h on E such that

$$\sup_X \left|\sqrt{-1}\Lambda_\omega F(h) - \lambda \cdot \mathrm{Id}_E\right|_h < \varepsilon.$$

In [4, Theorem 2] it was shown that a holomorphic vector bundle E over a compact Kähler manifold (X, ω) is semistable if and only if it admits approximate Hermitian–Einstein metrics. This generalizes a result of Kobayashi [5, p. 234, Theorem 10.13] for complex projective manifolds.

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It is an analogue, for semistable bundles, of the classical Hitchin–Kobayashi correspondence, which is given by the famous theorem of Donaldson, Uhlenbeck and Yau. This theorem relates polystable bundles to (exact) solutions of the Hermitian–Einstein equation, and was first proven for curves by Narasimhan and Seshadri [6], then for algebraic surfaces by Donaldson [3], and finally in arbitrary dimension by Uhlenbeck and Yau [7].

Our aim here is to generalize the above result of [4] to principal G-bundles over X, where G is a connected reductive linear algebraic group defined over $\mathbb{C}$. Fix a maximal compact subgroup $K \subset G$. A holomorphic principal G-bundle E_G over X is said to admit approximate Hermitian–Einstein structures if for every $\varepsilon > 0$, there exists a C^∞ reduction of structure group $E_K \subset E_G$ to K and an element λ in the center of $\mathrm{Lie}(G)$ such that

$$\sup_X \bigl|\Lambda_\omega F\bigl(\nabla^{E_K}\bigr) - \lambda\bigr|_{h_{E_K}} < \varepsilon,$$

where $F(\nabla^{E_K})$ is the curvature form of the Chern connection of E_K, and h_{E_K} is the Hermitian metric on $\mathrm{ad}(E_G)$ induced by E_K (this Hermitian metric h_{E_K} is described in (2.4)).

We prove the following:

Theorem 1. *A holomorphic principal G-bundle E_G over X is semistable if and only if it admits approximate Hermitian–Einstein structures.*

2. Preliminaries

Let (X, ω) be a compact connected Kähler manifold of complex dimension n, and let E be a holomorphic vector bundle over X.

Recall that the *degree* of a torsion-free coherent analytic sheaf $\mathcal{F}$ on X is defined to be

$$\deg(\mathcal{F}) := \int_X c_1(\mathcal{F}) \wedge \omega^{n-1};$$

if $\mathrm{rank}(\mathcal{F}) > 0$, the *slope* of $\mathcal{F}$ is

$$\mu(\mathcal{F}) := \frac{\deg(\mathcal{F})}{\mathrm{rank}(\mathcal{F})}.$$

Definition 2. A holomorphic vector bundle E is called *semistable* if $\mu(\mathcal{F}) \leqslant \mu(E)$ for every nonzero coherent analytic subsheaf $\mathcal{F}$ of E.

Definition 3. A holomorphic vector bundle E is said to admit *approximate Hermitian–Einstein metrics* if for every $\varepsilon > 0$, there exists a Hermitian metric h on E such that

$$\sup_X \bigl|\sqrt{-1}\Lambda_\omega F(h) - \lambda \cdot \mathrm{Id}_E\bigr|_h < \varepsilon.$$

Here Λ_ω is the adjoint of the wedge product with ω, $F(h)$ is the curvature form of the Chern connection for h, and λ is given by

$$\lambda = \frac{2\pi \cdot \mu(E)}{(n-1)! \cdot \mathrm{vol}(X)},$$

where $\mathrm{vol}(X)$ denotes the volume of X with respect to the Kähler form ω.

In [4], the following was proved:

Theorem 4. *(See [4, Theorem 2].) A holomorphic vector bundle E over X is semistable if and only if it admits approximate Hermitian–Einstein metrics.*

Now let G be a connected reductive linear algebraic group defined over $\mathbb{C}$, and let E_G be a holomorphic principal G-bundle over X.

Definition 5. E_G is called *semistable* if for every triple (P, U, σ), where

- $P \subset G$ is a maximal proper parabolic subgroup,
- $U \subset X$ is a dense open subset such that the complement $X \setminus U$ is a complex analytic subset of X of codimension at least 2, and
- $\sigma : U \to E_G/P$ is a holomorphic reduction of structure group, over U, of E_G to the subgroup P, satisfying the condition that the pullback $\sigma^* T_{\mathrm{rel}}$, which is a holomorphic vector bundle over U, extends to X as a coherent analytic sheaf (here T_{rel} is the relative tangent bundle over E_G/P for the natural projection $E_G/P \to X$),

the inequality

$$\deg\bigl(\sigma^* T_{\mathrm{rel}}\bigr) \geqslant 0$$

holds. The degree of $\sigma^* T_{\mathrm{rel}}$ is

$$\deg\bigl(\sigma^* T_{\mathrm{rel}}\bigr) := \int_X c_1\bigl(\iota_* \sigma^* T_{\mathrm{rel}}\bigr) \wedge \omega^{n-1},$$

where $\iota : U \to X$ is the inclusion map.

We will now define approximate Hermitian–Einstein structures on E_G. Let

$$0 \longrightarrow \mathrm{ad}(E_G) \longrightarrow \mathrm{At}(E_G) \xrightarrow{q} TX \longrightarrow 0 \tag{2.1}$$

be the Atiyah exact sequence for E_G (see [2] for the construction of (2.1)). Recall that a *complex connection* on E_G is a C^∞ splitting $D : TX \to \mathrm{At}(E_G)$ of this exact sequence, meaning $q \circ D = \mathrm{Id}_{TX}$. Note that (2.1) is a short exact sequence of sheaves of Lie algebras. The *curvature form* of a connection D,

$$F(D) \in H^0\bigl(X, \Lambda^{1,1} T^* X \otimes \mathrm{ad}(E_G)\bigr),$$

measures the obstruction of the homomorphism D to be Lie algebra structure preserving; see [2] for the details.

Fix a maximal compact subgroup

$$K \subset G.$$

A *Hermitian structure* on E_G is a smooth reduction of structure group E_K of E_G to K. Given a Hermitian structure E_K on E_G, there is a unique complex connection on E_G which is induced by a connection on E_K. This connection on E_G is called the *Chern connection* of the Hermitian structure E_K, and it will be denoted by ∇^{E_K}. The connection on E_K inducing the Chern connection on E_G will also be called the Chern connection. Let

$$F(\nabla^{E_K}) \in H^0(X, \Lambda^{1,1}T^*X \otimes \mathrm{ad}(E_G)) \tag{2.2}$$

be the curvature of ∇^{E_K}. Note that $F(\nabla^{E_K})$ lies in the image of $\Lambda^2(T^{\mathbb{R}}X)^* \otimes \mathrm{ad}(E_K)$, where $T^{\mathbb{R}}X$ is the real tangent bundle.

Let $\mathfrak{g}$ be the Lie algebra of G. Consider the adjoint representation

$$\rho : G \longrightarrow \mathrm{GL}(\mathfrak{g}). \tag{2.3}$$

Fix a maximal compact subgroup $\widetilde{K} \subset \mathrm{GL}(\mathfrak{g})$ containing $\rho(K)$. Let

$$E_{\mathrm{GL}(\mathfrak{g})} = E \times^G \mathrm{GL}(\mathfrak{g}) \longrightarrow X$$

be the principal $\mathrm{GL}(\mathfrak{g})$-bundle obtained by extending the structure group of E_G to $\mathrm{GL}(\mathfrak{g})$ using the homomorphism ρ in (2.3). We note that the vector bundle associated to the principal $\mathrm{GL}(\mathfrak{g})$-bundle $E_{\mathrm{GL}(\mathfrak{g})}$ for the standard action of $\mathrm{GL}(\mathfrak{g})$ on $\mathfrak{g}$ is identified with the adjoint bundle $\mathrm{ad}(E_G)$.

Given a Hermitian structure $E_K \subset E_G$, we obtain a reduction of structure group

$$E_K(\widetilde{K}) = E_K \times^K \widetilde{K} \subset E_{\mathrm{GL}(\mathfrak{g})} \tag{2.4}$$

of $E_{\mathrm{GL}(\mathfrak{g})}$ to $\widetilde{K}$. This reduction corresponds to a Hermitian metric on the adjoint vector bundle $\mathrm{ad}(E_G)$.

Let $\mathfrak{z}$ be the center of the Lie algebra $\mathfrak{g}$. Since the adjoint action of G on $\mathfrak{z}$ is trivial, an element $\lambda \in \mathfrak{z}$ defines a smooth section of $\mathrm{ad}(E_G)$, which will also be denoted by λ.

Definition 6. A holomorphic principal G-bundle E_G over X is said to admit *approximate Hermitian–Einstein structures* if for every $\varepsilon > 0$, there exists a Hermitian structure $E_K \subset E_G$ and an element $\lambda \in \mathfrak{z}$, such that

$$\sup_X \left|\Lambda_\omega F(\nabla^{E_K}) - \lambda\right|_{h_{E_K}} < \varepsilon,$$

where $F(\nabla^{E_K})$ is the curvature form of the Chern connection of E_K (see (2.2)), and h_{E_K} is the Hermitian metric on $\mathrm{ad}(E_G)$ induced by E_K (see (2.4)).

3. Proof of Theorem 1

We will first show that it is enough to prove the theorem under the assumption that G is semisimple.

Let $Z_0(G)$ be the connected component of the center of G which contains the identity element. The normal subgroup $[G,G] \subset G$ is semisimple because G is reductive. We have a natural surjective homomorphism

$$G \longrightarrow (G/Z_0(G)) \times (G/[G,G])$$

whose kernel is a finite group contained in the center of G. In particular, the induced homomorphism of Lie algebras is an isomorphism.

Let $\rho : A \to B$ be a homomorphism of Lie groups such that the induced homomorphism of Lie algebras

$$d\rho : \mathrm{Lie}(A) \longrightarrow \mathrm{Lie}(B)$$

is an isomorphism, and kernel(ρ) is contained in the center of A. Let E_A be a principal A-bundle, and let $E_B := E_A \times^{\rho} B$ be the principal B-bundle obtained by extending the structure group of E_A to B using ρ. The isomorphism of Lie algebras $d\rho$ produces an isomorphism

$$\widetilde{\rho} : \mathrm{ad}(E_A) \longrightarrow \mathrm{ad}(E_B)$$

between the adjoint bundles. There is a natural bijective correspondence between the connections on E_A and the connections on E_B. To construct this bijection, first note that ρ induces a map

$$\widehat{\rho} : E_A \longrightarrow E_B$$

that sends $z \in E_A$ to the element of E_B given by (z, e) (recall that E_B is a quotient of $E_A \times B$). This map $\widehat{\rho}$ intertwines the actions of A, with A acting on E_B through ρ. Since kernel(ρ) is a finite group contained in the center of A, any A-invariant vector field on $E_A|_U$, where U is some open subset of the base manifold, produces a B-invariant vector field on $E_B|_U$. This way we get an isomorphism of $\mathrm{At}(E_A)$ with $\mathrm{At}(E_B)$. This identification of $\mathrm{At}(E_A)$ with $\mathrm{At}(E_B)$ produces a bijection between the connections on E_A and the connections on E_B. The curvature of a connection on E_B is given by the curvature of the corresponding connection on E_A using the isomorphism $\widetilde{\rho}$.

Therefore, to prove the theorem, it suffices to prove it for $G/Z_0(G)$ and $G/[G,G]$ separately. Since $G/[G,G]$ is a product of copies of $\mathbb{C}^*$, in this case the theorem follows immediately from Theorem 4. Since $G/Z_0(G)$ is semisimple, it is enough to prove the theorem under the assumption that G is semisimple.

Henceforth, we will assume that G is semisimple. This implies that the center $\mathfrak{z}$ of its Lie algebra $\mathfrak{g}$ is trivial, and thus the constant λ in Definition 6 is zero. The Killing form on $\mathfrak{g}$, being G-invariant, produces a holomorphic bilinear form on the fibers of $\mathrm{ad}(E_G)$. Since the Killing form is nondegenerate (as G is semisimple), this bilinear form on the fibers of $\mathrm{ad}(E_G)$ is nondegenerate. Hence we get a trivialization

$$\det\bigl(\mathrm{ad}(E_G)\bigr) := \bigwedge^{\mathrm{top}} \mathrm{ad}(E_G) \xrightarrow{\sim} \mathcal{O}_X. \tag{3.1}$$

Therefore, $\deg(\mathrm{ad}(E_G)) = \deg(\det(\mathrm{ad}(E_G))) = 0$, or equivalently

$$\mu\bigl(\mathrm{ad}(E_G)\bigr) = 0. \tag{3.2}$$

First assume that the principal bundle E_G admits approximate Hermitian–Einstein structures. Given $\varepsilon > 0$, we thus obtain a Hermitian structure $E_K \subset E_G$ satisfying the condition that

$$\sup_X \bigl|\Lambda_\omega F\bigl(\nabla^{E_K}\bigr)\bigr|_{h_{E_K}} < \varepsilon. \tag{3.3}$$

The Hermitian structure E_K on E_G induces a Hermitian metric h_{E_K} on $\mathrm{ad}(E_G)$ (see (2.4)). The Chern connection on $\mathrm{ad}(E_G)$ for h_{E_K} coincides with the connection ∇^{ad} on $\mathrm{ad}(E_G)$ induced by ∇^{E_K}. The curvature forms of ∇^{E_K} and ∇^{ad} are related by

$$F\bigl(\nabla^{\mathrm{ad}}\bigr) = \mathrm{ad}\bigl(F\bigl(\nabla^{E_K}\bigr)\bigr), \tag{3.4}$$

where

$$\mathrm{ad} : \mathrm{ad}(E_G) \longrightarrow \mathrm{End}\bigl(\mathrm{ad}(E_G)\bigr) = \mathrm{ad}(E_G) \otimes \mathrm{ad}(E_G)^* \tag{3.5}$$

is the homomorphism of vector bundles induced by the homomorphism of Lie algebras $\mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}^*$ given by the adjoint action of G on $\mathfrak{g}$.

Since the Hermitian metric h_{E_K} on $\mathrm{ad}(E_G)$ is induced by a Hermitian structure on E_G, there is a real number $c_0 > 0$ such that $\frac{1}{c_0} \cdot \mathrm{ad}$ is an isometry, where ad is the homomorphism in (3.5). Now from (3.4) it follows that

$$\begin{aligned}\big|\sqrt{-1}\Lambda_\omega F\big(\nabla^{\mathrm{ad}}\big)\big|^2_{h_{E_K}} &= \mathrm{tr}\big(\Lambda_\omega F\big(\nabla^{\mathrm{ad}}\big) \circ \big(\Lambda_\omega F\big(\nabla^{\mathrm{ad}}\big)\big)^*\big) \\ &= \mathrm{tr}\big(\mathrm{ad}\big(\Lambda_\omega F\big(\nabla^{E_K}\big)\big) \circ \big(\mathrm{ad}\big(\Lambda_\omega F\big(\nabla^{E_K}\big)\big)\big)^*\big) \\ &= c_0^2 \cdot h_{E_K}\big(\Lambda_\omega F\big(\nabla^{E_K}\big), \Lambda_\omega F\big(\nabla^{E_K}\big)\big) \\ &= c_0^2 \cdot \big|\Lambda_\omega F\big(\nabla^{E_K}\big)\big|^2_{h_{E_K}},\end{aligned}$$

where "$*$" denotes the adjoint with respect to h_{E_K}. From this and (3.3) we conclude that

$$\sup_X \big|\sqrt{-1}\Lambda_\omega F\big(\nabla^{\mathrm{ad}}\big)\big|_{h_{E_K}} < c_0 \cdot \varepsilon.$$

Therefore, $\mathrm{ad}(E_G)$ admits approximate Hermitian–Einstein metrics, and is semistable by Theorem 4. A holomorphic principal G-bundle F_G over X is semistable if and only if its adjoint vector bundle $\mathrm{ad}(F_G)$ is semistable [1, Proposition 2.10]. Therefore, a principal G-bundle admitting approximate Hermitian–Einstein structures is semistable.

For the converse direction, assume that E_G is semistable. As we have stated, this is equivalent to the vector bundle $\mathrm{ad}(E_G)$ being semistable. Let $\mathcal{H}(\mathrm{ad}(E_G))$ be the space of all C^∞ Hermitian metrics h on $\mathrm{ad}(E_G)$ satisfying the following condition: the isomorphism in (3.1) takes the Hermitian metric on $\det(\mathrm{ad}(E_G))$ induced by h to the constant Hermitian metric on $\mathcal{O}_X$ given by the absolute value. For any initial $h \in \mathcal{H}(\mathrm{ad}(E_G))$, we can evolve the metric by the following parabolic equation, which we call the Donaldson heat flow

$$h^{-1}\partial_t h = -\big(\sqrt{-1}\Lambda_\omega F(h) - \lambda \cdot \mathrm{Id}_{\mathrm{ad}(E_G)}\big).$$

Here $F(h)$ is the curvature of the Chern connection on $\mathrm{ad}(E_G)$ for h, and

$$\lambda = \frac{2\pi \cdot \mu(\mathrm{ad}(E_G))}{(n-1)! \cdot \mathrm{vol}(X)}.$$

Since $\mu(\mathrm{ad}(E_G)) = 0$ (see (3.2)), for us the Donaldson heat flow is given by the simpler expression

$$h^{-1}\partial_t h = -\sqrt{-1}\Lambda_\omega F(h). \tag{3.6}$$

As shown in (2.4), a Hermitian structure on E_G produces a Hermitian metric on $\mathrm{ad}(E_G)$. Such a Hermitian metric on $\mathrm{ad}(E_G)$ always lies in $\mathcal{H}(\mathrm{ad}(E_G))$. Let

$$\mathcal{H}(E_G) \subset \mathcal{H}\big(\mathrm{ad}(E_G)\big)$$

be the subspace corresponding to the Hermitian structures on E_G.

Lemma 7. *The Donaldson heat flow on $\mathcal{H}(\mathrm{ad}(E_G))$ preserves $\mathcal{H}(E_G)$.*

Proof. Let $E_K \subset E_G$ be a C^∞ reduction of structure group to K. The element of $\mathcal{H}(\mathrm{ad}(E_G))$ given by the Hermitian structure E_K will be denoted by h. The Chern connection $\nabla(h)$ on $\mathrm{ad}(E_G)$ for h is given by the Chern connection ∇^{E_K} on E_K. In particular, the curvature of $\nabla(h)$ coincides with that of ∇^{E_K}. Therefore, the curvature $F(h)$ in (3.6), which is a priori a real two-form with values in $\mathrm{End}(\mathrm{ad}(E_G))$, is actually a real two-form with values in $\mathrm{ad}(E_K)$ (the adjoint bundle of E_K). Consequently, $\Lambda_\omega F(h)$ is a C^∞ section of $\mathrm{ad}(E_K)$ (the operator Λ_ω takes real two forms to real valued functions). This implies that the Donaldson heat flow on $\mathcal{H}(\mathrm{ad}(E_G))$ preserves $\mathcal{H}(E_G)$. □

Fix a Hermitian metric $h_0 \in \mathcal{H}(\mathrm{ad}(E_G))$, and consider the Donaldson heat flow with initial metric h_0. The space $\mathcal{H}(\mathrm{ad}(E_G))$ was defined so that h_0 satisfies the normalization

$$c_1\big(\mathrm{ad}(E_G), h_0\big) = \frac{\sqrt{-1}}{2\pi}\,\mathrm{tr}\big(F(h_0)\big) = 0.$$

This guarantees that

$$\det\big(h_0^{-1}h\big) = 1$$

along the flow. From this and the semistability of $\mathrm{ad}(E_G)$, it follows that approximate Hermitian–Einstein metrics on $\mathrm{ad}(E_G)$ are realized along the flow for sufficiently large time (see the proof of Theorem 4 in [4] for details). Consequently, taking h_0 to be an element of $\mathcal{H}(E_G)$, from Lemma 7 we conclude that E_G admits approximate Hermitian–Einstein structures. This completes the proof of Theorem 1.

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