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Boundary version of a twin region convergence theorem for continued fractions

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Abstract

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The starting point of the present paper is a result by Thron (1959) on twin convergence regions, dealing with continued fractions $\mathrm{K}(c_n^2/1)$, where $|c_{2m-1}| \leqslant \rho < 1$ and $|c_{2m} \pm \mathrm{i}| \geqslant \rho$. In the present paper these regions are replaced by their boundaries, and the sets of continued fraction values are determined.

Keywords: Continued fractions, convergence.

For continued fractions

$$\overset{\infty}{\underset{n=1}{\mathrm{K}}}\left(\frac{a_n}{1}\right)=\frac{a_1|}{|1}+\frac{a_2|}{|1}+\cdots+\frac{a_n|}{|1}+\cdots, \tag{1}$$

Thron [3] proved the following theorem.

Theorem 1. *If, in the continued fraction* (1), *the elements are subject to the conditions*

$$a_{2n-1}=c_{2n-1}^2, \qquad a_{2n}=c_{2n}^2, \tag{2}$$

where

$$|c_{2n-1}| \leqslant \rho, \quad |c_{2n} \pm \mathrm{i}| \geqslant \rho, \qquad \rho<1, \tag{3}$$

then the continued fraction (1) *converges, and its value is in the disk* $|w| \leqslant \rho$. (See also [2, Theorem 4.46].)

In the present paper we shall study what happens to the set of values when the conditions (3) are replaced by

$$|c_{2n-1}| = \rho, \quad |c_{2n} + \mathrm{i}| = \rho, \qquad \rho<1. \tag{4}$$

The same type of problem, but for *simple* regions, is discussed in [4,5].

In the first lemma we shall study approximants

$$\mathop{\mathrm{K}}_{n=1}^{1+q}\left(\frac{c_n^2}{1}\right) \quad \text{and} \quad \mathop{\mathrm{K}}_{n=2}^{2+q}\left(\frac{c_n^2}{1}\right) \tag{5}$$

of the continued fractions

$$\frac{c_1^2|}{|1} + \frac{c_2^2|}{|1} + \cdots + \frac{c_n^2|}{|1} + \cdots \quad \text{and} \quad \frac{c_2^2|}{|1} + \frac{c_3^2|}{|1} + \cdots + \frac{c_n^2|}{|1} + \cdots,$$

respectively. Observe that for any m the sets of possible values of

$$\mathop{\mathrm{K}}_{n=2m-1}^{2m-1+q}\left(\frac{c_n^2}{1}\right) \quad \text{and} \quad \mathop{\mathrm{K}}_{n=2m}^{2m+q}\left(\frac{c_n^2}{1}\right)$$

coincide with the ones of the approximants (5). The lemma is partly overlapped by results coming out of Theorem 1, but is included here in the form given below, partly for completeness, partly because it represents a convenient first step in establishing the main result.

Lemma 2. *Let ρ be a positive number < 1, and $\mathrm{K}(c_n^2/1)$ a continued fraction where*

$$|c_{2m-1}| = \rho \quad \textit{and} \quad |c_{2m} + \mathrm{i}| = \rho, \tag{6}$$

for all $m \geqslant 1$. Then

$$\left|\mathop{\mathrm{K}}_{n=1}^{q+1}\left(\frac{c_n^2}{1}\right)\right| \leqslant \rho, \quad \textit{for all } q \geqslant 0, \tag{7}$$

and

$$\left|\mathop{\mathrm{K}}_{n=2}^{2+q}\left(\frac{c_n^2}{1}\right) + 1\right| \geqslant \rho, \quad \textit{for all } q \geqslant 0. \tag{8}$$

Proof. Since $\rho^2 < \rho$, the inequality (7) obviously holds for $q = 0$. For $q = 0$ the left-hand side of (8) is

$$|(-\mathrm{i} + \rho\, \mathrm{e}^{\mathrm{i}\alpha})^2 + 1| = |-2\rho\mathrm{i}\, \mathrm{e}^{\mathrm{i}\alpha} + \rho^2\, \mathrm{e}^{2\mathrm{i}\alpha}| = |2 + \rho\mathrm{i}\, \mathrm{e}^{\mathrm{i}\alpha}|\, \rho > \rho.$$

Thus (8) is established for $q = 0$.

Next, let $q \geqslant 0$ be such that (8) holds. We map the region $|w + 1| \geqslant \rho$ by

$$\omega = \frac{c_1^2}{1 + w} = \frac{\rho^2\, \mathrm{e}^{2\mathrm{i}\alpha}}{1 + w}, \tag{9}$$

and take the union over all $\alpha \in [0, 2\pi)$, from which follows $|\omega| \leqslant \rho$. This shows, that if $q \geqslant 0$ is such that (8) holds, then (7) holds for $q + 1$.

Assume now that $q \geqslant 0$ is such that (7) holds. We map the region $|w| \leqslant \rho$ by

$$\omega = \frac{c_2^2}{1 + w} = \frac{(-\mathrm{i} + \rho\, \mathrm{e}^{\mathrm{i}\alpha})^2}{1 + w}, \tag{10}$$

and take the union over all $\alpha \in [0, 2\pi)$. First observe that the mapping

$$w \to \frac{1}{1 + w}$$

maps the disk $|w| \leqslant \rho$ onto the disk centered at the point

$$w = \frac{1}{1-\rho^2}$$

and with radius

$$\frac{\rho}{1-\rho^2}.$$

An arbitary c_2^2 can be written as

$$(-\mathrm{i} + \rho\, \mathrm{e}^{\mathrm{i}\alpha})^2 = -(1 + \rho\, \mathrm{e}^{\mathrm{i}\beta})^2,$$

where we have replaced $\alpha + \frac{1}{2}\pi$ by β. For a fixed β-value the image of the disk $|w| \leqslant \rho$ by (10) is the disk

$$\left| \omega + \frac{(1 + \rho\, \mathrm{e}^{\mathrm{i}\beta})^2}{1-\rho^2} \right| \leqslant \frac{\rho}{1-\rho^2} |1 + \rho\, \mathrm{e}^{\mathrm{i}\beta}|^2. \tag{11}$$

We need to find the union of these disks for $\beta \in [0, 2\pi)$ and relate it to the point $\omega = -1$. The distance from $\omega = -1$ to the center of (11) is

$$\left| -1 + \frac{(1 + \rho\, \mathrm{e}^{\mathrm{i}\beta})^2}{1-\rho^2} \right| = \frac{\rho}{1-\rho^2} |\rho + 2\, \mathrm{e}^{\mathrm{i}\beta} + \rho\, \mathrm{e}^{2\mathrm{i}\beta}| = \frac{2\rho}{1-\rho^2}(1 + \rho \cos\beta). \tag{12}$$

On the other hand, the radius of (11) is

$$\frac{\rho}{1-\rho^2} |1 + \rho\, \mathrm{e}^{\mathrm{i}\beta}|^2 = \frac{\rho}{1-\rho^2}(1 + 2\rho\cos\beta + \rho^2). \tag{13}$$

Hence the distance from the point $\omega = -1$ to the closest point on the disk (11) is

$$\frac{2\rho}{1-\rho^2}(1 + \rho\cos\beta) - \frac{\rho}{1-\rho^2}(1 + 2\rho\cos\beta + \rho^2) = \rho. \tag{14}$$

Hence, if $q \geqslant 0$ is such that (7) holds, then (8) holds for $q+1$, and Lemma 2 is thus proved by induction. □

It follows immediately, that the value of

$$\mathop{\mathrm{K}}_{n=1}^{\infty} \frac{c_n^2}{1}$$

must lie in $|\omega| \leqslant \rho$, and that the value of

$$\mathop{\mathrm{K}}_{n=2}^{\infty} \frac{c_n^2}{1}$$

must lie in $|\omega + 1| \geqslant \rho$. The first of these statements is contained in Theorem 1. We can squeeze more information out of the proof of Lemma 2.

Lemma 3. *Under the conditions of Lemma 2 we have*

$$\left| \operatorname*{K}_{n=1}^{\infty} \frac{c_n^2}{1} \right| \geqslant \rho \frac{1-\rho}{3+\rho} \tag{15}$$

and

$$\operatorname*{K}_{n=2}^{\infty} \frac{c_n^2}{1} \in A, \tag{16}$$

where A is the union of the disks (11) for $0 \leqslant \beta \leqslant 2\pi$. The circle

$$|\omega + 1| = \rho \tag{16a}$$

is part of the boundary of A. Another part is part of the curve

$$\begin{aligned} x &= \operatorname{Re} \omega = \frac{-1}{1-\rho^2}(1-\rho \cos \gamma)(4\rho^2 \cos^2\gamma - 2\rho \cos \gamma + 1 - 3\rho^2), \\ y &= \operatorname{Im} \omega = \frac{\rho \sin \gamma}{1-\rho^2}(4\rho^2 \cos^2\gamma - 6\rho \cos \gamma + 3 - \rho^2), \quad 0 \leqslant \gamma < 2\pi. \end{aligned} \tag{16b}$$

Proof. The largest distance from $\omega = -1$ to a point on the disk (11) is (from (12) and (13))

$$\frac{2\rho}{1-\rho^2}(1+\rho \cos \beta) + \frac{\rho}{1-\rho^2}(1+2\rho \cos \beta + \rho^2) = \frac{\rho}{1-\rho^2}(3+4\rho \cos \beta + \rho^2),$$

which takes its maximum for $\beta = 0$. This maximum is

$$\frac{\rho}{1-\rho^2}(3+4\rho+\rho^2) = \frac{\rho(3+\rho)}{1-\rho}.$$

Hence

$$\left| \operatorname*{K}_{n=2}^{\infty} \frac{c_n^2}{1} + 1 \right| \leqslant \frac{\rho(3+\rho)}{1-\rho},$$

from which follows that

$$\left| \operatorname*{K}_{n=1}^{\infty} \frac{c_n^2}{1} \right| \geqslant \frac{\rho^2}{\rho(3+\rho)/(1-\rho)} = \rho \frac{(1-\rho)}{3+\rho},$$

and (15) is established.

It remains to describe the set A. First some preliminary remarks. With

$$\Omega = \omega + 1 \tag{17}$$

and β replaced by $\pi + \gamma$ we can rewrite (11) in the following way:

$$\left| \Omega - \frac{2\rho}{1-\rho^2}(1-\rho \cos \gamma)\, \mathrm{e}^{\mathrm{i}\gamma} \right| \leqslant \frac{2\rho}{1-\rho^2}\left(\tfrac{1}{2}(1+\rho^2) - \rho \cos \gamma\right). \tag{11'}$$

When γ varies, the center moves along the curve (in polar coordinates),

$$r = \frac{2\rho}{1-\rho^2}(1-\rho \cos \gamma),$$

where $r_{\max} = 2\rho/(1-\rho)$ and $r_{\min} = 2\rho/(1+\rho)$.

The radius is

$$\frac{2\rho}{1-\rho^2}\left(\tfrac{1}{2}(1+\rho^2)-\rho\cos\gamma\right),$$

with maximal value $\rho(1+\rho)/(1-\rho)$, and minimal value $\rho(1-\rho)/(1+\rho)$. We have already seen, in the proof of Lemma 2, that the circle $|\Omega|=\rho$, i.e., $|\omega+1|=\rho$, is part of the boundary of A, and that no point inside of this circle is in A. The circle $|\Omega|=\rho$ is part of the envelope of the circles bounding the disks in (11′) , we shall here refer to it as the *inner envelope* of the family of circles. We have also seen that A is contained in the disk

$$|\omega+1|\leqslant\rho\frac{3+\rho}{1-\rho}.$$

It remains to determine the rest of the envelope, here to be referred to as the *outer envelope*.

In the equations for the circles in (11′) we introduce the new variable

$$U=\frac{\Omega}{2\rho}(1-\rho^2), \tag{18}$$

with u and v as real and imaginary parts:

$$\left(u-(1-\rho\cos\gamma)\cos\gamma\right)^2+\left(v-(1-\rho\cos\gamma)\sin\gamma\right)^2=\left(\tfrac{1}{2}(1+\rho^2)-\rho\cos\gamma\right)^2. \tag{19}$$

In order to determine the envelope of this family of circles we have to eliminate between (19) and its partial derivative with respect to the parameter γ. See, for instance, [1, Theorem 1, p.53]. Partial differentiation of (19) and some rearrangement leads to

$$(-2\rho\sin 2\gamma+2\sin\gamma)u+(2\rho\cos 2\gamma-2\cos\gamma)v+\rho(1-\rho^2)\sin\gamma=0. \tag{20}$$

We know that the pair (u, v), where

$$u=\tfrac{1}{2}(1-\rho^2)\cos\gamma,\qquad v=\tfrac{1}{2}(1-\rho^2)\sin\gamma$$

satisfies (19) and (20) simultaneously, and introduction of

$$\tilde{u}=u-\tfrac{1}{2}(1-\rho^2)\cos\gamma,\qquad \tilde{v}=v-\tfrac{1}{2}(1-\rho^2)\sin\gamma \tag{21}$$

leads to a simplification:

$$\tilde{u}^2+\tilde{v}^2-2\tilde{u}\left(\tfrac{1}{2}(1+\rho^2)-\rho\cos\gamma\right)\cos\gamma-2\tilde{v}\left(\tfrac{1}{2}(1+\rho^2)-\rho\cos\gamma\right)\sin\gamma=0, \tag{19′}$$

$$(\rho\cos 2\gamma-\cos\gamma)\tilde{v}=(\rho\sin 2\gamma-\sin\gamma)\tilde{u}. \tag{20′}$$

In addition to the already known solution $\tilde{u}=\tilde{v}=0$ we have the solution

$$\tilde{u}=(\rho\cos 2\gamma-\cos\gamma)(\rho\cos\gamma-1), \tag{22}$$

$$\tilde{v}=(\rho\sin 2\gamma-\sin\gamma)(\rho\cos\gamma-1). \tag{23}$$

From (22) and (23), together with

$$\operatorname{Re}\omega = x = -1 + \frac{2\rho}{1-\rho^2}\left(\tilde{u} + \tfrac{1}{2}(1-\rho^2)\cos\gamma\right),$$

$$\operatorname{Im}\omega = y = \frac{2\rho}{1-\rho^2}\left(\tilde{v} + \tfrac{1}{2}(1-\rho^2)\sin\gamma\right)$$

we get (16b). An elementary, but tedious discussion of how x and y vary with varying γ leads to the shape of the curve (16b). Crucial in the discussion is the ρ-value $\frac{1}{2}\sqrt{3}$. For $\rho < \frac{1}{2}\sqrt{3}$ the equation $y = 0$ has *no* solution for $0 < \gamma < \pi$, for $\rho = \frac{1}{2}\sqrt{3}$ it has *one* solution, $\gamma = \operatorname{Arccos}\frac{1}{2}\sqrt{3} = \frac{1}{6}\pi$, for $\rho > \frac{1}{2}\sqrt{3}$ it has the *two* solutions

$$\gamma_{\mp} = \operatorname{Arccos}\frac{3 \mp \sqrt{4\rho^2 - 3}}{4\rho}.$$

In the first two cases the whole curve (16b), $0 \leqslant \gamma \leqslant 2\pi$, is part of the boundary of A, the rest being the circle $|w+1| = \rho$. In the last case the "boundary part" of (16b) consists of two closed curves, defined by the parameter intervals $\gamma_- \leqslant \gamma \leqslant 2\pi - \gamma_-$ and $0 \leqslant \gamma \leqslant \gamma_+$, $2\pi - \gamma_+ \leqslant \gamma \leqslant 2\pi$.

Two questions remain: the question of sharpness and the question of surjectivity. The question of sharpness is essentially to ask whether or not in (16a) ρ can be replaced by a smaller number. Since, however, the continued fraction

$$\frac{-\rho^2|}{|1} + \frac{-(1-\rho)^2|}{|1} + \frac{-\rho^2|}{|1} + \frac{-(1-\rho)^2|}{|1} + \cdots,$$

with coefficients satisfying the conditions (4), had the value $-\rho$, it follows that ρ cannot be replaced by any smaller number. But more can be concluded. For continued fractions

$$\underset{n=1}{\overset{\infty}{\mathrm{K}}}\frac{c_n^2}{1},$$

where the elements satisfy the condition (4), the following property holds. If $\omega = \omega_0$ is a value, taken by such a continued fraction, then all values on $|\omega| = |\omega_0|$ are taken. Hence, all values on $|\omega| = \rho$ are possible values of such continued fractions. This implies that all possible values of continued fractions

$$\underset{n=2}{\overset{\infty}{\mathrm{K}}}\left(\frac{c_n^2}{1}\right), \quad \text{satisfying (4)},$$

is the union of all circles bounding the disks in (11), and hence the union of all the disks (11), i.e., the set A, and in turn that the set of all values of continued fractions

$$\underset{n=2}{\overset{\infty}{\mathrm{K}}}\frac{c_n^2}{1}, \quad \text{satisfying (4)},$$

is the set between

$$|\omega| = \frac{\rho^2}{\rho(3+\rho)/(1-\rho)} = \rho\frac{1-\rho}{3+\rho}$$

and

$$|\omega| = \frac{\rho^2}{\rho} = \rho. \qquad \square$$

In conclusion we have the following result.

Theorem 4. *For a fixed* ρ, $0 < \rho < 1$, *let* $\mathscr{F}_\rho^{(1)}$ *and* $\mathscr{F}_\rho^{(2)}$ *be the families of continued fractions*

$$\mathop{\mathrm{K}}_{n=1}^{\infty}\left(\frac{c_n^2}{1}\right) \quad \text{and} \quad \mathop{\mathrm{K}}_{n=2}^{\infty}\left(\frac{c_n^2}{1}\right),$$

where for all $n \geqslant 1$

$$|c_{2n-1}| = \rho, \qquad |c_{2n} + \mathrm{i}| = \rho. \tag{4}$$

Then the set of all possible values of continued fractions in $\mathscr{F}_\rho^{(1)}$ *is the annulus*

$$\rho\frac{1-\rho}{3+\rho} \leqslant |\omega| \leqslant \rho, \tag{24}$$

and the set of all possible values of contined fractions in $\mathscr{F}_\rho^{(2)}$ *is the set of all points on and between the circle*

$$|\omega + 1| = \rho \tag{16a}$$

and the curve

$$\begin{aligned} x = \operatorname{Re} \omega &= \frac{-1}{1-\rho^2}(1 - \rho\cos\gamma)(4\rho^2\cos^2\gamma - 2\rho\cos\gamma + 1 - 3\rho^2), \\ y = \operatorname{Im} \omega &= \frac{\rho \sin\gamma}{1-\rho^2}(4\rho^2\cos^2\gamma - 6\rho\cos\gamma + 3 - \rho^2), \end{aligned} \tag{16b}$$

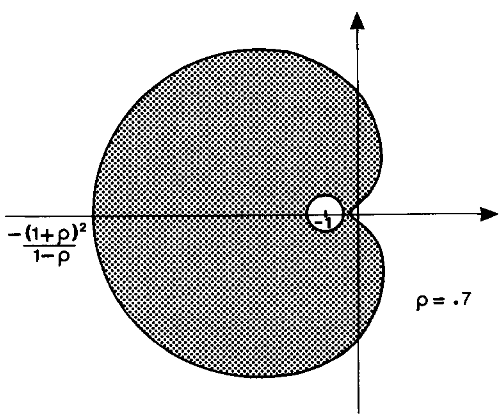

Fig. 1.

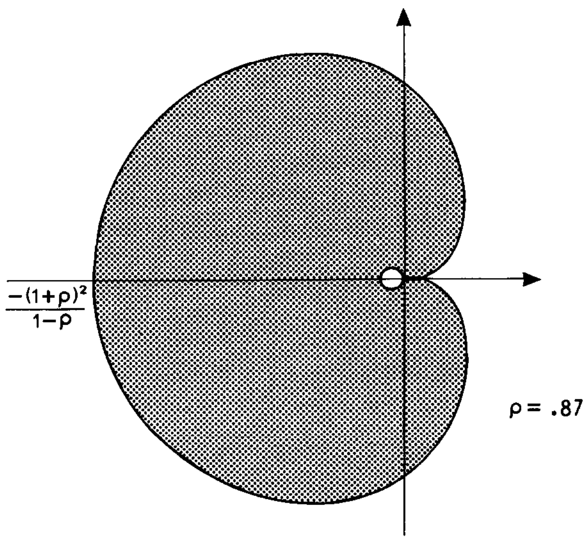

Fig. 2.

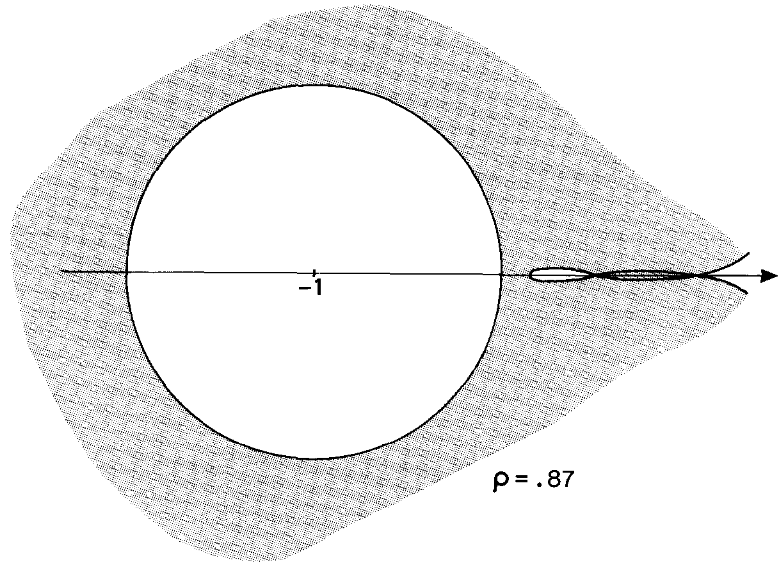

Fig. 3.

$0 \leqslant \gamma \leqslant 2\pi$, *if* $0 < \rho \leqslant \frac{1}{2}\sqrt{3}$, *and the curves* (16b), $\gamma_- \leqslant \gamma \leqslant 2\pi - \gamma_-$ *and* $0 \leqslant \gamma \leqslant \gamma_+$, $2\pi - \gamma_+ \leqslant \gamma \leqslant 2\pi$, *where*

$$\gamma_{\pm} = \operatorname{Arccos} \frac{3 \pm \sqrt{4\rho^2 - 3}}{4\rho},$$

if $\frac{1}{2}\sqrt{3} < \rho < 1$.

Remark. If the conditions (4) are replaced by the conditions (3), then the annulus (24) is replaced by the disk $|\omega| \leqslant \rho$, and the set described by (16) is replaced by the set $|\omega + 1| \geqslant \rho$.

The illustrations show, in two cases, the set of values of continued fractions in $\mathscr{F}_\rho^{(2)}$. In Fig. 1 the ρ-value, $\rho = 0.7$, is smaller than $\frac{1}{2}\sqrt{3}$, whereas in Fig. 2 we have $\rho = 0.87 > \frac{1}{2}\sqrt{3}$. Figure 3 shows, in enlarged form, a detail from the last case, a detail which cannot be seen in Fig. 2.

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