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Niche Number Four

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Abstract—It was shown previously that the minimum-order graph with niche number four has eleven or fewer vertices. We prove here that the smallest graph with niche number four has ten vertices. The proof combines a 10-vertex graph of niche number four with extensive computations which show that no graph with nine or fewer vertices has niche number four.

Keywords—Niche graph, Niche number, Competition graph.

1. INTRODUCTION

A finite simple graph $G = (V, E)$ with vertex set V and edge set E is *niche-like* if there is a finite set X disjoint from V and a directed graph $D = (V \cup X, \prec)$ with no cycles, vertex set $V \cup X$, and directed edge relation $\prec$ such that, for all $a, b \in V \cup X$,

$$\{a, b\} \in E \Leftrightarrow [\{a \prec x, b \prec x\} \text{ or } \{x \prec a, x \prec b\} \text{ for some } x \in V \cup X].$$

When this holds, we say that D *induces* G and define the niche number η of G by

$$\eta(G) = \min \{|X| : D = (V \cup X, \prec) \text{ induces } G\}.$$

By convention, $\eta(G) = \infty$ if G is not niche-like. The smallest G with $\eta(G) = \infty$ is the 4-vertex star $(\{a, b, c, d\}, \{\{a, b\}, \{a, c\}, \{a, d\}\})$. In what follows, $n = |V|$, the order of G.

Cable, Jones, Lundgren and Seager [1] introduced niche-like graphs as extensions of competition graphs [2], and Anderson [3] offers a recent survey of the topic. The seminal paper of Cable *et al.* identified graphs with $\eta \in \{0, 1, 2, \infty\}$ but left open the question of whether $\eta(G)$ can be in $\{3, 4, \dots\}$. Fishburn and Gehrlein [4] constructed graphs with $\eta = 3$ and $\eta = 4$ with 14 and 11 vertices, respectively, and proved that there are graphs with arbitrarily large finite niche numbers. We also showed that $n \leq 7 \Rightarrow \eta(G) \in \{0, 1, 2, \infty\}$ and then proved in [5] that all niche-like graphs on eight vertices have $\eta \leq 3$. Moreover, there are precisely two 8-vertex graphs for which $\eta(G) = 3$.

The present note concludes our study of small niche numbers by proving that $n = 10$ for the smallest graph with niche number four. The next section presents a 10-vertex G with $\eta(G) = 4$. Section 3 then describes a computer-intensive procedure which shows that $\eta(G) \leq 4 \Rightarrow \eta(G) \leq 3$, for every niche-like G with nine vertices.

2. EXAMPLE

Figure 1 pictures a 10-vertex G along with a 14-vertex D that induces G. The four added vertices in X for D are shown as open circles, and $x \prec y$ in D if, and only if, there is a line joining x and y that goes from x up to y (and contains no intervening vertex). Inspection shows that vertices u and v in D have another vertex w for which $\{u \prec w, v \prec w\}$ or $\{w \prec u, w \prec v\}$ if, and only if, $\{u, v\}$ is one of the 16 edges in G. Hence, $\eta(G) \leq 4$.

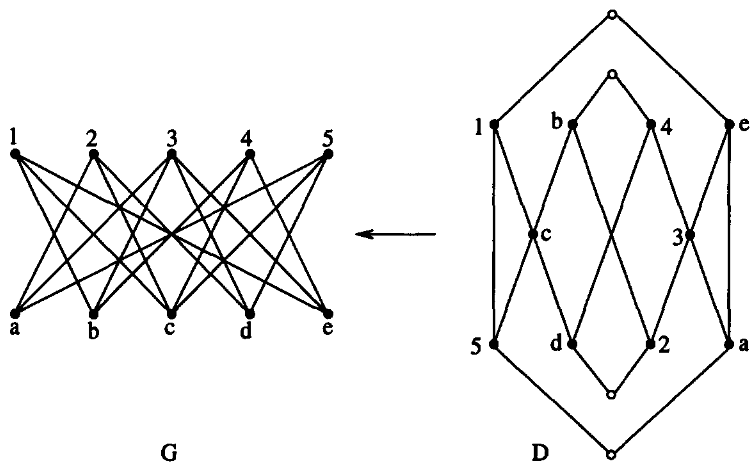

Figure 1. A 10-vertex G with $\eta(G) = 4$.

In [4, Theorem 1], it is implied that if graph G' is niche-like, triangle-free (no 3-vertex clique), and has at least $m^2 - 1$ edges incident to every vertex, then $\eta(G') \geq 2m$. Because G satisfies these hypotheses with $m = 2$, $\eta(G) \geq 4$, and we conclude that $\eta(G) = 4$.

The construction of G was guided by [4, Theorem 1] to have the properties just noted. We know of no other 10-vertex graph with niche number four.

3. NICHE NUMBERS FOR $n = 9$

The procedure used to show that $\eta(G) \neq 4$, for every niche-like G with $n = 9$ is similar to the procedure for $n = 8$ in [5], modified to reduce processing time necessitated by the larger n. In the ensuing description, we follow the usual practice of referring to a directed graph without cycles as an acyclic digraph.

Let $\mathcal{L}_k$ be the set of all 9-vertex unlabeled graphs without isolated vertices that have niche number k, for $k = 0, 1, 2, \ldots$. Our objective is to show that $|\mathcal{L}_4| = 0$. Let $\mathcal{D}_k$ be a set of acyclic digraphs of order $9 + k$ whose members necessarily induce every G in $\mathcal{L}_k$. (A procedure for generating a suitable $\mathcal{D}_k$ is described shortly.) We say that $D \in \mathcal{D}_k$ is *niche minimal* if it induces a $G \in \mathcal{L}_k$ and no $D' \in \mathcal{D}_j$, for $j < k$ also induces G. We do not presume at the start that any particular D in $\mathcal{D}_k$ is niche minimal, so we consider the graph G' induced on all $9 + k$ vertices by D. If G' has a 9-vertex induced subgraph G without isolated vertices, plus k isolated vertices, then $G \in \mathcal{L}_0 \cup \mathcal{L}_1 \cup \cdots \cup \mathcal{L}_k$.

For listing and isomorphism checks, we label vertices so that $V = \{1, 2, \ldots, 9\}$ and $X = \{10, 11, \ldots\}$, and store G' and G in 0-1 matrix form. The same labels are used later in this section to describe initial restrictions on acyclic digraphs that might be niche minimal.

Our $\mathcal{D}_k$ use restrictions that limit the number of acyclic digraphs that need to be considered as potentially niche minimal. The most important restrictions are listed here.

(R1) If $D \in \mathcal{D}_k$, then its inverse can be excluded from $\mathcal{D}_k$ since it induces the same G'.

(R2) No $D \in \mathcal{D}_k$ has an isolated vertex. More generally, every $D \in \mathcal{D}_k$ is connected, for if a D is not connected, its components can be stacked vertically with single directed edges from one minimal vertex of a component to one maximal vertex of the next lower component. Such additions do not affect G', and if an augmented connected D is potentially niche minimal, then this D or an equivalent graph inducer will be in $\mathcal{D}_k$.

(R3) All X vertices must be maximal or minimal in $D \in \mathcal{D}_k$, see [6, Lemma 2.1] or [4, Section 1].

(R4) For $D \in \mathcal{D}_k$, every X vertex must be adjacent to at least one vertex in $V = \{1, 2, \dots, 9\}$ that is not adjacent to any other vertex in X, see [6, Lemma 3.3].

(R5) Every X vertex must induce an edge for G' not induced by a V vertex (else the X vertex is redundant), and no pair of X vertices is dominated by ($\prec$) another vertex, or dominates ($\succ$) another vertex (else X vertices will not be isolated in G').

These restrictions provide initial conditions for $D \in \mathcal{D}_k$, $k \geq 1$, in the form of directed edges between vertices in $X = \{10, 11, \dots\}$ and in $V = \{1, 2, \dots, 9\}$ that are imposed on all digraphs in $\mathcal{D}_k$. Sets of initial conditions for $\mathcal{D}_1$ through $\mathcal{D}_4$ are shown, respectively, on Figures 2a–2d. For example, every constructed acyclic digraph in $\mathcal{D}_4$ contains one of the ten subconfigurations in Figure 2d.

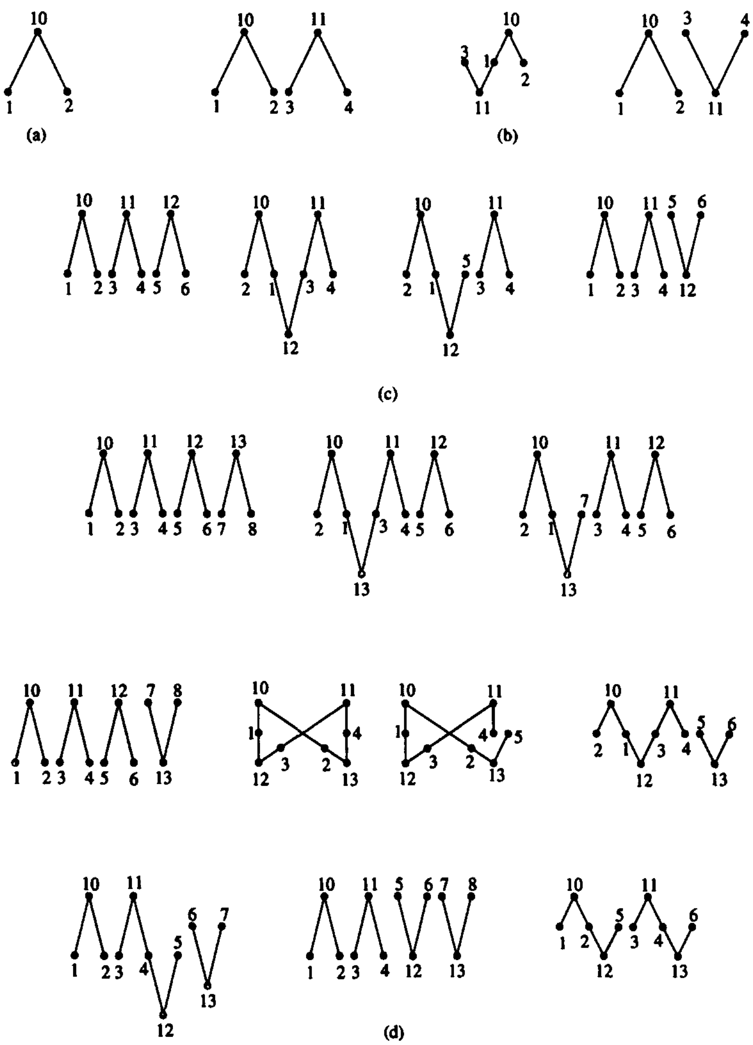

Figure 2. Initial conditions for digraph constuction.

To avoid unnecessary consideration of label permutations that do not affect G', we adopted the convention for distinct vertices i and j in V that are not in the initial condition, that $i > j$ whenever $i \prec j$.

To show that $|\mathcal{L}_4| = 0$, we began by generating $\mathcal{D}_4$ and found that $|\mathcal{D}_4| = 7,400$. We would like to show that everything in $\mathcal{D}_4$ is in a $\mathcal{D}_k$ for $k < 4$. It was not possible to compile a list of graphs for $\mathcal{D}_2$ and $\mathcal{D}_3$ since $|\mathcal{D}_2|$ and $|\mathcal{D}_3|$ are very large, and it was not possible to perform all the repeated checks for isomorphism on the induced graphs in a reasonable amount of computation time. To circumvent this problem, we defined $T_3 = \mathcal{D}_4 \setminus (\mathcal{D}_3 \cap \mathcal{D}_4)$, and obtained T_3 by sequentially generating $D \in \mathcal{D}_3$ and removing the associated digraphs from $\mathcal{D}_4$, if the corresponding niche-like graphs were isomorphic. We found $|T_3| = 442$. Hence, if there is a 9-vertex graph with niche number four, then its inducing digraph D must be among these 442.

Given T_3, we then defined T_2 by $T_2 = T_3 \setminus (\mathcal{D}_2 \cap T_3)$, but attempts to enumerate T_2 failed, because $|\mathcal{D}_2|$ is so large that it could not be enumerated in a reasonable amount of computation time. We did obtain a partial listing $\mathcal{D}_2'$ of $\mathcal{D}_2$ that contained all graphs starting with the first two configurations in Figure 2b. At that point, $T_2' = T_3 \setminus (\mathcal{D}_2' \cap T_3)$ had $|T_2'| = 57$. An attempt was made to directly calculate η for the 57 graphs in T_2'. We verified $\eta \leq 3$ for 13 of them, but could not obtain η for the remaining set of 44, which we denote by T_2''.

Graphs in $\mathcal{D}_2$ which start with the third configuration in Figure 2b were enumerated by adding a sixth restriction to our (R) list to limit the number of digraphs that had to be considered for entry into $\mathcal{D}_2$. To describe this restriction, we define the set $M(a)$ for $a \in V$ by

$$M(a) = \{b \in V \cup X : b \prec a\}.$$

Our new restriction is defined on digraphs starting from the third configuration in Figure 2b as follows.

(R6) For odd $a \in V$ with $a \leq 7$ and $(a+1) \not\prec a$, $|M(a)| \geq |M(a+1)|$. (Otherwise the same G' could be obtained by the isomorphic D with vertices a and $(a+1)$ interchanged.)

This restriction follows from the initial condition that $i > j$ whenever $i \prec j$. Using (R6), the enumeration of members of $\mathcal{D}_2$ starting with the third configuration in Figure 2b eliminated all graphs in T_2''. Thus, $|\mathcal{L}_4| = 0$ for $n = 9$.

It should be noted that our proof does not imply that $\eta(G) \leq 4$ for every niche-like G with $n = 9$. Reference [6, Theorem 3.4] tells us that $\eta(G) \leq 6$ for all such G, so we have not excluded the possibility that some 9-point niche-like G has niche number five or six.

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