Another generalization of weighted Ostrowski type inequality for mappings of bounded variation

Zheng Liu
Institute of Applied Mathematics, School of Science, University of Science and Technology Liaoning, Anshan 114051, Liaoning, China

**Abstract**
A new generalization of weighted Ostrowski type inequality for mappings of bounded variation with a unified sharp bound is established.

© 2011 Elsevier Ltd. All rights reserved.

1. Introduction

The following theorem contains an integral inequality which is known in the literature as the Ostrowski inequality [1, p. 469].

**Theorem 1.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a differentiable mapping on \((a, b)\) whose derivative is bounded on \((a, b)\) and denote \( \|f'\|_\infty = \sup_{t \in (a, b)} |f'(t)| < \infty \). Then for all \( x \in [a, b] \), we have the inequality

\[
\left| \int_a^b f(t) \, dt - f(x)(b - a) \right| \leq \frac{1}{4} (b - a)^2 + \left( x - \frac{a + b}{2} \right)^2 \|f'\|_\infty.
\]

The constant \( \frac{1}{4} \) is sharp in the sense that it cannot be replaced by a smaller one.

The first generalization of the Ostrowski inequality for mappings of bounded variation was given by Dragomir in [2] as follows.

**Theorem 2.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a mapping with bounded variation on \([a, b]\). Then for all \( x \in [a, b] \), we have the inequality

\[
\left| \int_a^b f(t) \, dt - f(x)(b - a) \right| \leq \frac{1}{2} (b - a) + \left( x - \frac{a + b}{2} \right) \sqrt{\mathcal{V}(f)}.
\]

where \( \sqrt{\mathcal{V}(f)} \) denotes the total variation of \( f \) on the interval \([a, b]\). The constant \( \frac{1}{2} \) is the best possible one.

In [3], Kuei-Lin Tseng et al. have proved the following generalization of weighted Ostrowski type inequality for mappings of bounded variation.

**Theorem 3.** Let \( f : [a, b] \rightarrow \mathbb{R} \) be a mapping of bounded variation, \( g : [a, b] \rightarrow [0, \infty) \) continuous and positive on \((a, b)\) and \( h : [a, b] \rightarrow \mathbb{R} \) be differentiable such that \( h'(t) = g(t) \) on \([a, b]\). Let \( 0 \leq \alpha \leq 1, c = h^{-1}((1 - \frac{\alpha}{2})h(a) + \frac{\alpha}{2}h(b)) \) and...
\[ d = h^{-1}(\frac{\alpha}{2} h(a) + (1 - \frac{\alpha}{2}) h(b)). \] Then, for all \( x \in [c, d] \), we have

\[
\left| \int_a^b f(t) g(t) \, dt - \left[ (1 - \alpha)f(x) + \alpha \cdot \frac{f(a) + f(b)}{2} \right] \int_a^b g(t) \, dt \right| \leq K \cdot \sqrt{b_a(f),}
\] (1)

where

\[
K = \begin{cases} 
\frac{1}{2} - \alpha + \frac{1}{2} \int_a^b g(t) \, dt + \left| h(x) - \frac{h(a) + h(b)}{2} \right|, & 0 \leq \alpha \leq \frac{1}{2}, \\
\alpha - \frac{1}{2} \int_a^b g(t) \, dt, & \frac{1}{2} < \alpha < \frac{3}{2}, \\
\frac{\alpha}{2} \int_a^b g(t) \, dt, & \frac{3}{2} \leq \alpha \leq 1
\end{cases}
\]

and \( \sqrt{b_a(f)} \) denotes the total variation of \( f \) on the interval \([a, b]\). In (1), the constant \( \frac{1}{2} - \alpha \) for \( 0 \leq \alpha \leq \frac{1}{2} \) and the constant \( \frac{\alpha}{2} \) for \( \frac{3}{2} \leq \alpha \leq 1 \) are the best possible.

If we take \( \alpha = 0 \), then for all \( x \in [a, b] \) the inequality (1) reduces to the following inequality:

\[
\left| \int_a^b f(t) g(t) \, dt - f(x) \cdot \int_a^b g(t) \, dt \right| \leq \left[ \frac{1}{2} \int_a^b g(t) \, dt + \left| h(x) - \frac{h(a) + h(b)}{2} \right| \right] \cdot \sqrt{b_a(f),}
\] (2)

which is the “weighted Ostrowski” inequality.

In this work, we will provide another generalization of the above weighted Ostrowski type inequality which has a unified sharp bound.

2. The results

**Theorem 4.** Let \( f : [a, b] \to \mathbb{R} \) be a mapping of bounded variation, \( g : [a, b] \to [0, \infty) \) continuous and positive on \((a, b)\). Then for any \( x \in [a, b] \) and \( \alpha \in [0, 1] \) we have

\[
\left| \int_a^b f(t) g(t) \, dt - \left( (1 - \alpha)f(x) \int_a^b g(t) \, dt + \alpha \left[ f(a) \int_a^x g(t) \, dt + f(b) \int_x^b g(t) \, dt \right] \right) \right| \\
\leq \left[ \frac{1}{2} + \frac{1}{2} - \alpha \right] \left[ \frac{1}{2} \int_a^b g(t) \, dt + \left| \int_a^x g(t) \, dt - \frac{1}{2} \int_a^b g(t) \, dt \right| \right] \sqrt{b_a(f),}
\] (3)

where \( \sqrt{b_a(f)} \) denotes the total variation of \( f \) on the interval \([a, b]\). The constant \( \frac{1}{2} + \frac{1}{2} - \alpha \) is the best possible.

**Proof.** Integrating by parts produces the identity

\[
\int_a^b S_x(t) f(t) \, dt = f(x) \left[ \int_a^b g(t) \, dt + \alpha \left[ f(a) \int_a^x g(t) \, dt + f(b) \int_x^b g(t) \, dt \right] \right] - \int_a^b f(t) g(t) \, dt,
\] (4)

where

\[
S_x(t) = \begin{cases} 
(1 - \alpha) \int_a^t g(s) \, ds + \alpha \int_x^t g(s) \, ds, & t \in [a, x), \\
(1 - \alpha) \int_a^b g(s) \, ds + \alpha \int_x^b g(s) \, ds, & t \in [x, b].
\end{cases}
\] (5)

It is well known [4, p. 159] that if \( u, v : [a, b] \to \mathbb{R} \) are such that \( u \) is continuous on \([a, b]\) and \( v \) is of bounded variation on \([a, b]\), then \( \int_a^b u(t) \, d\nu(t) \) exists and [4, p. 177]

\[
\left| \int_a^b u(t) \, d\nu(t) \right| \leq \sup_{t \in [a, b]} |u(t)| \sqrt{b_a(f),}
\] (6)

Now, using (4) and (6), we have

\[
\left| \int_a^b f(t) g(t) \, dt - \left( (1 - \alpha)f(x) \int_a^b g(t) \, dt + \alpha \left[ f(a) \int_a^x g(t) \, dt + f(b) \int_x^b g(t) \, dt \right] \right) \right| \\
\leq \sup_{t \in [a, b]} |S_x(t)| \sqrt{b_a(f),}
\] (7)
For brevity, we put
\[ p(t) = (1 - \alpha) \int_a^t g(s) \, ds + \alpha \int_a^t g(s) \, ds, \quad t \in [a, x], \]
\[ q(t) = (1 - \alpha) \int_b^t g(s) \, ds + \alpha \int_b^t g(s) \, ds, \quad t \in [x, b]. \]

It is easy to find that \( p(t) \) is increasing on the interval \([a, x]\) and \( q(t) \) is increasing on the interval \([x, b]\), since \( p'(t) = q'(t) = g(t) > 0 \). So, we get
\[
\sup_{t \in [a, x]} |S_g(t)| = \max \left\{ (1 - \alpha) \int_a^x g(t) \, dt, \alpha \int_a^x g(t) \, dt \right\} = \left[ \frac{1}{2} + \left| \frac{1}{2} - \alpha \right| \right] \int_a^x g(t) \, dt
\]
and
\[
\sup_{t \in [x, b]} |S_g(t)| = \max \left\{ (1 - \alpha) \int_x^b g(t) \, dt, \alpha \int_x^b g(t) \, dt \right\} = \left[ \frac{1}{2} + \left| \frac{1}{2} - \alpha \right| \right] \int_x^b g(t) \, dt.
\]

Thus
\[
\sup_{t \in [a, b]} |S_g(t)| = \max \left\{ \sup_{t \in [a, x]} |S_g(t)|, \sup_{t \in [x, b]} |S_g(t)| \right\} = \left[ \frac{1}{2} + \left| \frac{1}{2} - \alpha \right| \right] \left[ \frac{1}{2} \int_a^b g(t) \, dt + \int_a^x g(t) \, dt + \frac{1}{2} \int_a^b g(t) \, dt \right]. \tag{8}
\]

By (7) and (8), we obtain (3).

Suppose \( 0 \leq \alpha \leq \frac{1}{2} \). We assume that the inequality (3) holds with a constant \( C_1 > 0 \), i.e.,
\[
\left| \int_a^b f(t) g(t) \, dt - \left\{ (1 - \alpha)f(x) \int_a^x g(t) \, dt + \alpha \left[ f(a) \int_a^x g(t) \, dt + f(b) \int_x^b g(t) \, dt \right] \right\} \right| \leq C_1 \left[ \frac{1}{2} \int_a^b g(t) \, dt + \int_a^x g(t) \, dt + \frac{1}{2} \int_a^b g(t) \, dt \right] \sqrt{f}. \tag{9}
\]

It is clear that the function \( \int_a^t g(s) \, ds \) is strictly increasing and continuous on \([a, b]\) and so there exists unique \( t_0 \) such that \( \int_a^{t_0} g(s) \, ds = \frac{1}{2} \int_a^b g(s) \, ds \). Let
\[
f(t) = \begin{cases} 0, & t \in [a, b] \setminus \{t_0\}, \\ \frac{1}{2}, & t = t_0. \end{cases}
\]
Then \( f \) is with bounded variation on \([a, b]\), and
\[
\int_a^b f(t) g(t) \, dt = 0, \quad \sqrt{f} = 1
\]
and for \( x = t_0 \), we get in (9)
\[ 1 - \alpha \leq C_1, \]
which implies that the constant \( 1 - \alpha \) is the best possible.

Suppose \( \frac{1}{2} \leq \alpha \leq 1 \). We assume that the inequality (3) holds with a constant \( C_2 > 0 \), i.e.,
\[
\left| \int_a^b f(t) g(t) \, dt - \left\{ (1 - \alpha)f(x) \int_a^x g(t) \, dt + \alpha \left[ f(a) \int_a^x g(t) \, dt + f(b) \int_x^b g(t) \, dt \right] \right\} \right| \leq C_2 \left[ \frac{1}{2} \int_a^b g(t) \, dt + \int_a^x g(t) \, dt - \frac{1}{2} \int_a^b g(t) \, dt \right] \sqrt{f}. \tag{10}
\]
Let 
\[ f(t) = \begin{cases} 
0, & t \in [a, b), \\
\frac{1}{2}, & t = b. 
\end{cases} \]

Then \( f \) is with bounded variation on \([a, b]\), and
\[
\int_a^b f(t)g(t) \, dt = 0, \quad \sqrt{b_a} = 1
\]
and for \( x = a \), we get in (10)
\[
\alpha \leq C_2,
\]
which implies that the constant \( \alpha \) is the best possible. Consequently, we can conclude that the constant \( \frac{1}{2} + |\frac{1}{2} - \alpha| \) is the best possible.

This completes the proof. \( \square \)

**Remark 1.** It is clear that \( \alpha = \frac{1}{2} \) is the best possible case in (3).

If we take \( \alpha = 0 \) in (3), we get the sharp ”weighted Ostrowski” inequality
\[
\left| \int_a^b f(t)g(t) \, dt - f(x) \int_a^b g(t) \, dt \right| \leq \left[ \frac{1}{2} \int_a^b g(t) \, dt + \left( \int_a^x f(t) \, dt - \frac{1}{2} \int_a^b g(t) \, dt \right) \right] \sqrt{b_a} \]
which is just the same of (2).

**Remark 2.** If we take \( g(t) = 1 \) on \([a, b]\) in (3), then we get the following inequality
\[
\left| \int_a^b f(t) \, dt - (b - a) \left( (1 - \alpha)f(x) + \alpha \left( \frac{x - a}{b - a} f(a) + \frac{b - x}{b - a} f(b) \right) \right) \right|
\leq \left[ \frac{1}{2} + \frac{1}{2} - \alpha \right] \left[ \frac{1}{2} + \frac{x - a + b}{2} \right] \sqrt{b_a} \]
which has been proved in Theorem 22 of [5]. Moreover, if we take \( x = \frac{a + b}{2} \) in (11) then we have
\[
\left| \int_a^b f(t) \, dt - (b - a) \left( (1 - \alpha)f\left( \frac{a + b}{2} \right) + \alpha \frac{f(a) + f(b)}{2} \right) \right| \leq \frac{b - a}{2} \left[ \frac{1}{2} + |\frac{1}{2} - \alpha| \right] \sqrt{b_a}. \quad (12)
\]

It should be noticed that (12) provides a unified treatment of some classical quadrature rules and shows the averaged midpoint-trapezoid rule is optimal in the current situation.

**Corollary 1.** Let \( 0 \leq \alpha \leq 1, f : [a, b] \to \mathbb{R} \) be absolutely continuous on \([a, b]\). Then we have the inequality
\[
\left| \int_a^b f(t)g(t) \, dt - \left( (1 - \alpha)f(x) \int_a^x g(t) \, dt + \alpha \left( f(a) \int_a^x g(t) \, dt + f(b) \int_x^b g(t) \, dt \right) \right) \right|
\leq \left[ \frac{1}{2} + \frac{1}{2} - \alpha \right] \left[ \frac{1}{2} \int_a^b g(t) \, dt + \left( \int_a^x g(t) \, dt - \frac{1}{2} \int_a^b g(t) \, dt \right) \|f'\|_1 \right] \sqrt{b_a}
\]
for all \( x \in [a, b] \), where \( \| \cdot \|_1 \) is the \( L_1 \)-norm, namely
\[
\|f'\|_1 = \int_a^b |f'(t)| \, dt.
\]

**Corollary 2.** Let \( 0 \leq \alpha \leq 1, f : [a, b] \to \mathbb{R} \) be a Lipschitzian mapping with the constant \( L > 0 \). Then we have the inequality
\[
\left| \int_a^b f(t)g(t) \, dt - \left( (1 - \alpha)f(x) \int_a^x g(t) \, dt + \alpha \left( f(a) \int_a^x g(t) \, dt + f(b) \int_x^b g(t) \, dt \right) \right) \right|
\leq \left[ \frac{1}{2} + \frac{1}{2} - \alpha \right] \left[ \frac{1}{2} \int_a^b g(t) \, dt + \left( \int_a^x g(t) \, dt - \frac{1}{2} \int_a^b g(t) \, dt \right) L(b - a) \right]
\]
for all \( x \in [a, b] \).
**Corollary 3.** Let $0 \leq \alpha \leq 1$, $f : [a, b] \to \mathbb{R}$ be a monotonic mapping. Then we have the inequality

$$
\left| \int_a^b f(t)g(t) \, dt - \left( 1 - \alpha \right)f(x) \int_a^b g(t) \, dt + \alpha \left[ f(a) \int_a^x g(t) \, dt + f(b) \int_x^b g(t) \, dt \right] \right| 
\leq \left[ \frac{1}{2} + \frac{1}{2} - \alpha \right] \left[ \frac{1}{2} \int_a^b g(t) \, dt + \left| \int_a^x g(t) \, dt - \frac{1}{2} \int_a^b g(t) \, dt \right| \right] |f(b) - f(a)|
$$

for all $x \in [a, b]$.

**References**


