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ENUMERATION OF INTERSECTING FAMILIES

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It is shown that the logarithm to the base 2 of the number of maximal intersecting families on m elements is asymptotically equal to $\binom{m-1}{n-1}$ where $n=[\frac{1}{2}m]$.

1. Introduction

For a natural number m, let $\boldsymbol{m}=\{1,2,\ldots,m\}$. An intersecting family on $\boldsymbol{m}$ is a set $\mathscr{A}$ of sets such that $\bigcup\mathscr{A}\subseteq\boldsymbol{m}$ and any two members of $\mathscr{A}$ have non-empty intersection. We let $\mathscr{I}_m$ be the set of all maximal intersecting families on $\boldsymbol{m}$. We are concerned with estimating $|\mathscr{I}_m|$.

In Section 2 we obtain a lower bound by elementary counting methods. In Section 3 we obtain an upper bound using a result of Kleitman and Markowsky on the number of monotone Boolean functions.

Notice that if in the definition of intersecting families, the requirement that any two members of A have non-empty intersection is raised to any three members, the problem becomes trivial. Indeed, by [1, Remark 7.5] any maximal intersecting family would be an ultrafilter; that is it would consist of all subsets of $\boldsymbol{m}$ containing some singleton.

2. A lower bound and statistical remarks

We observe that an intersecting family $\mathscr{A}$ on $\boldsymbol{m}$ is maximal if and only if for every $A\subseteq\boldsymbol{m}$, either $A\in\mathscr{A}$ or $\boldsymbol{m}\setminus A\in\mathscr{A}$. Observe also that if $\mathscr{A}$ is an intersecting family on $\boldsymbol{m}$, $A\in\mathscr{A}$, and $A\subseteq B\subseteq\boldsymbol{m}$, then $B\in\mathscr{A}$.

2.1. Definition. A subset $\mathscr{B}$ of $\mathscr{P}(\boldsymbol{m})$ is a free choice family on $\boldsymbol{m}$ if and only if whenever $\mathscr{C}\subseteq\mathscr{B}$, $\mathscr{C}\cup\{\boldsymbol{m}\setminus B\colon B\in\mathscr{B}\setminus\mathscr{C}\}$ is an intersecting family.

We denote by $[A]^k$ the set of k-element subsets of A.

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2.2. Lemma. *Let $n=[\frac{1}{2}m]$ and let $\mathscr{B}=\{A\in[\boldsymbol{m}]^n\colon\ 1\in A\}$. Then $\mathscr{B}$ is a free choice family on $\boldsymbol{m}$ with largest possible cardinality.*

Proof. Trivially any two members of $\mathscr{B}$ have non-empty intersection. Distinct members of $\{\boldsymbol{m}\setminus B;\ B\in\mathscr{B}\}$ must meet because of their size. Likewise, if $B, C\in\mathscr{B}$ and $B\cap(\boldsymbol{m}\setminus C)=\emptyset$, then $B=C$. Thus $\mathscr{B}$ is a free choice family on m with cardinality $\binom{m-1}{n-1}$.

By Theorem 1 of [1], if $\mathscr{C}$ is an intersecting family on $\boldsymbol{m}$, each $A\in\mathscr{C}$ has $|A|\leqslant n$, and whenever A and B are distinct members of $\mathscr{C}$ neither $A\subseteq B$ nor $B\subseteq A$, then $|\mathscr{C}|\leqslant\binom{m-1}{n-1}$. Given a free choice family $\mathscr{D}$ on $\boldsymbol{m}$, let $\mathscr{C}=\{A\in\mathscr{D}\colon |A|\leqslant n\}\cup\{\boldsymbol{m}\setminus A\colon\ A\in\mathscr{D}$ and $|A|>n\}$. Then $\mathscr{C}$ satisfies the conditions above, so $\mathscr{C}$ (and hence $\mathscr{D}$) has at most $\binom{m-1}{n-1}$ elements. $\square$

Lemma 2.2 yields immediately a lower bound of $2^{\binom{m-1}{n-1}}$ for $|\mathscr{I}_m|$. As we shall see this is an asymptotically correct value in the exponent. However we do manage to raise the lower bound somewhat by considering free choices which remain given a particular choice from $\mathscr{B}$.

2.3. Theorem. *Let $n=[\frac{1}{2}m]$.*

(a) *If $m=2n$, then $|\mathscr{I}_m|\geqslant 2^{\binom{m-1}{n-1}+\binom{m}{n-1}/2^{n+1}}$*

(b) *If $m=2n+1$, then $|\mathscr{I}_m|\geqslant 2^{\binom{m-1}{n-1}+\binom{m-2}{n-1}/2^n}$.*

Proof. Let $\mathscr{B}=\{A\in[\boldsymbol{m}]^n\colon\ 1\in A\}$. Given $\mathscr{F}\subseteq\mathscr{B}$, let $\mathscr{C}(\mathscr{F})=\mathscr{F}\cup\{\boldsymbol{m}\setminus B\colon B\in\mathscr{B}\setminus\mathscr{F}\}$ (so that $\mathscr{C}(\mathscr{F})$ is the choice induced by $\mathscr{F}$). If $\mathscr{F}\subseteq\mathscr{B}$ and $m=2n$, let

$$\mathscr{D}(\mathscr{F})=\{A\in[\boldsymbol{m}]^{n+1}\colon \text{ for all } B\in\mathscr{C}(\mathscr{F}), B\setminus A\neq\emptyset\}.$$

If $\mathscr{F}\subseteq\mathscr{B}$ and $m=2n+1$, let

$$\mathscr{D}(\mathscr{F})=\{A\in[\boldsymbol{m}]^{n+1}\colon \{1,2\}\subseteq A \text{ and for all } B\in\mathscr{C}(\mathscr{F}), B\setminus A\neq\emptyset\}.$$

For any $\mathscr{F}\subseteq\mathscr{B}$, let $d(\mathscr{F})=|\mathscr{D}(\mathscr{F})|$. We claim that

(*) If $\mathscr{F}\subseteq\mathscr{B}$ and $\mathscr{G}\subseteq\mathscr{D}(\mathscr{F})$, then $\mathscr{C}(\mathscr{F})\cup\mathscr{G}\cup\{\boldsymbol{m}\setminus A\colon\ A\in\mathscr{D}(\mathscr{F})\setminus\mathscr{G}\}$ is an intersecting family.

To see (*) note that $\mathscr{D}(\mathscr{F})$ was defined so that whenever $B\in\mathscr{C}(\mathscr{F})$ and $A\in\mathscr{D}(\mathscr{F})$, both $B\cap A$ and $B\cap(\boldsymbol{m}\setminus A)$ are non-empty. Also if $A, B\in\mathscr{D}(\mathscr{F})$ and $A\neq B$, then $B\cap A$ and $B\cap(\boldsymbol{m}\setminus A)$ are non-empty by virtue of their sizes. (If one had $B\cap(\boldsymbol{m}\setminus A)=\emptyset$ one would have $B=A$). Consequently we need only show that if $A, B\in\mathscr{D}(\mathscr{F})$ and $A\neq B$, then $(\boldsymbol{m}\setminus A)\cap(\boldsymbol{m}\setminus B)\neq\emptyset$. If $m=2n+1$, then $\{1,2\}\subseteq A\cap B$ and hence $|A\cup B|\leqslant 2n$ so we can assume $m=2n$. Suppose $(\boldsymbol{m}\setminus A)\cap(\boldsymbol{m}\setminus B)=\emptyset$. Then $A\cup B=\boldsymbol{m}$ so $|A\cap B|=2$. Pick $x, y\in\boldsymbol{m}$ such that $A\cap B=\{x, y\}$. Then either $\{x\}\cup(A\setminus B)$ or $\{y\}\cup(B\setminus A)$ is in $\mathscr{C}(\mathscr{F})$ and we may assume the former. Then since $\{x\}\cup(A\setminus B)\subseteq A$ we have $A\notin\mathscr{D}(\mathscr{F})$, a contradiction.

Since (*) holds, we have $|\mathscr{I}_m|\geqslant\sum_{\mathscr{F}\subseteq\mathscr{B}}2^{d(\mathscr{F})}$

Let $G=\{(\mathscr{F}, A)\colon \mathscr{F}\subseteq\mathscr{B} \text{ and } A\in\mathscr{D}(\mathscr{F})\}$. We count G in two ways. On the one hand $|G|=\sum_{\mathscr{F}\subseteq\mathscr{B}} d(\mathscr{F})$. Given $A\in[\boldsymbol{m}]^{n+1}$ (with $\{1,2\}\subseteq A$ if $m=2n+1$) and $\mathscr{F}\subseteq\mathscr{B}$, we have $A\in\mathscr{D}(\mathscr{F})$ if and only if no subset of A is in $\mathscr{C}(\mathscr{F})$. Assume now $m=2n$ and $A\in[\boldsymbol{m}]^{n+1}$. There are $n+1$ n-element subsets of A and $\binom{m-1}{n-1}$ elements of $\mathscr{B}$ so $|\{\mathscr{F}\subseteq\mathscr{B}\colon A\in\mathscr{D}(\mathscr{F})\}|=2^{\binom{m-1}{n-1}-n-1}$. Since $|[\boldsymbol{m}]^{n+1}|=\binom{m}{n+1}=\binom{m}{n-1}$ we have

$$|G|=\binom{m}{n-1}\cdot 2^{\binom{m-1}{n-1}-n-1}.$$

Now assume $m=2n+1$ and $A\in[\boldsymbol{m}]^{n+1}$ with $\{1,2\}\subseteq A$. Any subset B of A which is in $\mathscr{C}(\mathscr{F})$ must in fact be in $\mathscr{F}$ and hence must have $1\in B$. There are n such n-elements subsets so $|\{\mathscr{F}\subseteq\mathscr{B}\colon A\in\mathscr{D}(\mathscr{F})\}|=2^{\binom{m-1}{n-1}-n}$. Since $|\{A\in[\boldsymbol{m}]^{n+1}\colon \{1,2\}\subseteq A\}|=\binom{m-2}{n-1}$ we have

$$|G|=\binom{m-2}{n-1}\cdot 2^{\binom{m-1}{n-1}-n}.$$

Let $\bar{d}=(\sum_{\mathscr{F}\subseteq\mathscr{B}} d(\mathscr{F}))/|\mathscr{P}(\mathscr{B})|$. (Thus $\bar{d}$ is the mean value of the $d(\mathscr{F})$'s.) We have then

$$|\mathscr{I}_m|\geqslant\sum_{\mathscr{F}\subseteq\mathscr{B}} 2^{d(\mathscr{F})}\geqslant\sum_{\mathscr{F}\subseteq\mathscr{B}} 2^{\bar{d}}=2^{\binom{m-1}{n-1}+\bar{d}}.$$

Inserting the value for $\bar{d}$ obtained by our double counting of G we have the desired result. $\square$

We now restrict our attention to the simpler case when $m=2n$ and discuss the distribution of $\{d(\mathscr{F})\colon \mathscr{F}\subseteq\mathscr{B}\}$. We obtained above the value $\binom{m}{n-1}/2^{n+1}$ for the mean by counting twice the set $\{(\mathscr{F}, A)\colon \mathscr{F}\subseteq\mathscr{B} \text{ and } A\in\mathscr{D}(\mathscr{F})\}$, we can also compute the variance by counting twice the set

$$\{(\mathscr{F}, A, B)\colon \mathscr{F}\subseteq\mathscr{B}, A\in\mathscr{D}(\mathscr{F}), \text{ and } B\in\mathscr{D}(\mathscr{F})\}.$$

(In this computation we consider separately pairs (A, B) where $A=B$, $|A\cap B|=n$, and $3\leqslant|A\cap B|\leqslant n-1$. As we saw earlier if $|A\cap B|=2$, then for no $\mathscr{F}\subseteq\mathscr{B}$ do we have $A\in\mathscr{D}(\mathscr{F})$ and $B\in\mathscr{D}(\mathscr{F})$.) This computation yields the result (for $n\geqslant 3$)

$$\sum_{\mathscr{F}\subseteq\mathscr{B}}(d(\mathscr{F})-\bar{d})^2=\binom{m}{n-1}2^{\binom{m-1}{n-1}-n-1}(1+(n^2-n-4)2^{-n-2})$$

so that the variance is $\binom{m}{n-1}/2^{n+1}\cdot(1+o(1))$.

3. An upper bound

Let $\mathscr{I}_m$ be the set of antichains in $\mathscr{P}(\boldsymbol{m})$. (A set $\mathscr{A}\subseteq\mathscr{P}(\boldsymbol{m})$ is an anti-chain provided that whenever $A, B\in\mathscr{A}$ with $A\subseteq B$ one has $A=B$). It was shown in [3], improving an earlier result [2], that there is a constant c such that $|\mathscr{I}_m|<2^{(1+c\log m/m)\binom{m}{n}}$ where $n=[\frac{1}{2}m]$. We show in this section that $|\mathscr{I}_m|\leqslant|\mathscr{I}_{m-1}|$.

3.1. Definition. Define a function g on $\mathcal{I}_m$ by $g(\mathcal{A})=\{A\subseteq \boldsymbol{m-1}\colon A=B\cap \boldsymbol{m-1}$ for some $B\in\mathcal{A}$ and there does not exist $C\in\mathcal{A}$ such that $C\cap\boldsymbol{m-1}\subsetneq A\}$.

3.2. Lemma. *Let $\mathcal{A}\in\mathcal{I}_m$ and let $A\subseteq\boldsymbol{m-1}$.*

(a) *$A\cup\{m\}\in\mathcal{A}$ if and only if there exists $B\in g(\mathcal{A})$ such that $B\subseteq A$.*

(b) *$A\in\mathcal{A}$ if and only if there exists $B\in g(\mathcal{A})$ such that $B\subseteq A$ and there does not exist $C\in g(\mathcal{A})$ such that $C\cap A=\emptyset$.*

Proof. (a) Assume $A\cup\{m\}\in\mathcal{A}$. Pick $D\in\mathcal{A}$ such that $D\subseteq A\cup\{m\}$ and $|D\cap\boldsymbol{m-1}|$ is minimal among all such members of $\mathcal{A}$. Let $B=D\cap\boldsymbol{m-1}$. Then $B\in g(\mathcal{A})$ and $B\subseteq A$.

Now assume we have $B\in g(\mathcal{A})$ such that $B\subseteq A$. Pick $D\in\mathcal{A}$ such that $D\cap\boldsymbol{m-1}=B$. Then $D\subseteq A\cup\{m\}$ so $A\cup\{m\}\in\mathcal{A}$.

(b) Assume $A\in\mathcal{A}$. Then $A\cup\{m\}\in\mathcal{A}$ so by (a) we have some $B\in g(\mathcal{A})$ such that $B\subseteq A$. Suppose we have $C\in g(\mathcal{A})$ such that $C\cap A=\emptyset$. Pick $D\in\mathcal{A}$ such that $D\cap\boldsymbol{m-1}=C$. Then $D\cap A=\emptyset$, a contradiction.

Finally assume we have some $B\in g(\mathcal{A})$ such that $B\subseteq A$ and have no $C\in g(\mathcal{A})$ such that $C\cap A=\emptyset$. By (a) we have $A\cup\{m\}\in\mathcal{A}$. Suppose that $A\notin\mathcal{A}$ so that $\boldsymbol{m}\setminus A\in\mathcal{A}$. Again by (a) pick $C\in g(\mathcal{A})$ such that $C\subseteq\boldsymbol{m}\setminus A$. Then $C\cap A=\emptyset$, a contradiction. □

3.3. Theorem. *The function g is one-to-one and takes $\mathcal{I}_m$ to $\mathcal{S}_{m-1}$.*

Proof. By Lemma 3.2, $\mathcal{A}$ is completely determined by $g(\mathcal{A})$ so g is one-to-one. Let $\mathcal{A}\in\mathcal{I}_m$. To see that $g(\mathcal{A})\in\mathcal{S}_{m-1}$ suppose instead we have $B, C\in g(\mathcal{A})$ with $C\subsetneq B$. Pick $D, F\in\mathcal{A}$ such that $D\cap\boldsymbol{m-1}=B$ and $E\cap\boldsymbol{m-1}=C$. Then $E\cap\boldsymbol{m-1}\subsetneq B$ so $B\notin g(\mathcal{A})$. □

3.4. Corollary. *Let $n=[\frac{1}{2}m]$. There is a constant c such that:*

(a) *If $m=2n$, then*

$$|\mathcal{I}_m|\leqslant 2^{(1+c\log m/m)\binom{m-1}{n-1}}.$$

(b) *If $m=2n+1$, then*

$$|\mathcal{I}_m|\leqslant 2^{(1+c\log m/m)\binom{m-1}{n}}.$$

Proof. By Theorem 3.3, $|\mathcal{I}_m|\leqslant|\mathcal{S}_{m-1}|$ so the theorem of Kleitman and Markowsky cited above applies. □

3.5. Corollary. $\log_2|\mathcal{I}_m|$ *is asymptotically equal to* $\binom{m-1}{n}$ *where* $n=[\frac{1}{2}m]$.

Proof. Theorem 2.3 and Corollary 3.4. □

References

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[3] D. Kleitman and G. Markowsky, On Dedekind's problem: the number of isotone Boolean functions II, Trans. Amer. Math. Soc. 213 (1975) 373–390.