Cospectral Graphs and Digraphs with Given Automorphism Group

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Communicated by W. T. Tutte

Received April 22, 1974

1. Introduction

The characteristic polynomial of a digraph is the characteristic polynomial of the adjacency matrix of the digraph. Two (nonisomorphic) digraphs with the same characteristic polynomial are called cospectral. Mowshowitz [8] and the authors [7] proved the following.

Theorem 1. For each of the following connectedness types and for any positive integer k, there exist k cospectral digraphs.

(i) Weak but not unilateral,
(ii) Unilateral but not strong,
(iii) Strong but not symmetric,
(iv) Symmetric.

We shall refer to the automorphism group of a digraph D simply as the group of the digraph and denote it by \( \Gamma(D) \). A symmetric digraph will be called a graph and denoted by G. The existence of graphs with given group and graph-theoretic properties has been investigated by Frucht [2], Sabidussi [9], Izbicki [6], and others. We are particularly interested in the fact that there are regular graphs of degree \( n \geq 3 \) with given group.

The main aim of this paper is to show that Theorem 1 can be strengthened to assert, for any positive integer \( k \), the existence of \( k \) cospectral digraphs of each of the above types, with given automorphism group. Whereas for the first three types, the construction of these digraphs is based on the patterns in [7, 8] for the symmetric case, it is derived from the discovery of some new graphs with an arbitrarily large number of

* Research supported by the financial assistance of the Council of Scientific and Industrial Research, India.
strictly pseudosimilar points. (Two points $u$ and $v$ of a graph $G$ are said to be strictly pseudosimilar if $G \cup u \cong G \cup v$ but there exists no automorphism of $G$ taking $u$ to $v$. For a discussion of such points and their bearing on Ulam’s conjecture see Harary and Palmer [3].)

2. Cospectral Digraphs with Given Group

2.1. Weakly Connected but Not Unilateral Digraphs

See the digraph $D_i$ of Fig. 1a in which all arrows except the $i$th one are in one direction and the $i$th is in the opposite direction. By taking $i = 1, 2, 3, \ldots, k$, we get $k$ digraphs $D_i$, each with characteristic polynomial $\lambda^{k+1}$.

We shall denote the characteristic polynomial of a digraph $D$ by $\phi_\lambda(D)$ and call it simply the polynomial of $D$, since we will have no occasion to refer to any other polynomial of $D$ in this paper.

Now take any graph $G$ with given group $\Gamma(G) = K$, the existence of which is well known, and obtain digraphs $G_i$ from the $D_i$ by joining $v_{k+1}$ of $D_i$ to every point of $G$. The resulting graph, for the case when $G = K_3$, is shown in Fig. 1b.

We thus get $k$ digraphs $G_i$ which are clearly nonisomorphic and are of connectedness type (i).

Further, since $G$ is symmetric, while $D_i$ is not, no automorphism of $G_i$ can take a point of $G$ to a point of $D_i$ and vice versa. Thus every automorphism of $G_i$ fixes $v_{k+1}$ and hence the whole of $D_i$. The automorphisms of $G_i$ are therefore obtained by concatenation of the automorphisms of $G$ with the identity permutation on the vertices of $D_i$. Thus

$$\Gamma(G_i) \cong \Gamma(G) \cong K.$$
Since the polynomial of a digraph is the product of the polynomials of its strong components, we have

$$\phi_d(G_i) = \phi_d(G) \cdot \lambda^{k+1} = \phi_d(G_j).$$

Thus the $G_i$'s are $k$ nonisomorphic, cospectral digraphs of type (i) with given automorphism group $K$.

It may be observed that the $D_i$'s used here are simpler than those in [8] and serve as well the purpose of Theorem 3 there.

2.2. Unilateral but Not Strong Digraphs

In Fig. 2, the portion $G$ is as above, and the point $v_i$ of the digraph $D$ has been joined to each point of $G$ by a line directed both ways. Here, as elsewhere in this paper, an undirected line represents two lines directed in opposite ways. By shifting the point of attachment of $G$ to $D$, to $v_1, v_2, ..., v_k$, we get $k$ digraphs $G_i$ of type (ii) which are clearly non-isomorphic to each other.

As in Section 2.1, the restriction of $\Gamma(G_i)$ to $D$ is the identity permutation and hence $\Gamma(G_i) \cong K \cong \Gamma(G_j)$. Using standard notation for an induced subgraph, we see that the strong components of $G_i$ are $\langle V(G) \cup v_i \rangle$ and the $(k - 1)$ points $v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_k$. Thus $\phi_d(G_i) = \lambda^{k-1}\phi_d(\langle V(G) \cup v_i \rangle)$. Similarly, $\phi_d(G_j) = \lambda^{k-1}\phi_d(\langle V(G) \cup v_j \rangle)$ and since $\langle V(G) \cup v_i \rangle \cong \langle V(G) \cup v_j \rangle$ we have $\phi_d(G_i) = \phi_d(G_j)$.

Thus the $G_i$'s are $k$ cospectral digraphs of type (ii) with given automorphism group $K$.

2.3. Strong but Not Symmetric Digraphs

The typical digraph $G_i$ for this case is shown in Fig. 3. The other graphs of this series are obtained by shifting the point of attachment of $G$ to $v_2$ through $v_{k+1}$. The other arguments being as above, here we show that the graphs are cospectral.

The following theorem is due to Harary et al. [5, Theorem 2].
THEOREM 2. If $u$ is a pendant vertex (point of degree one) of a graph $G$ and $v$ is the point of $G$ to which it is adjacent, then

$$\phi_{\lambda}(G) = \lambda \cdot \phi_{\lambda}(G - u) - \phi_{\lambda}(G - u - v).$$

The following theorem can be proved in the same way.

THEOREM 3. If $u$ is a point of a digraph $D$ such that $(u, v)$ and $(v, u)$ are the only directed lines containing $u$, then

$$\phi_{\lambda}(D) = \lambda \cdot \phi_{\lambda}(D - u) - \phi_{\lambda}(D - u - v).$$

Applying this theorem to the digraph $G_i$, and taking $v_{k+2}$ as $u$, we get

$$\phi_{\lambda}(G_i) = \lambda \cdot \phi_{\lambda}(G_i - v_{k+2}) - \phi_{\lambda}(G_i - v_{k+2} - v_1),$$

$$\phi_{\lambda}(G_j) = \lambda \cdot \phi_{\lambda}(G_j - v_{k+2}) - \phi_{\lambda}(G_j - v_{k+2} - v_1).$$

Since $G_i - v_{k+2} \cong G_j - v_{k+2}$ and

$$\phi_{\lambda}(G_i - v_{k+2} - v_1) = \lambda^{k-1} \cdot \phi_{\lambda}(G \cup v_{i+1}),$$

$$= \lambda^{k-1} \cdot \phi_{\lambda}(G \cup v_{j+1}),$$

$$= \phi_{\lambda}(G_j - v_{k+2} - v_1),$$

we get $\phi_{\lambda}(G_i) = \phi_{\lambda}(G_j)$.

The case where $G = K_1$ (a single point) requires special mention since only about half the number of nonisomorphic digraphs are obtained in this case, because of similarity of the attachments (two $K_2$'s) at two points of the directed cycle $D$. However, this difficulty can be overcome by increasing the length of the directed cycle or by replacing $K_1$ by any other graph with identity group.
3. **Graphs with Arbitrarily Large Number of Strictly Pseudosimilar Points**

**Definition.** Two points $u$, $v$ of a graph $G$ are said to be pseudosimilar if $G - u \cong G - v$, and strictly pseudosimilar if $G - u \cong G - v$ and $u$ is not similar to $v$.

For the definition of similar points and the existence of pseudosimilar points, refer to [3].

We prove the following.

**Theorem 4.** For any positive integer $k$, there exists a graph $G$ with at least $k$ strictly pseudosimilar points.

**Proof.** We will construct a family of graphs $\{G_m | m = 1, 2, 3, \ldots\}$ such that $G_m$ has exactly $2^m$ strictly pseudosimilar points. Then for any positive integer $k$, the graph $G$ of the theorem is $G_m$ where $m$ is chosen such that $2^{m-1} < k \leq 2^m$.

The graph $G_1$ is shown in Fig. 4. If $v_1$ were to be similar to $v_2$ there should exist an automorphism $\sigma \in \Gamma(G_1)$ such that $\sigma(v_1) = v_2$. But this results in the following chain of implications.

$$\sigma(v_1) = v_2 \Rightarrow \sigma(u_1) = u_2 \Rightarrow \sigma(w_1) = s_1 \Rightarrow \sigma(w_2) = s_2$$

$$\Rightarrow \sigma(u_2) = u_3 \text{ or } t_2 .$$

The last result, however, is untenable since $u_2$, $u_3$, and $t_2$ have different degrees. Thus $v_1$ and $v_2$ are not similar. To show that $G_1 - v_2 \cong G_1 - v_1$ we use the isomorphism $\sigma: (G_1 - v_2) \rightarrow (G_1 - v_1)$ defined by $\sigma(v_1) \rightarrow v_2$, $\sigma(u_1) = u_2$, $\sigma(w_1) = s_1$, etc. We denote this by $R_1$, rotation of order 1 (in the anticlockwise direction) and it is clear that $R_1$ is the required isomorphism. Thus $v_1$ and $v_2$ are strictly pseudosimilar.
To describe the construction of $G_2$ and the higher-order graphs, we employ the following abbreviated notation for the graphs. Figure 5a represents the graph 5b and Fig. 5c represents the graph 5d. In this notation $G_1$ will be represented by the figure in 5c. The arrow in the "triangle" denotes the anticlockwise direction representing the anticlockwise rotation $R_1$ described above. It is apparent that an arrow along one of the edges of the triangle is enough to indicate this sense of rotation.

$G_2$ is obtained by adjoining to the vertices of 5a, in the manner shown in 5c, two copies of $G_1$ and one copy of $G_1 - v_1$. See the representation in Fig. 6. The two copies of $G_1$ in $G_2$ are denoted by $G_{2,1}$ and $G_{2,2}$ and the copy of $G_1 - v_1$ by $G_{2,3}$, with the numbering of these subgraphs in the anticlockwise direction indicated by the central triangle. The pendant vertices in $G_{2,1}$ and $G_{2,2}$ are, respectively, named $v_1$, $v_2$, $v_3$, and $v_4$ (as in $G_1$). The claim is that these four points are strictly pseudosimilar in $G_2$.

To see that $v_1$ is not similar to $v_2$ we observe that the only possible automorphism that can take $v_1$ to $v_2$ is the $R_1$ in $G_{2,1}$ and, as explained for
the case of $G_1$, this is inadequate for the purpose. Similarly, $v_3$ is not similar to $v_4$. Now if there were an automorphism $\sigma$ taking $v_1$ to $v_3$, we should have $\sigma(u_1) = u_3$, $\sigma(c_1) = c_2$, $\sigma(u_2) = u_4$, $\sigma(v_2) = v_3$, etc. This requires that $G_{3,2}$ should go to $G_{2,3}$ under $\sigma$, which is obviously impossible. Thus $v_1$ is not similar to $v_3$, hence not to $v_4$ also.

To see that $G_2 - v_1 \cong G_2 - v_2$, we use the $R_1$ on $G_{3,1}$ and to see that $G_2 - v_3 \cong G_2 - v_4$, we use $R_1$ on $G_{3,2}$. To see that $G_2 - v_1 \cong G_2 - v_3$ we use the $\sigma$ described above, which is adequate for the purpose because both $G_2 - v_1$ and $G_2 - v_3$ are obtained by attaching one copy of $G_1$ and two copies of $G_1 - v_1$ to a central triangle. Since the effect of $\sigma$ is to rotate the "triangular" attachments at $u_1$, $u_2$, $u_3$ around the central triangle and also possibly make rotations about the central points $c_1$, $c_2$, $c_3$ in the graphs $G_{2,i}$, $i = 1, 2, 3$, we call it a second-order rotation and denote it by $R_2$.

The rest of the graphs $G_m$ are obtained by an inductive construction. $G_m$ is obtained by "central attachment" of two copies of $G_{m-1}$ and one copy of $G_{m-1} - v_i$, where $v_i$ is one of the strictly pseudosimilar points of $G_{m-1}$ which we are considering. These subgraphs of $G_m$ are named $G_{m,1}$, $G_{m,2}$, $G_{m,3}$ in the anticlockwise sense indicated by the direction along the central triangle. The strictly pseudosimilar points of $G_m$ are those of $G_{m,1}$ and $G_{m,2}$ and are therefore $2^m$ in number. These are denoted by $v_1 \cdots v_{2^m}$. Rotations up to $m$th order can, obviously, be defined in $G_m$ and, to show that a pair $G_m - v_i$ and $G_m - v_j$ are isomorphic, we use an appropriate combination of these rotations. We can show that $v_i$ is not similar to $v_j$ in $G_m$, since if we have an automorphism $\sigma \in \Gamma(G_m)$ such that $\sigma(v_i) = v_j$, as in the case of $G_2$, $\sigma$ invariably takes some point of degree 2 to some point of degree 3 or 4. This completes the proof of the theorem.

Remarks. (1) Similar lines have been defined [4, p. 171]. Define two lines $e$, $e'$ of $G$ to be pseudosimilar if $G - e \cong G - e'$, and strictly pseudosimilar if they are pseudosimilar but not similar. The lines $(v_i, u_i)$ indicated in the graphs $G_m$ are strictly pseudosimilar. Thus the theorem for pseudosimilar lines corresponding to Theorem 4 is verified by the same set of graphs, $\{G_m\}$.

(2) The automorphism group of each $G_m$ is the identity group.

4. COSPECTRAL GRAPHS WITH GIVEN GROUP

Resuming our discussion of cospectral graphs, we show that for any positive integer $k$, there are $k$ nonisomorphic graphs with given group and the same characteristic polynomial.
Consider the $2^m$ graphs $H_i$ obtained from $G_m$ as follows. Add a point $c_0$ and join it to the vertices of the inner triangle of the central triangle. Take any regular graph $G$ with $\Gamma(G) = K$, the given group, (the existence of $G$ is guaranteed by Sabidussi [9]). Join $c_0$ to every point of $G$. Take another point $u$ and join it to $v_i$. All the new lines are bidirected. By changing $i$ we get the $2^m$ graphs $H_i$. Figure 7 gives $H_1$ derived from $G_2$ where $G = K_3$.

We claim that the graphs $H_i$ are nonisomorphic and cospectral and have automorphism group $K$.

If $H_i \cong H_j$ for $i \neq j$, this isomorphism, say $\sigma$, should take $v_i$ to $v_j$. But then the restriction of $\sigma$ to $H_i - u - c_0 - G$ (which is same as $H_j - u - c_0 - G = G_m$) should be an automorphism of $G_m$, which contradicts the fact that $v_i$ and $v_j$ are not similar points in $G_m$. Thus the $H_i$’s are nonisomorphic.

Since $G$ is regular, there cannot be an automorphism of $H_i$ taking a point of $G$ to any point of $H_i - G$. Also, every automorphism of $H_i$ fixes $c_0$ and through that every point of $G_m \cup u$. Thus the automorphism group of $H_i$ is isomorphic to $K$.

To compute the polynomial of $H_i$, we use Theorem 2. Taking $u$ for $u$ and $v_i$ for $v$, we get

$$\phi_\lambda(H_i) = \phi_\lambda(H_i - (u, v_i)) - \phi_\lambda(H_i - u - v_i)$$
$$= \lambda \cdot \phi_\lambda(H_i - u) - \phi_\lambda(H_i - u - v_i).$$
But \( H_i - u = G_m \cup c_0 \cup G = H_j - u \) and

\[
H_i - u - v_i = (G_m - v_i) \cup c_0 \cup G \\
\cong (G_m - v_j) \cup c_0 \cup G = H_j - u - v_j,
\]

since \( v_i \) and \( v_j \) are pseudosimilar. This implies \( \phi(H_i) = \phi(H_j) \). That is, the \( H_i \)'s are cospectral.

For a given integer \( k \), we choose \( m \) such that \( 2^{m-1} < k \leq 2^m \). Combining the results of Sections 2 and 4 we have

**Theorem 5.** For any positive integer \( k \) and for any of the following types of connectedness there are \( k \) cospectral digraphs with given automorphism group.

(i) Weak but not unilateral,
(ii) Unilateral but not strong,
(iii) Strong but not symmetric,
(iv) Symmetric.

**Remark.** The construction in Section 4 is not unique. Consider \( G \) and \( G_m \) as above. Join each point of \( G \) to \( v_i \) with bidirected lines. This gives \( H_i, i = 1, 2, \ldots, 2^m \). These \( H_i \) also will serve our purpose.

5. Conclusion

The spectral multiplicity of a graph \( G \) may be defined as the number of nonisomorphic graphs which have the same characteristic polynomial as \( G \). Theorems 1 and 5 only assert the existence of graphs with large spectral multiplicities. Existence of graphs with specified spectral multiplicity and the enumeration of graphs in terms of spectral multiplicity appear to be difficult unsolved problems.

![Figure 8](image-url)
It may be observed that the graphs in Fig. 8 provide a simpler counterexample to a conjecture of Bondy relating to Ulam's conjecture (see [1, p. 287]). The graphs of Fig. 9 give another counterexample with six pendant vertices.

**REFERENCES**