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Cospectral Graphs and Digraphs with Given Automorphism Group

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1. INTRODUCTION

The characteristic polynomial of a digraph is the characteristic polynomial of the adjacency matrix of the digraph. Two (nonisomorphic) digraphs with the same characteristic polynomial are called cospectral. Mowshowitz [8] and the authors [7] proved the following.

THEOREM 1. *For each of the following connectedness types and for any positive integer k, there exist k cospectral digraphs.*

(i) *Weak but not unilateral,*

(ii) *Unilateral but not strong,*

(iii) *Strong but not symmetric,*

(iv) *Symmetric.*

We shall refer to the automorphism group of a digraph D simply as the group of the digraph and denote it by $\Gamma(D)$. A symmetric digraph will be called a graph and denoted by G. The existence of graphs with given group and graph-theoretic properties has been investigated by Frucht [2], Sabidussi [9], Izbicki [6], and others. We are particularly interested in the fact that there are regular graphs of degree n ($\geqslant 3$) with given group.

The main aim of this paper is to show that Theorem 1 can be strengthened to assert, for any positive integer k, the existence of k cospectral digraphs of each of the above types, with given automorphism group. Whereas for the first three types, the construction of these digraphs is based on the patterns in [7, 8] for the symmetric case, it is derived from the discovery of some new graphs with an arbitrarily large number of

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strictly pseudosimilar points. (Two points u and v of a graph G are said to be strictly pseudosimilar if $G - u \simeq G - v$ but there exists no automorphism of G taking u to v. For a discussion of such points and their bearing on Ulam's conjecture see Harary and Palmer [3].)

2. Cospectral Digraphs with Given Group

2.1. *Weakly Connected but Not Unilateral Digraphs*

See the digraph D_i of Fig. 1a in which all arrows except the ith one are in one direction and the ith is in the opposite direction. By taking $i = 1, 2, 3,..., k$, we get k digraphs D_i, each with characteristic polynomial λ^{k+1}.

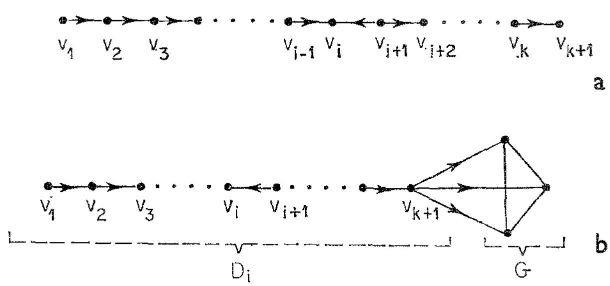

Figure 1

We shall denote the characteristic polynomial of a digraph D by $\phi_\lambda(D)$ and call it simply the polynomial of D, since we will have no occasion to refer to any other polynomial of D in this paper.

Now take any graph G with given group $\Gamma(G) = K$, the existence of which is well known, and obtain digraphs G_i from the D_i by joining v_{k+1} of D_i to every point of G. The resulting graph, for the case when $G = K_3$, is shown in Fig. 1b.

We thus get k digraphs G_i which are clearly nonisomorphic and are of connectedness type (i).

Further, since G is symmetric, while D_i is not, no automorphism of G_i can take a point of G to a point of D_i and vice versa. Thus every automorphism of G_i fixes v_{k+1} and hence the whole of D_i. The automorphisms of G_i are therefore obtained by concatenation of the automorphisms of G with the identity permutation on the vertices of D_i. Thus

$$\Gamma(G_i) \simeq \Gamma(G) \simeq K.$$

Since the polynomial of a digraph is the product of the polynomials of its strong components, we have

$$\phi_\lambda(G_i) = \phi_\lambda(G) \cdot \lambda^{k+1} = \phi_\lambda(G_j).$$

Thus the G_i's are k nonisomorphic, cospectral digraphs of type (i) with given automorphism group K.

It may be observed that the D_i's used here are simpler than those in [8] and serve as well the purpose of Theorem 3 there.

2.2. *Unilateral but Not Strong Digraphs*

In Fig. 2, the portion G is as above, and the point v_i of the digraph D has been joined to each point of G by a line directed both ways. Here, as elsewhere in this paper, an undirected line represents two lines directed

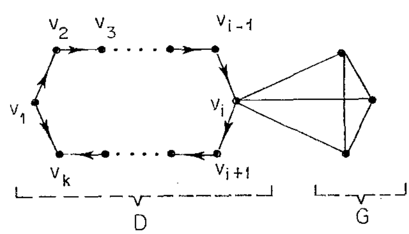

FIGURE 2

in opposite ways. By shifting the point of attachment of G to D, to $v_1, v_2, \ldots, v_k$, we get k digraphs G_i of type (ii) which are clearly non-isomorphic to each other.

As in Section 2.1, the restriction of $\Gamma(G_i)$ to D is the identity permutation and hence $\Gamma(G_i) \cong K \cong \Gamma(G_j)$. Using standard notation for an induced subgraph, we see that the strong components of G_i are $\langle V(G) \cup v_i \rangle$ and the $(k-1)$ points $v_1, v_2, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k$. Thus $\phi_\lambda(G_i) = \lambda^{k-1}\phi_\lambda(\langle V(G) \cup v_i \rangle)$. Similarly, $\phi_\lambda(G_j) = \lambda^{k-1}\phi_\lambda(\langle V(G) \cup v_j \rangle)$ and since $\langle V(G) \cup v_i \rangle \cong \langle V(G) \cup v_j \rangle$ we have $\phi_\lambda(G_i) = \phi_\lambda(G_j)$.

Thus the G_i's are k cospectral digraphs of type (ii) with given automorphism group K.

2.3. *Strong but Not Symmetric Digraphs*

The typical digraph G_i for this case is shown in Fig. 3. The other graphs of this series are obtained by shifting the point of attachment of G to v_2 through v_{k+1}. The other arguments being as above, here we show that the graphs are cospectral.

The following theorem is due to Harary *et al.* [5, Theorem 2].

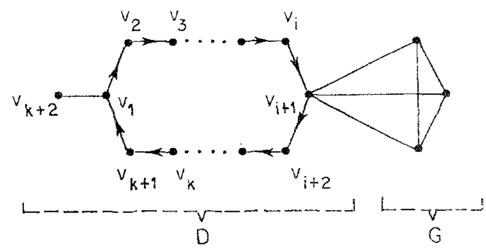

FIGURE 3

THEOREM 2. *If u is a pendant vertex (point of degree one) of a graph G and v is the point of G to which it is adjacent, then*

$$\phi_\lambda(G) = \lambda \cdot \phi_\lambda(G - u) - \phi_\lambda(G - u - v).$$

The following theorem can be proved in the same way.

THEOREM 3. *If u is a point of a digraph D such that* (u, v) *and* (v, u) *are the only directed lines containing u, then*

$$\phi_\lambda(D) = \lambda \cdot \phi_\lambda(D - u) - \phi_\lambda(D - u - v).$$

Applying this theorem to the digraph G_i, and taking v_{k+2} as u, we get

$$\phi_\lambda(G_i) = \lambda \cdot \phi_\lambda(G_i - v_{k+2}) - \phi_\lambda(G_i - v_{k+2} - v_1),$$
$$\phi_\lambda(G_j) = \lambda \cdot \phi_\lambda(G_j - v_{k+2}) - \phi_\lambda(G_j - v_{k+2} - v_1).$$

Since $G_i - v_{k+2} \cong G_j - v_{k+2}$ and

$$\begin{aligned}\phi_\lambda(G_i - v_{k+2} - v_1) &= \lambda^{k-1} \cdot \phi_\lambda(\langle G \cup v_{i+1}\rangle)\\ &= \lambda^{k-1} \cdot \phi_\lambda(\langle G \cup v_{j+1}\rangle)\\ &= \phi_\lambda(G_j - v_{k+2} - v_1),\end{aligned}$$

we get $\phi_\lambda(G_i) = \phi_\lambda(G_j)$.

The case where $G = K_1$ (a single point) requires special mention since only about half the number of nonisomorphic digraphs are obtained in this case, because of similarity of the attachments (two K_2's) at two points of the directed cycle D. However, this difficulty can be overcome by increasing the length of the directed cycle or by replacing K_1 by any other graph with identity group.

3. Graphs with Arbitrarily Large Number of Strictly Pseudosimilar Points

Definition. Two points u, v of a graph G are said to be pseudosimilar if $G - u \cong G - v$, and strictly pseudosimilar if $G - u \cong G - v$ and u is not similar to v.

For the definition of similar points and the existence of pseudosimilar points, refer to [3].

We prove the following.

Theorem 4. *For any positive integer k, there exists a graph G with at least k strictly pseudosimilar points.*

Proof. We will construct a family of graphs $\{G_m/m = 1, 2, 3,...\}$ such that G_m has exactly 2^m strictly pseudosimilar points. Then for any positive integer k, the graph G of the theorem is G_m where m is chosen such that $2^{m-1} < k \leqslant 2^m$.

The graph G_1 is shown in Fig. 4. If v_1 were to be similar to v_2 there should

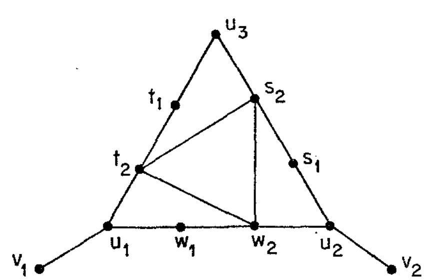

Figure 4

exist an automorphism $\sigma \in \Gamma(G_1)$ such that $\sigma(v_1) = v_2$. But this results in the following chain of implications.

$$\sigma(v_1) = v_2 \Rightarrow \sigma(u_1) = u_2 \Rightarrow \sigma(w_1) = s_1 \Rightarrow \sigma(w_2) = s_2$$
$$\Rightarrow \sigma(u_2) = u_3 \text{ or } t_2 .$$

The last result, however, is untenable since u_2, u_3, and t_2 have different degrees. Thus v_1 and v_2 are not similar. To show that $G_1 - v_2 \cong G_1 - v_1$ we use the isomorphism $\sigma\colon (G_1 - v_2) \to (G_1 - v_1)$ defined by $\sigma(v_1) = v_2$, $\sigma(u_1) = u_2$, $\sigma(w_1) = s_1$, etc. We denote this by R_1, rotation of order 1 (in the anticlockwise direction) and it is clear that R_1 is the required isomorphism. Thus v_1 and v_2 are strictly pseudosimilar.

To describe the construction of G_2 and the higher-order graphs, we employ the following abbreviated notation for the graphs. Figure 5a represents the graph 5b and Fig. 5c represents the graph 5d. In this notation G_1 will be represented by the figure in 5e. The arrow in the "triangle" denotes the anticlockwise direction representing the anticlockwise rotation R_1 described above. It is apparent that an arrow along one of the edges of the triangle is enough to indicate this sense of rotation.

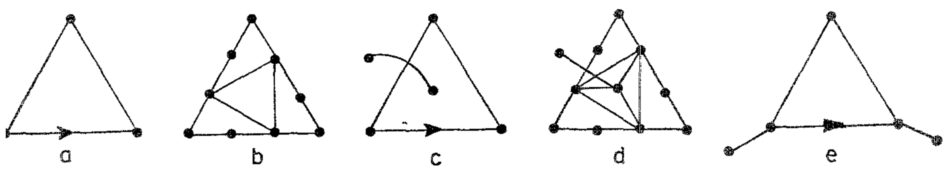

FIGURE 5

G_2 is obtained by adjoining to the vertices of 5a, in the manner shown in 5c, two copies of G_1 and one copy of $G_1 - v_1$. See the representation in Fig. 6. The two copies of G_1 in G_2 are denoted by $G_{2,1}$ and $G_{2,2}$ and the copy of $G_1 - v_1$ by $G_{2,3}$, with the numbering of these subgraphs in the anticlockwise direction indicated by the central triangle. The pendant vertices in $G_{2,1}$ and $G_{2,2}$ are, respectively, named v_1, v_2, v_3, and v_4 (as in G_1). The claim is that these four points are strictly pseudosimilar in G_2.

To see that v_1 is not similar to v_2 we observe that the only possible automorphism that can take v_1 to v_2 is the R_1 in $G_{2,1}$ and, as explained for

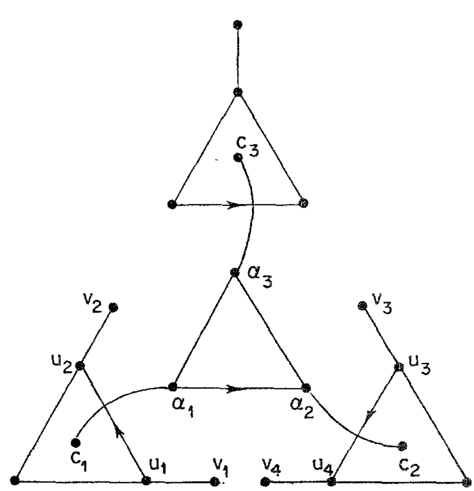

FIGURE 6

the case of G_1, this is inadequate for the purpose. Similarly, v_3 is not similar to v_4. Now if there were an automorphism σ taking v_1 to v_3, we should have $\sigma(u_1) = u_3$, $\sigma(c_1) = c_2$, $\sigma(\alpha_1) = \alpha_2$, $\sigma(\alpha_2) = \alpha_3$, etc. This requires that $G_{2,2}$ should go to $G_{2,3}$ under σ, which is obviously impossible. Thus v_1 is not similar to v_3, hence not to v_4 also.

To see that $G_2 - v_1 \cong G_2 - v_2$, we use the R_1 on $G_{2,1}$ and to see that $G_2 - v_3 \cong G_2 - v_4$, we use R_1 on $G_{2,2}$. To see that $G_2 - v_1 \cong G_2 - v_3$ we use the σ described above, which is adequate for the purpose because both $G_2 - v_1$ and $G_2 - v_3$ are obtained by attaching one copy of G_1 and two copies of $G_1 - v_1$ to a central triangle. Since the effect of σ is to rotate the "triangular" attachments at α_1, α_2, α_3 around the central triangle and also possibly make rotations about the central points c_1, c_2, c_3 in the graphs $G_{2,i}$, $i = 1, 2, 3$, we call it a second-order rotation and denote it by R_2.

The rest of the graphs G_m are obtained by an inductive construction. G_m is obtained by "central attachment" of two copies of G_{m-1} and one copy of $G_{m-1} - v_i$, where v_i is one of the strictly pseudosimilar points of G_{m-1} which we are considering. These subgraphs of G_m are named $G_{m,1}$, $G_{m,2}$, $G_{m,3}$ in the anticlockwise sense indicated by the direction along the central triangle. The strictly pseudosimilar points of G_m are those of $G_{m,1}$ and $G_{m,2}$ and are therefore 2^m in number. These are denoted by $v_1 \cdots v_{2m}$. Rotations up to mth order can, obviously, be defined in G_m and, to show that a pair $G_m - v_i$ and $G_m - v_j$ are isomorphic, we use an appropriate combination of these rotations. We can show that v_i is not similar to v_j in G_m, since if we have an automorphism $\sigma \in \Gamma(G_m)$ such that $\sigma(v_i) = v_j$, as in the case of G_2, σ invariably takes some point of degree 2 to some point of degree 3 or 4. This completes the proof of the theorem.

Remarks. (1) Similar lines have been defined [4, p. 171]. Define two lines e, e' of G to be pseudosimilar if $G - e \cong G - e'$, and strictly pseudosimilar if they are pseudosimilar but not similar. The lines (v_i, u_i) indicated in the graphs G_m are strictly pseudosimilar. Thus the theorem for pseudosimilar lines corresponding to Theorem 4 is verified by the same set of graphs, $\{G_m\}$.

(2) The automorphism group of each G_m is the identity group.

4. Cospectral Graphs with Given Group

Resuming our discussion of cospectral graphs, we show that for any positive integer k, there are k nonisomorphic graphs with given group and the same characteristic polynomial.

Consider the 2^m graphs H_i obtained from G_m as follows. Add a point c_0 and join it to the vertices of the inner triangle of the central triangle. Take any regular graph G with $\Gamma(G) = K$, the given group, (the existence of G is guaranteed by Sabidussi [9]). Join c_0 to every point of G. Take another point u and join it to v_i. All the new lines are bidirected. By changing i we get the 2^m graphs H_i. Figure 7 gives H_1 derived from G_2 where $G = K_3$.

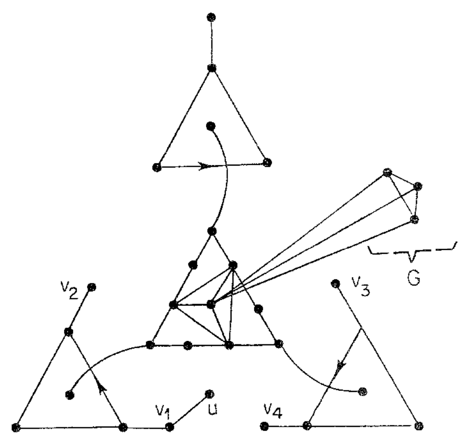

FIGURE 7

We claim that the graphs H_i are nonisomorphic and cospectral and have automorphism group K.

If $H_i \cong H_j$ for $i \neq j$, this isomorphism, say σ, should take v_i to v_j. But then the restriction of σ to $H_i - u - c_0 - G$ (which is same as $H_j - u - c_0 - G = G_m$) should be an automorphism of G_m, which contradicts the fact that v_i and v_j are not similar points in G_m. Thus the H_i's are nonisomorphic.

Since G is regular, there cannot be an automorphism of H_i taking a point of G to any point of $H_i - G$. Also, every automorphism of H_i fixes c_0 and through that every point of $G_m \cup u$. Thus the automorphism group of H_i is isomorphic to K.

To compute the polynomial of H_i we use Theorem 2. Taking u for u and v_i for v, we get

$$\begin{aligned} \phi_\lambda(H_i) &= \phi_\lambda(H_i - (u, v_i)) - \phi_\lambda(H_i - u - v_i) \\ &= \lambda \cdot \phi_\lambda(H_i - u) - \phi_\lambda(H_i - u - v_i). \end{aligned}$$

But $H_i - u = G_m \cup c_0 \cup G = H_j - u$ and

$$\begin{aligned} H_i - u - v_i &= (G_m - v_i) \cup c_0 \cup G \\ &\cong (G_m - v_j) \cup c_0 \cup G = H_j - u - v_j, \end{aligned}$$

since v_i and v_j are pseudosimilar. This implies $\phi_\lambda(H_i) = \phi_\lambda(H_j)$. That is, the H_i's are cospectral.

For a given integer k, we choose m such that $2^{m-1} < k \leqslant 2^m$. Combining the results of Sections 2 and 4 we have

THEOREM 5. *For any positive integer k and for any of the following types of connectedness there are k cospectral digraphs with given automorphism group.*

(i) *Weak but not unilateral,*

(ii) *Unilateral but not strong,*

(iii) *Strong but not symmetric,*

(iv) *Symmetric.*

Remark. The construction in Section 4 is not unique. Consider G and G_m as above. Join each point of G to v_i with bidirected lines. This gives H_i, $i = 1, 2, \ldots, 2^m$. These H_i also will serve our purpose.

5. CONCLUSION

The spectral multiplicity of a graph G may be defined as the number of nonisomorphic graphs which have the same characteristic polynomial as G. Theorems 1 and 5 only assert the existence of graphs with large spectral multiplicities. Existence of graphs with specified spectral multiplicity and the enumeration of graphs in terms of spectral multiplicity appear to be difficult unsolved problems.

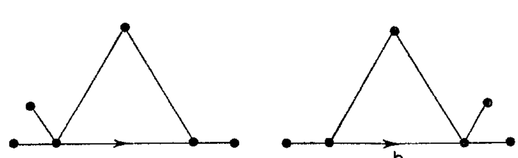

FIGURE 8

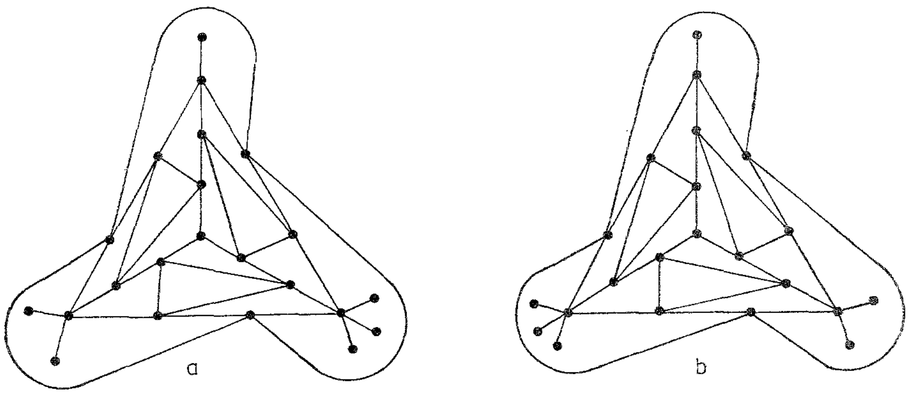

FIGURE 9

It may be observed that the graphs in Fig. 8 provide a simpler counterexample to a conjecture of Bondy relating to Ulam's conjecture (see [1, p. 287]). The graphs of Fig. 9 give another counterexample with six pendant vertices.

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