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# The Brown–Colbourn conjecture on zeros of reliability polynomials is false

Gordon Royle[a] and Alan D. Sokal[b]

[a] Department of Computer Science & Software Engineering, University of Western Australia, 35 Stirling Highway, Crawley, WA 6009, Australia
[b] Department of Physics, New York University, 4 Washington Place, New York, NY 10003, USA

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**Abstract**

We give counterexamples to the Brown–Colbourn conjecture on reliability polynomials, in both its univariate and multivariate forms. The multivariate Brown–Colbourn conjecture is false already for the complete graph $K_4$. The univariate Brown–Colbourn conjecture is false for certain simple planar graphs obtained from $K_4$ by parallel and series expansion of edges. We show, in fact, that a graph has the multivariate Brown–Colbourn property if and only if it is series–parallel.


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## 1. Introduction

Let us consider a connected (multi)graph[1] $G = (V, E)$ as a communications network with unreliable communication channels, in which edge $e$ is operational with probability $p_e$ and failed with probability $1 - p_e$, independently for each edge.

*E-mail addresses:* gordon@csse.uwa.edu.au (G. Royle), sokal@nyu.edu (A.D. Sokal).

[1] Henceforth we omit the prefix "multi". In this paper a "graph" is allowed to have loops and/or multiple edges unless explicitly stated otherwise.

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Let $R_G(\mathbf{p})$ be the probability that every node is capable of communicating with every other node (this is the so-called *all-terminal reliability*). Clearly we have

$$R_G(\mathbf{p}) = \sum_{\substack{A \subseteq E \\ (V,A) \text{ connected}}} \prod_{e \in A} p_e \prod_{e \in E \setminus A} (1 - p_e), \tag{1.1}$$

where the sum runs over all connected spanning subgraphs of $G$, and we have written $\mathbf{p} = \{p_e\}_{e \in E}$. We call $R_G(\mathbf{p})$ the (multivariate) *reliability polynomial* [7] for the graph $G$; it is a multiaffine polynomial, i.e. of degree at most 1 in each variable separately. If the edge probabilities $p_e$ are all set to the same value $p$, we write the corresponding univariate polynomial as $R_G(p)$, and call it the univariate reliability polynomial. We are interested in studying the zeros of these polynomials when the variables $p_e$ (or $p$) are taken to be *complex* numbers.

Brown and Colbourn [5] studied a number of examples and made the following conjecture:

*Univariate Brown–Colbourn conjecture.* For any graph $G$, the zeros of the univariate reliability polynomial $R_G(p)$ all lie in the closed disc $|p-1| \leqslant 1$. In other words, if $|p-1| > 1$, then $R_G(p) \neq 0$.

Subsequently, one of us [16] proposed a multivariate extension of the Brown–Colbourn conjecture:

*Multivariate Brown–Colbourn conjecture.* For any graph $G$, if $|p_e - 1| > 1$ for all edges $e$, then $R_G(\mathbf{p}) \neq 0$.

Not long ago, Wagner [18] proved, using an ingenious and complicated construction, that the univariate Brown–Colbourn conjecture holds for all series–parallel graphs.[2] Subsequently, one of us [16, Remark 3 in Section 4.1] showed, by a two-line induction, that the multivariate Brown–Colbourn conjecture holds for all series–parallel graphs.[3] Both the univariate and multivariate conjectures remained open for general graphs, but most workers in the field suspected that they would be true. (At least the present authors did.)

In this short note we would like to report that both the univariate and multivariate Brown–Colbourn conjectures are false! The multivariate conjecture is false already for the simplest non series–parallel graph, namely the complete graph $K_4$. As a corollary we will deduce that the univariate conjecture is false for a 4-vertex, 16-edge planar graph that can be obtained from $K_4$ by adding parallel edges, and for a 1512-vertex, 3016-edge simple planar graph that can be obtained from $K_4$ by adding

[2] Unfortunately, there seems to be no completely standard definition of "series–parallel graph"; a plethora of slightly different definitions can be found in the literature [9,7,13,14,4]. So let us be completely precise about our own usage: we shall call a loopless graph *series–parallel* if it can be obtained from a forest by a finite sequence of series and parallel extensions of edges (i.e. replacing an edge by two edges in series or two edges in parallel). We shall call a general graph (allowing loops) series–parallel if its underlying loopless graph is series–parallel. Some authors write "obtained from a tree", "obtained from $K_2$" or "obtained from $C_2$" in place of "obtained from a forest"; in our terminology these definitions yield, respectively, all *connected* series–parallel graphs, all connected series–parallel graphs whose blocks form a path, or all 2-*connected* series–parallel graphs. See [4, Section 11.2] for a more extensive bibliography.

[3] This proof is reproduced here as Theorem 5.6(c) ⇒ (a).

parallel edges and then subdividing edges. So the Brown–Colbourn conjecture is not true even for simple planar graphs.

Furthermore, for the multivariate property we are able to obtain a complete characterization: a graph has the multivariate Brown–Colbourn property *if and only if* it is series–parallel.

It is convenient to restate the Brown–Colbourn conjectures in terms of the generating polynomial for connected spanning subgraphs,

$$C_G(\mathbf{v}) = \sum_{\substack{A \subseteq E \\ (V,A) \text{ connected}}} \prod_{e \in A} v_e, \tag{1.2}$$

where we have written $\mathbf{v} = \{v_e\}_{e \in E}$. This is clearly related to the reliability polynomial by

$$R_G(\mathbf{p}) = \left[\prod_{e \in E} (1 - p_e)\right] C_G\left(\frac{\mathbf{p}}{\mathbf{1} - \mathbf{p}}\right), \tag{1.3}$$

$$C_G(\mathbf{v}) = \left[\prod_{e \in E} (1 + v_e)\right] R_G\left(\frac{\mathbf{v}}{\mathbf{1} + \mathbf{v}}\right), \tag{1.4}$$

where $\mathbf{1}$ denotes the vector with all entries 1, and division of vectors is understood componentwise. The multivariate Brown–Colbourn conjecture then states that if $G$ is a *loopless* graph and $|1 + v_e| < 1$ for all edges $e$, then $C_G(\mathbf{v}) \neq 0$. Loops must be excluded because a loop $e$ multiplies $C_G$ by a factor $1 + v_e$ but leaves $R_G$ unaffected. Some workers also prefer to use the failure probabilities $q_e = 1 - p_e$ as the variables.

The plan of this paper is as follows: In Section 2, we show that the multivariate Brown–Colbourn conjecture fails for the complete graph $K_4$. In Section 3, we review the series and parallel reduction formulae for the reliability polynomial. In Section 4, we show that the univariate Brown–Colbourn conjecture fails for certain graphs that are obtained from $K_4$ by adding parallel edges and then optionally subdividing edges. In Section 5, we complete these results by showing that a graph has the multivariate Brown–Colbourn property if and only if it is series–parallel.

## 2. The multivariate Brown–Colbourn conjecture is false for $K_4$

For the complete graph $K_4$, the univariate polynomial $C_G(v)$ is

$$C_{K_4}(v) = 16v^3 + 15v^4 + 6v^5 + v^6. \tag{2.1}$$

The roots of this polynomial all lie outside the disc $|1 + v| < 1$, so the univariate Brown–Colbourn conjecture is true for $K_4$.

Let us now consider the bivariate situation, in which the six edges receive two different weights $a$ and $b$. There are five cases:

(a) One edge receives weight $a$ and the other five receive weight $b$:

$$C_{K_4}(a, b) = (8b^3 + 5b^4 + b^5) + (8b^2 + 10b^3 + 5b^4 + b^5)a. \tag{2.2}$$

(b) A pair of nonintersecting edges receive weight $a$ and the other four edges receive weight $b$:

$$C_{K_4}(a,b) = (4b^3 + b^4) + (8b^2 + 8b^3 + 2b^4)a + (4b + 6b^2 + 4b^3 + b^4)a^2. \tag{2.3}$$

(c) A pair of intersecting edges receive weight $a$ and the other four edges receive weight $b$:

$$C_{K_4}(a,b) = (3b^3 + b^4) + (10b^2 + 8b^3 + 2b^4)a + (3b + 6b^2 + 4b^3 + b^4)a^2. \tag{2.4}$$

(d) A 3-star receives weight $a$ and the complementary triangle receives weight $b$:

$$C_{K_4}(a,b) = (9b^2 + 3b^3)a + (6b + 9b^2 + 3b^3)a^2 + (1 + 3b + 3b^2 + b^3)a^3. \tag{2.5}$$

(e) A three-edge path receives weight $a$ and the complementary three-edge path receives weight $b$:

$$C_{K_4}(a,b) = b^3 + (7b^2 + 3b^3)a + (7b + 9b^2 + 3b^3)a^2 + (1 + 3b + 3b^2 + b^3)a^3. \tag{2.6}$$

We have plotted the roots $a$ when $b$ traces out the circle $|1+b| = 1$, and vice versa. In cases (b) and (d) it turns out that the roots can enter the "forbidden discs" $|1+a|<1$ and $|1+b|<1$. This is shown in Fig. 1 for case (b); blow-ups of the crucial regions are shown in Fig. 2 both for case (b) and for case (d). As a result, counterexamples to the multivariate Brown–Colbourn conjecture can be obtained in these two cases: indeed, for any $a$ lying in the region $A_+$ (resp. $A_-$), there exists $b \in B_-$ (resp. $B_+$) such that $C_{K_4}(a,b) = 0$, and conversely.

Let us note for future reference that the endpoint of the region $A_\pm$ (resp. $B_\pm$) lies at $a = -1 + e^{\pm 2\pi i\alpha}$ (resp. $b = -1 + e^{\pm 2\pi i\beta}$), where $\alpha \approx 0.120692$ and $\beta \approx 0.164868$ in case (b), and $\alpha \approx 0.110198$ and $\beta \approx 0.030469$ in case (d).

We can understand this behavior analytically as follows: For each of the five cases, let us solve the equation $C_{K_4}(a,b) = 0$ for $a$ in terms of $b$, expanding in power series for $b$ near 0. We obtain:

(a) $a = -b + \frac{5}{8}b^2 + O(b^3)$,
(b) $a = -b \pm \frac{1}{2}b^{3/2} + O(b^2)$,
(c) $a = -\frac{1}{3}b + \frac{1}{8}b^2 + O(b^3)$ and $a = -3b + \frac{31}{8}b^2 + O(b^3)$,
(d) $a = -3b \pm i\sqrt{3}b^{3/2} + O(b^2)$ and $a = 0$,
(e) $a = -b + \frac{3}{4}b^2 + O(b^3)$ and $a = (-3 \pm 2\sqrt{2})b + \frac{9}{16}(10 \mp 7\sqrt{2})b^2 + O(b^3)$.

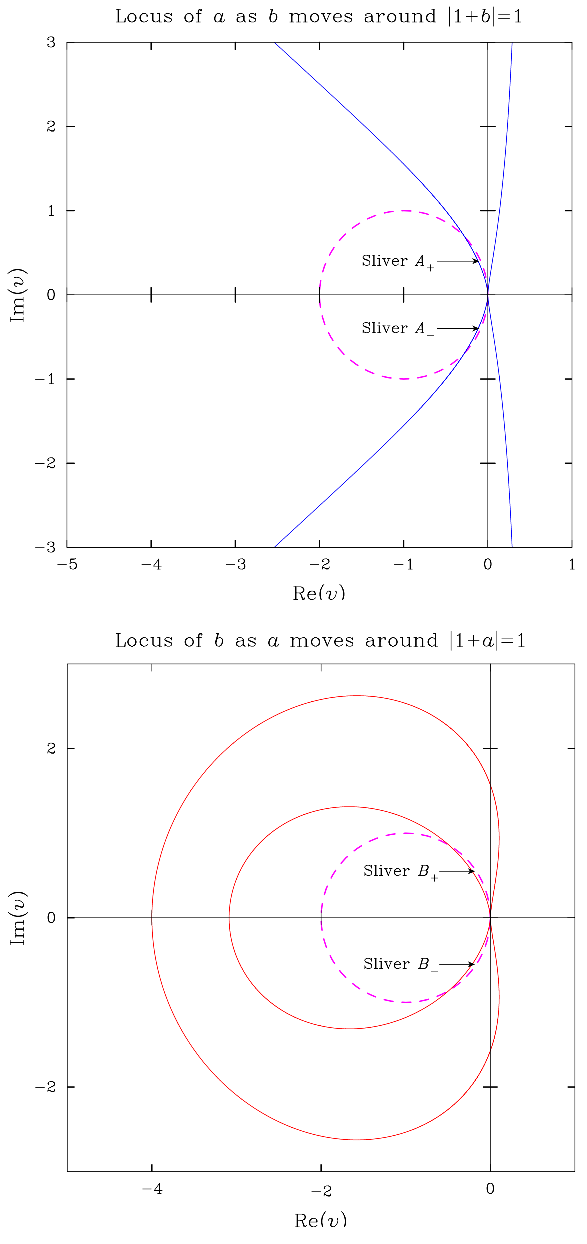


Fig. 1. Curves for case (b). First plot shows the $a$-plane; second plot shows the $b$-plane. Dashed magenta curve is the circle $|1+v|=1$; solid blue curve is the locus of root $a$; solid red curve is the locus of root $b$.

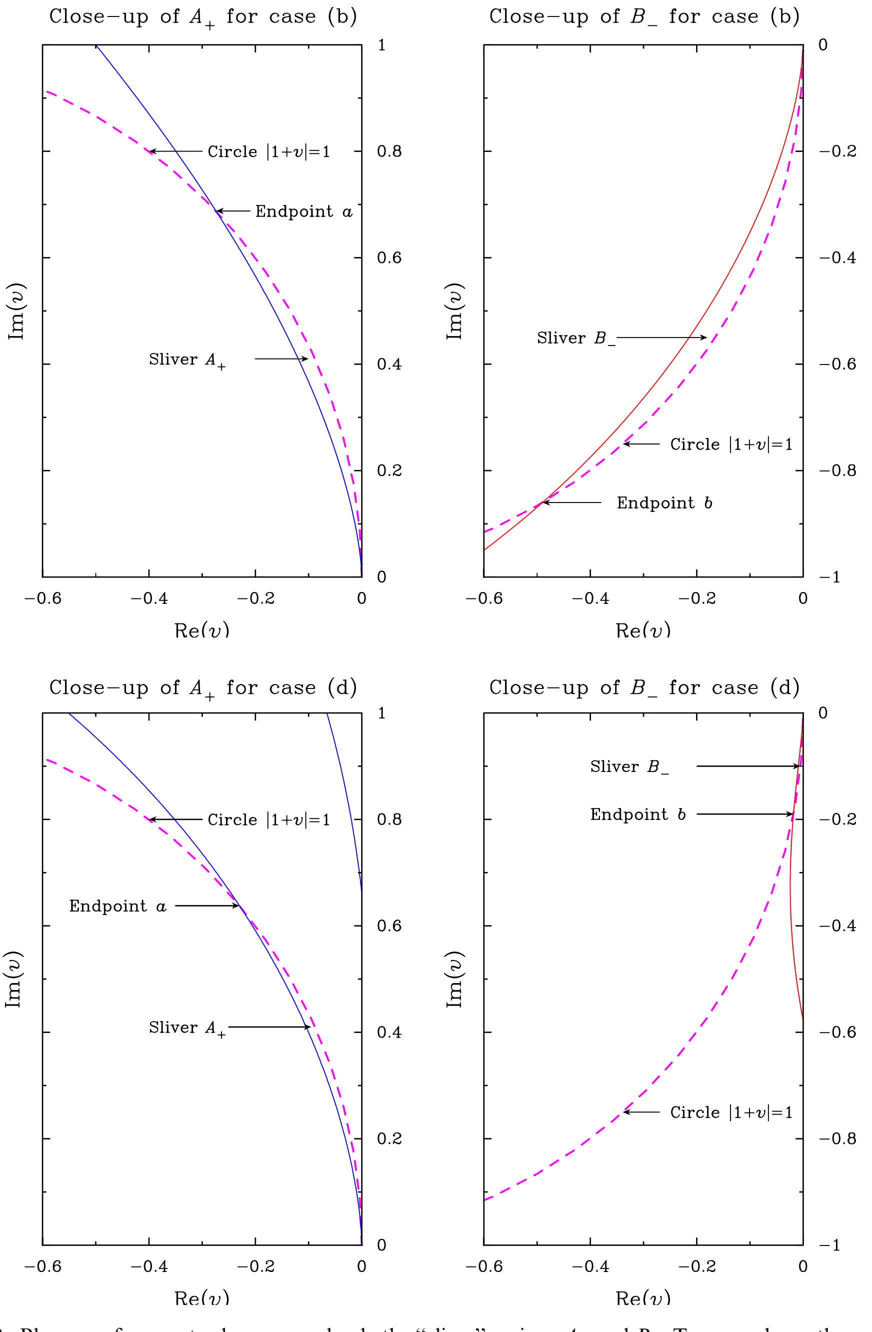


Fig. 2. Blow-up of curves to show more clearly the "sliver" regions $A_+$ and $B_-$. Top row shows the $a$- and $b$-planes for case (b); bottom row shows the $a$- and $b$-planes for case (d).

The behavior is thus different in cases (a,c,e) on the one hand, and cases (b,d) on the other:

*Cases* (*a,c,e*): Here the solution is of the form

$$a = \gamma_1 b + \gamma_2 b^2 + O(b^3) \tag{2.7}$$

with $\gamma_1, \gamma_2$ real. Therefore, if we set $b = -1 + e^{i\theta}$ and expand in powers of $\theta$, we obtain

$$|1 + a|^2 = 1 + (\gamma_1^2 - \gamma_1 - 2\gamma_2)\theta^2 + O(\theta^4). \tag{2.8}$$

Provided that $\gamma_1^2 - \gamma_1 - 2\gamma_2 > 0$—as indeed holds for all the roots in cases (a,c,e)—we have $|1 + a| \geqslant 1$ for small $\theta$, so no counterexample is found (at least for small $\theta$).

*Cases* (*b,d*): Here, by contrast, the solution is of the form

$$a = \delta_1 b + \delta_2 b^{3/2} + O(b^2), \tag{2.9}$$

with $\delta_1 < 0$ and $\delta_2 \neq 0$. Therefore, if we set $b = -1 + e^{i\theta}$ and expand as before, we obtain

$$a = i\delta_1\theta + e^{\pm 3\pi i/4}\delta_2\theta^{3/2} + O(\theta^2). \tag{2.10}$$

Since $\mathrm{Re}(e^{\pm 3\pi i/4}\delta_2) < 0$ for at least one of the roots, we have $\mathrm{Re}\, a \propto -|\mathrm{Im}\, a|^{3/2}$ for small $\theta$; in particular, we have $|1 + a| < 1$ for small $\theta \neq 0$.

In fact, more can be said: suppose that we fix *any* $\lambda > 0$ and set $b = \lambda(-1 + e^{i\theta})$. Then we have

$$a = i\delta_1\lambda\theta + e^{\pm 3\pi i/4}\delta_2\lambda^{3/2}\theta^{3/2} + O(\theta^2), \tag{2.11}$$

so that once again $\mathrm{Re}\, a \propto -|\mathrm{Im}\, a|^{3/2}$ for small $\theta$. In particular, we will have $|\lambda + a| < \lambda$ for small $\theta \neq 0$, irrespective of how small $\lambda$ was chosen. This observation will play a crucial role in Section 5 (see Proposition 5.5).

## 3. Series and parallel reduction formulae

Suppose that $G$ contains edges $e_1, \ldots, e_n$ (with corresponding weights $v_1, \ldots, v_n$) in parallel between the same pair of vertices $x, y$. Then it is easy to see that the edges $e_1, \ldots, e_n$ can be replaced by a single edge of weight

$$v_1 || v_2 || \cdots || v_n \equiv \prod_{i=1}^{n} (1 + v_i) - 1 \tag{3.1}$$

without changing the value of $C_G(\mathbf{v})$. (Reason: $x$ is connected to $y$ via this "super-edge" if and only if $x$ is connected to $y$ by at least one of the edges $e_1, \ldots, e_n$.)

Suppose next that $G$ contains edges $e_1, \ldots, e_n$ (with corresponding weights $v_1, \ldots, v_n$) in series between the pair of vertices $x, y$: this means that the edges $e_1, \ldots, e_n$ form a path in which all the vertices except possibly the endvertices $x$ and $y$ have degree 2 in $G$. Let $G'$ be the graph in which the edges $e_1, \ldots, e_n$ are replaced by a single edge $e_*$ from $x$ to $y$. Then it is not hard to see that

$$C_G(\mathbf{v}) = \left(\sum_{j=1}^{n} \prod_{i \neq j} v_i\right) C_{G'}(\mathbf{v}'), \tag{3.2}$$

where the edge $e_*$ is given weight

$$v_*{}' = v_1 \bowtie v_2 \bowtie \cdots \bowtie v_n \equiv \frac{1}{\sum_{i=1}^{n} 1/v_i} \tag{3.3}$$

and all edges other than $e_1, \ldots, e_n, e_*$ are given weight $v_e{}' = v_e$. (Reason: a connected spanning subgraph of $G$ can omit at most one of the edges $e_1, \ldots, e_n$, for otherwise at least one of the internal vertices of the path would be disconnected from both $x$ and $y$. Moreover, $x$ is connected to $y$ via the "super-edge" $e_*$ if and only if none of the edges $e_1, \ldots, e_n$ are omitted. The relative weight of the cases with and without $x$ connected to $y$ via $e_*$ is thus $(\prod_{i=1}^{n} v_i)/(\sum_{j=1}^{n} \prod_{i \neq j} v_i) = v_*$; and there is an overall normalization factor $\sum_{j=1}^{n} \prod_{i \neq j} v_i$. See also [7, p. 35] for an equivalent formula.)

The formula for series reduction can be applied immediately to handle arbitrary subdivisions of a graph $G$. Given a finite graph $G = (V, E)$ and a family of integers $\mathbf{s} = \{s_e\}_{e \in E} \geqslant 1$, we define $G^{\bowtie \mathbf{s}}$ to be the graph in which each edge $e$ of $G$ is subdivided into $s_e$ edges in series. If $s \geqslant 1$ is an integer, we define $G^{\bowtie s}$ to be the graph in which each edge of $G$ is subdivided into $s$ edges in series. All the edges in $G^{\bowtie \mathbf{s}}$ or $G^{\bowtie s}$ obtained by subdividing the edge $e \in E$ are assigned the same weight $v_e$ as was assigned to $e$ in the original graph $G$. It follows immediately from (3.2)/(3.3) that

$$C_{G^{\bowtie \mathbf{s}}}(\mathbf{v}) = \left(\prod_{e \in E} s_e v_e^{s_e - 1}\right) C_G(\mathbf{v}/\mathbf{s}), \tag{3.4}$$

where $(\mathbf{v}/\mathbf{s})_e \equiv v_e/s_e$.

**Remarks.** 1. Series and parallel reduction formulae can be derived in the more general context of the $q$-state Potts model (also known as the multivariate Tutte polynomial): see e.g. [17, Section 2]. Parallel reduction is always given by (3.1) independently of the value of the parameter $q$. Series reduction is given by

$$v_1 \bowtie v_2 \bowtie \cdots \bowtie v_n = \frac{q}{\prod_{i=1}^{n} (1 + q/v_i) - 1}. \tag{3.5}$$

Please note that (3.5) reduces to (3.3) when $q \to 0$, which is precisely the limit in which the multivariate Tutte polynomial $Z_G(q, \mathbf{v})$ tends (after division by $q$) to $C_G(\mathbf{v})$.

2. If one takes in $C_G(\mathbf{v})$ the further limit of $\mathbf{v}$ infinitesimal, one obtains the generating polynomial of *minimal* connected spanning subgraphs, i.e. spanning trees. Now, spanning trees are intimately related to linear electrical circuits, as was noticed by Kirchhoff in 1847 [10,12]. For $v$ infinitesimal, the parallel reduction formula (3.1) becomes

$$v_1 || v_2 || \cdots || v_n \equiv v_1 + v_2 + \cdots + v_n, \tag{3.6}$$

which is precisely the law for putting electrical conductances in parallel. And the series reduction formula (3.3) is precisely the law for putting electrical conductances in series!

## 4. The univariate Brown–Colbourn conjecture is false as well

Let $K_4^{(\mathrm{a},p_1,p_2)}$ be the graph obtained from $K_4$ by replacing one edge by $p_1$ parallel edges and replacing each of the other five edges by $p_2$ parallel edges. Let $K_4^{(\mathrm{b},p_1,p_2)}$ be the graph obtained from $K_4$ by replacing each of two nonintersecting edges by $p_1$ parallel edges and replacing each of the remaining four edges by $p_2$ parallel edges. Define in a similar manner $K_4^{(\mathrm{c},p_1,p_2)}$, $K_4^{(\mathrm{d},p_1,p_2)}$ and $K_4^{(\mathrm{e},p_1,p_2)}$ for the cases (c), (d) and (e) discussed in Section 2.

We saw in Section 2 that in cases (b) and (d) one can obtain a counterexample to the multivariate Brown–Colbourn conjecture by choosing the weight $a$ to lie anywhere in the region $A_+$; this leads to a root $b$ lying in the region $B_-$ (see Figs. 1 and 2). Note now that the $p$th power of the region $1 + A_+$ will overlap the region $1 + B_-$ whenever $p > (1-\beta)/\alpha$ (just choose any point $b \in B_-$ close enough to the endpoint $-1 + e^{-2\pi i\beta} = -1 + e^{2\pi i(1-\beta)}$; then one of the $p$th roots of $1 + b$ will lie in the region $1 + A_+$). And (3.1) tells us that $p$ edges in parallel, each with weight $v$, are equivalent to a single edge with weight $v_{\mathrm{eff}}$ satisfying $1 + v_{\mathrm{eff}} = (1+v)^p$. This reasoning suggests that counterexamples to the univariate Brown–Colbourn conjecture might be found for the graphs $K_4^{(\mathrm{b},1,p)}$ and $K_4^{(\mathrm{d},1,p)}$: for all $p > (1-\beta)/\alpha$ they should have a root $v \in A_+$.[4] Likewise, the graphs $K_4^{(\mathrm{b},p,1)}$ and $K_4^{(\mathrm{d},p,1)}$ are expected to have, for all $p > (1-\alpha)/\beta$, a root $v \in B_+$. These guesses are in fact correct, and we find the following counterexamples to the univariate Brown–Colbourn conjecture:

- $G = K_4^{(\mathrm{b},1,7)}$, 30 edges: $v \approx -0.269253 \pm 0.682304i$, $|1+v| \approx 0.999765$,
- $G = K_4^{(\mathrm{b},6,1)}$, 16 edges: $v \approx -0.405015 \pm 0.801589i$, $|1+v| \approx 0.998274$,
- $G = K_4^{(\mathrm{d},1,9)}$, 30 edges: $v \approx -0.220759 \pm 0.626655i$, $|1+v| \approx 0.999956$,
- $G = K_4^{(\mathrm{d},30,1)}$, 93 edges: $v \approx -0.017476 \pm 0.185846i$, $|1+v| \approx 0.999946$.

Counterexamples are also obtained for each larger $p$; some typical numbers are shown in Table 1. Please note that all these counterexample graphs are planar.

The graphs $G = K_4^{(\mathrm{b/d},p_1,p_2)}$ are, of course, nonsimple (except when $p_1 = p_2 = 1$); so one might cling to the hope that the univariate Brown–Colbourn conjecture is true at

[4] We do not claim that this is a proof, though we suspect that a suitable topological argument might be able to turn it into a proof.

Table 1
Minimum value of $|1+v|$ for a zero of $C_G(v)$ for selected graphs $G = K_4^{(\mathrm{b/d},p_1,p_2)}$

| Graph | Value of $p$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| $K_4^{(\mathrm{b},1,p)}$ | 1 | 0.999765 | 0.997818 | 0.996996 | 0.996734 | 0.996749 | 0.996897 | 0.997102 | 0.997326 | 0.997547 |
| $K_4^{(\mathrm{b},p,1)}$ | 0.998274 | 0.997234 | 0.997001 | 0.997083 | 0.997284 | 0.997519 | 0.997753 | 0.997971 | 0.998169 | 0.998345 |
| $K_4^{(\mathrm{d},1,p)}$ | 1 | 1 | 1 | 0.999956 | 0.999813 | 0.999746 | 0.999718 | 0.999713 | 0.999718 | 0.999730 |
| $K_4^{(\mathrm{d},p,1)}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |

For $1 \leqslant p \leqslant 5$ the value equals 1. A value strictly less than 1 indicates a counterexample to the univariate Brown–Colbourn conjecture. For $K_4^{(\mathrm{d},p,1)}$ a counterexample can be found for $p \geqslant 30$.

least for *simple* graphs (or, weaker yet, for *simple planar* graphs). But these hopes too are false. To see why this is the case, consider the following procedure:

(1) Choose $p_1, p_2$ so that the graph $K_4^{(\mathrm{b},p_1,p_2)}$ has a root $v_1$ satisfying $|1+v_1| < 1$.
(2) Choose any integer $s \geqslant 2$.
(3) Find an integer $k$ large enough so that $v_k \equiv -1 + (1+v_1)^{1/k}$—defined using the root with $|\arg[(1+v_1)^{1/k}]| \leqslant \pi/k$—lies in the disc $|1/s + v_k| < 1/s$. (It is always possible to find such a $k$, because the points $v_k$ lie on a logarithmic spiral that approaches the point $v = 0$ making a nonzero angle with the imaginary axis, while all the circles $|1/s + v| = 1/s$ pass through $v = 0$ tangent to the imaginary axis.)

Then $v_k$ is a root for the graph $K_4^{(\mathrm{b},kp_1,kp_2)}$, by the rules for parallel reduction; and $sv_k$ is a root for the graph $(K_4^{(\mathrm{b},kp_1,kp_2)})^{\bowtie s}$, by the rules for series reduction. And by construction we have $|1 + sv_k| < 1$. Therefore, the graph $(K_4^{(\mathrm{b},kp_1,kp_2)})^{\bowtie s}$, which is simple and planar, is the desired counterexample.

For example, if we take $(p_1, p_2) = (11, 1)$ and $s = 2$, counterexamples can be obtained for $k \geqslant 58$:

- $v_1 \approx -0.140\,970\,808\,664 + 0.507\,062\,767\,880i$, $|1+v_1| \approx 0.997\,518\,822\,949$,
- $v_{58} \approx -0.000\,085\,091\,565 + 0.009\,193\,226\,407i$, $|1+2v_{58}| \approx 0.999\,998\,862\,173$.

This shows that the graph $(K_4^{(\mathrm{b},638,58)})^{\bowtie 2}$, which has 1512 vertices and 3016 edges, is a counterexample to the univariate Brown–Colbourn conjecture. Similarly, if we take $(p_1, p_2) = (1, 12)$ and $s = 2$, counterexamples can be obtained for $k \geqslant 36$:

- $v_1 \approx -0.112\,358\,418\,620 + 0.453\,757\,934\,703i$, $|1+v_1| \approx 0.996\,897\,106\,175$,
- $v_{36} \approx -0.000\,172\,469\,038 + 0.013\,125\,252\,246i$, $|1+2v_{36}| \approx 0.999\,999\,665\,908$.

Therefore, the graph $(K_4^{(\mathrm{b},36,432)})^{\bowtie 2}$, which has 1804 vertices and 3600 edges, is a counterexample to the univariate Brown–Colbourn conjecture.

Smaller counterexamples of the forms $(K_4^{(\mathrm{b/d},p,1)})^{\bowtie(s,1)}$ or $(K_4^{(\mathrm{b/d},1,p)})^{\bowtie(1,s)}$ can probably be found by direct search. But the foregoing construction has the advantage that there is no need to compute the roots of extremely-high-degree polynomials; it suffices to compute the roots for the base case $K_4^{(\mathrm{b/d},p_1,p_2)}$ (for which the polynomials are large but not huge) and then make simple manipulations on them.

*Methodological remark*: In this work, we needed to compute accurately the roots of polynomials of fairly high degree (up to 93) with very large integer coefficients (up to about $10^{27}$). To do this we used the package MPSolve 2.0 developed by Bini and Fiorentino [2,3]. MPSolve is much faster than MATHEMATICA's `NSolve` for high-degree polynomials (this is reported in [3], and we confirm it); it gives guaranteed error bounds for the roots, based on rigorous theorems [3]; its algorithms are publicly documented [3]; and its source code is freely available [2].

Let us mention, finally, that counterexamples with smaller values of $|1+v|$ can be found. Consider, for example, the complete graph $K_6$ in which a pair of vertex-disjoint triangles receives weight $a$ and the remaining nine edges receive weight $b$. We have

$$\begin{aligned}
C_{K_6}(a,b) = {} & (81b^5+78b^6+36b^7+9b^8+b^9)\\
& + (324b^4+594b^5+480b^6+216b^7+54b^8+6b^9)a\\
& + (486b^3+1314b^4+1665b^5+1224b^6+540b^7+135b^8+15b^9)a^2\\
& + (324b^2+1188b^3+2160b^4+2376b^5+1656b^6+720b^7\\
& \quad + 180b^8+20b^9)a^3\\
& + (81b+432b^2+1134b^3+1800b^4+1854b^5+1254b^6+540b^7\\
& \quad + 135b^8+15b^9)a^4\\
& + (54b+216b^2+504b^3+756b^4+756b^5+504b^6+216b^7\\
& \quad + 54b^8+6b^9)a^5\\
& + (9b+36b^2+84b^3+126b^4+126b^5+84b^6+36b^7\\
& \quad + 9b^8+b^9)a^6.
\end{aligned} \tag{4.1}$$

If we then substitute $a=(1+v)^{p_1}-1$ and $b=(1+v)^{p_2}-1$, counterexamples to the univariate Brown–Colbourn conjecture can be found for many pairs $(p_1,p_2)$. For example, for $(p_1,p_2)=(1,6)$ we obtain a 60-edge nonplanar graph whose roots include $v\approx -0.357514\pm 0.713815\,i$, yielding $|1+v|\approx 0.960375$.

It would be interesting to know whether examples can be found in which $|1+v|$ is arbitrarily small. More generally, one can ask:

**Question 4.1.** *What is the closure of the set of all roots of the polynomials $C_G(v)$ as $G$ ranges over all graphs? Over all planar graphs? Over all simple planar graphs?*

Brown and Colbourn [5] pointed out that the graphs $G = C_n^{(p)}$ (the $n$-cycle with each edge replaced by $p$ parallel edges) have roots that, taken together, are dense in the region $|1+v| \geqslant 1$. We have shown here that roots can also enter the region $|1+v| < 1$. But how far into this latter region can they penetrate? Might the roots actually be dense in the whole complex plane? If this is indeed the case, it would mean that the univariate Brown–Colbourn conjecture is as false as it can possibly be.

*Note added* (*April 2004*): Building on the examples constructed in this section, Chang and Shrock [6, Sections 5.17 and 5.18] have recently devised families of strip graphs in which the limiting curve of zeros of $C_G(v)$, as the strip length tends to infinity, penetrates into the "forbidden region" $|1+v| < 1$. Some of these families consist of planar graphs.

## 5. Series–parallel is necessary and sufficient

In this section, we shall prove that a graph has the multivariate Brown–Colbourn property *if and only if* it is series–parallel.

Let us begin by defining a weakened version of the Brown–Colbourn property:

**Definition 5.1.** Let $G$ be a graph, and let $\lambda > 0$. We say that $G$

- has the *univariate property* $\mathrm{BC}_\lambda$ if $C_G(v) \neq 0$ whenever $|\lambda + v| < \lambda$;
- has the *multivariate property* $\mathrm{BC}_\lambda$ if $C_G(\mathbf{v}) \neq 0$ whenever $|\lambda + v_e| < \lambda$ for all edges $e$.

Properties $\mathrm{BC}_1$ are, of course, the original univariate and multivariate Brown–Colbourn properties; the properties $\mathrm{BC}_\lambda$ become increasingly weaker as $\lambda$ is decreased.

The properties $\mathrm{BC}_\lambda$ are intimately related to subdivisions:

**Lemma 5.2.** *Let $\lambda > 0$ and let $s$ be a positive integer. Then the following are equivalent for a graph $G$:*

1. *$G$ has the univariate property $BC_\lambda$.*
2. *$G^{\bowtie s}$ has the univariate property $BC_{s\lambda}$.*

**Lemma 5.3.** *Let $\lambda > 0$ and let $s$ be a positive integer. Then the following are equivalent for a graph $G$:*

1. *$G$ has the multivariate property $BC_\lambda$.*
2. *$G^{\bowtie s}$ has the multivariate property $BC_{s\lambda}$.*
3. *$G^{\bowtie \mathbf{s}}$ has the multivariate property $BC_{s\lambda}$ for all vectors $\mathbf{s}$ satisfying $s_e \geqslant s$ for all edges $e$.*

Indeed, Lemmas 5.2 and 5.3 are an immediate consequence of the formula (3.4) for subdivisions—which states that subdivision by $\mathbf{s}$ moves the nonzero roots from $\mathbf{v}$ to $\mathbf{sv}$—together with the fact that $|s\lambda + sv| < s\lambda$ is equivalent to $|\lambda + v| < \lambda$.

In the preceding section we have shown that not all graphs have the univariate property $BC_1$. It is nevertheless true—and virtually trivial—that every connected graph has the univariate property $BC_\lambda$ for *some* $\lambda > 0$. (Since a nonidentically-vanishing univariate polynomial has finitely many roots, it suffices to choose $\lambda$ small enough so that none of the roots of $C_G(v)$ lie in the disc $|\lambda + v| < \lambda$.) By Lemma 5.2, an equivalent assertion is that $G^{\bowtie s}$ has the univariate property $BC_1$ for all sufficiently large integers $s$.[5]

The situation is very different, however, when we consider the multivariate property $BC_\lambda$. We begin with a simple but important lemma:

**Lemma 5.4.** *Let* $\lambda > 0$, *and suppose that the connected graph G has the multivariate property* $BC_\lambda$. *Then every connected subgraph* $H \subseteq G$ *also has the multivariate property* $BC_\lambda$.

**Proof.** Consider first the case in which $H$ is a connected *spanning* subgraph (i.e. its vertex set is the same as that of $G$). Let us write $\mathbf{v} = (\mathbf{v}', \mathbf{v}'')$ where $\mathbf{v}' = \{v_e\}_{e \in E(H)}$ and $\mathbf{v}'' = \{v_e\}_{e \in E(H) \setminus E(G)}$. Then

$$C_H(\mathbf{v}') = C_G(\mathbf{v}', \mathbf{0}) = \lim_{\mathbf{v}'' \to \mathbf{0}} C_G(\mathbf{v}', \mathbf{v}''). \tag{5.1}$$

By hypothesis, $C_G(\mathbf{v}', \mathbf{v}'') \neq 0$ whenever $|\lambda + v_e| < \lambda$ for all $e \in E(G)$. Now take $\mathbf{v}'' \to \mathbf{0}$ from within this product of discs ($\mathbf{0}$ lies on its boundary). By Hurwitz's theorem,[6] either $C_H(\mathbf{v}')$ is nonvanishing whenever $|\lambda + v_e| < \lambda$ for all $e \in E(H)$, or else $C_H$ is identically zero. But the latter is impossible since $H$ is connected.

Now let $H$ be an arbitrary connected subgraph of $G$ (spanning or not). Construct a connected spanning subgraph $\widehat{H}$ of $G$ by hanging trees off some or all of the vertices of $H$ without creating any new circuits.[7] Let us write

[5] Brown and Colbourn [5, Proposition 4.4 and Theorem 4.5] have proven a result also for *nonuniform* subdivisions $G^{\bowtie \mathbf{s}}$: namely, for each graph $G$ there exists an integer $s$ such that $G^{\bowtie \mathbf{s}}$ has the univariate property $BC_1$ whenever $s_e \geqslant s$ for all $e$. This is significantly stronger than the just-mentioned trivial result, and it would be worth trying to understand it better. Brown and Colbourn's method looks very different from ours, at least at first glance; it would be interesting to try to translate it into our language. In particular, there may be a "partially multivariate" result hiding underneath their apparently univariate proof.

[6] Hurwitz's theorem states that if $D$ is a domain in $\mathbb{C}^n$ and $(f_k)$ are nonvanishing analytic functions on $D$ that converge to $f$ uniformly on compact subsets of $D$, then $f$ is either nonvanishing or else identically zero. Hurwitz's theorem for $n = 1$ is proved in most standard texts on the theory of analytic functions of a single complex variable (see e.g. [1, p. 176]). Surprisingly, we have been unable to find Hurwitz's theorem proven for general $n$ in any standard text on several complex variables (but see [11, p. 306] and [15, p. 337]). So here, for completeness, is the sketch of a proof: Suppose that $f(c) = 0$ for some $c = (c_1, \ldots, c_n) \in D$, and let $D' \subset D$ be a small polydisc centered at $c$. Applying the single-variable Hurwitz theorem, we conclude that $f(z_1, c_2, \ldots, c_n) = 0$ for all $z_1$ such that $(z_1, c_2, \ldots, c_n) \in D'$. Applying the same argument repeatedly in the variables $z_2, \ldots, z_n$, we conclude that $f$ is identically vanishing on $D'$ and hence, by analytic continuation, also on $D$.

[7] This can be done, for instance, by running breadth-first search with the vertices of $H$ initially on the queue.

$\mathbf{v} = \{v_e\}_{e \in E(\widehat{H})} = (\mathbf{v}', \mathbf{v}'')$ where $\mathbf{v}' = \{v_e\}_{e \in E(H)}$ and $\mathbf{v}'' = \{v_e\}_{e \in E(\widehat{H}) \setminus E(H)}$. Then

$$C_{\widehat{H}}(\mathbf{v}) = C_H(\mathbf{v}') \prod_{e \in E(\widehat{H}) \setminus E(H)} v_e. \tag{5.2}$$

Since $\widehat{H}$ has multivariate property $\mathrm{BC}_\lambda$, so does $H$. □

The following is the fundamental fact from which all else flows:

**Proposition 5.5.** *The complete graph $K_4$ does not have the multivariate property $BC_\lambda$ for any $\lambda > 0$.*

**Proof.** This is an almost immediate consequence of the observations made at the end of Section 2. In cases (b) and (d), for any $\lambda > 0$ there exists $b$ with $|\lambda + b| = \lambda$ for which at least one of the solutions to $C_{K_4}(a, b) = 0$ satisfies $|\lambda + a| < \lambda$. By slightly perturbing this pair, we can find a pair $(a, b)$ with $C_{K_4}(a, b) = 0$ satisfying $|\lambda + a| < \lambda$ and $|\lambda + b| < \lambda$. So $K_4$ does not even have the bivariate property $\mathrm{BC}_\lambda$. □

We can deduce from Lemma 5.4 and Proposition 5.5 a necessary and sufficient condition for $G$ to have various forms of the multivariate Brown–Colbourn property:

**Theorem 5.6.** *Let $G$ be a loopless connected graph. Then the following are equivalent*:

(a) *$G$ has the multivariate property $BC_1$.*
(b) *$G$ has the multivariate property $BC_\lambda$ for some $\lambda > 0$.*
(c) *$G$ is series–parallel.*

**Proof.** (a) ⇒ (b) is trivial.

(b) ⇒ (c): Let $G$ be a loopless connected graph that is not series–parallel. Then $G$ contains a subgraph $H$ that is a subdivision of $K_4$.[8] Suppose that $H = (K_4)^{\bowtie \mathbf{s}}$ with $\mathbf{s} = (s_1, \ldots, s_6)$, and define $s = \max(s_1, \ldots, s_6)$. Now fix any $\lambda > 0$; then, by Proposition 5.5 we can find a vector $\mathbf{v} = (v_1, \ldots, v_6)$ that is a zero of $C_{K_4}(\mathbf{v})$ and satisfies $|\lambda/s + v_i| < \lambda/s$ for $i = 1, \ldots, 6$. It then follows that the vector $\mathbf{v}' = ({v_1}', \ldots, {v_6}')$ defined by ${v_i}' = s_i v_i$ satisfies $C_H(\mathbf{v}') = 0$ and $|\lambda + {v_i}'| < \lambda$ for $i = 1, \ldots, 6$. Therefore, $H$ does not have the multivariate property $\mathrm{BC}_\lambda$. By Lemma 5.4, $G$ cannot have this property either.

(c) ⇒ (a): This is proven in [16, Remark 3 in Section 4.1], but for the convenience of the reader we repeat the proof here. Suppose that $G$ is a loopless connected series–parallel graph; this means that $G$ can be obtained from a tree by a finite sequence of series and parallel extensions of edges (i.e. replacing an edge by two edges in series or two edges in parallel). We will prove that $G$ has the multivariate property $\mathrm{BC}_1$, by

[8] The relevant fact is the following [8, Exercise 8.16 and Proposition 1.7.2]: $G$ is series–parallel ⇔ $G$ has no $K_4$ minor ⇔ $G$ has no $K_4$ topological minor. And the latter statement says precisely that $G$ contains no subgraph $H$ that is a subdivision of $K_4$. See also [9,13].

induction on the length of this sequence of series and parallel extensions. The base case is when $G$ is a tree: then $C_G(\mathbf{v}) = \prod_{e \in E(G)} v_e$ and $G$ manifestly has the multivariate property $BC_1$. Now suppose that $G$ is obtained from a smaller graph $G'$ by replacing an edge $e_*$ of $G'$ by two parallel edges $e_1, e_2$. Use the parallel reduction formula (3.1): since $|1 + v_1| < 1$ and $|1 + v_2| < 1$ imply $|1 + v_*| < 1$, we deduce that $G$ has the multivariate property $BC_1$ if $G'$ does. Suppose, finally, that $G$ is obtained from a smaller graph $G'$ by replacing an edge $e_*$ of $G'$ by two edges $e_1, e_2$ in series. Use the series reduction formula (3.2)/(3.3) and the fact that $|1 + v| < 1$ is equivalent to $\mathrm{Re}(1/v) < -1/2$: then $\mathrm{Re}(1/v_i) < -1/2$ for $i = 1, 2$ implies that $\mathrm{Re}(1/v_*) < -1 < -1/2$, and moreover the prefactor $v_1 + v_2$ is nonzero; so we deduce that $G$ has the multivariate property $BC_1$ if $G'$ does. □

For each graph $G$, let us define $\lambda_\star(G)$ to be the maximum $\lambda$ for which $G$ has the multivariate property $BC_\lambda$. Then Theorem 5.6 states a surprising (at first sight) dichotomy: either $\lambda_\star(G) = 0$ [when $G$ is not series–parallel] or else $\lambda_\star(G) \geqslant 1$ (when $G$ is series–parallel).

Some series–parallel graphs have $\lambda_\star(G) = 1$ exactly: for example, the graphs $K_2^{(n)}$ (a pair of vertices connected by $n$ parallel edges) have $C_{K_2^{(n)}}(v) = (1+v)^n - 1$ and hence even have univariate roots on the circle $|1 + v| = 1$. On the other hand, some series–parallel graphs have $\lambda_\star(G) > 1$: for example, the cycles $C_n$ have $\lambda_\star(G) = n/2$. (Proof: We have

$$C_{C_n}(\mathbf{v}) = \left(\prod_{i=1}^{n} v_i\right)\left(1 + \sum_{i=1}^{n} \frac{1}{v_i}\right), \tag{5.3}$$

which is nonvanishing if $\mathrm{Re}(1/v_i) < -1/n$ for all $i$. But this is equivalent to $|n/2 + v_i| < n/2$.) It is an interesting open problem to characterize the graphs that have $\lambda_\star(G) = 1$ or, more ambitiously, to find a simple graph-theoretic formula for $\lambda_\star(G)$.

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