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Operatorial approach to the non-Archimedean stability of a Pexider K-quadratic functional equation

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Abstract. We use the operatorial approach to obtain, in non-Archimedean spaces, the Hyers–Ulam stability of the Pexider K-quadratic functional equation

$$\sum_{k\in K} f(x+k\cdot y) = \kappa g(x) + \kappa h(y),\ \ x, y \in E,$$

where $f, g, h : E \to F$ are applications and K is a finite subgroup of the group of automorphisms of E and κ is its order.

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1. Introduction

The concept of the stability for functional equations was introduced for the first time by Ulam in 1940 [17]. Ulam started the stability by the following question

Given a group G, a metric group (G', d), a number $\delta > 0$ and a mapping $f : G \to G'$ which satisfies the inequality $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G$, does there exist an

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homomorphism $h : G \to G'$ and a constant $\gamma > 0$, depending only on G and G' such that $d(f(x), h(x)) \leqslant \gamma\delta$ for all x in G?

In 1941, Hyers [17] gave the partial solution to Ulam's question in Banach spaces. The result of Hyers was extended, for additive mappings by Aoki [1] and later, for linear mappings by Rassias [29]. For more information on the history of the concept see [4,5,7,11,15,18–20,23,25,32–34,36–38] and especially the recent developments of the stability in [6,7].

The first stability theorem for the K-quadratic functional equation was proved for $K = \{id\}$ by Hyers–Ulam (1941) [17] and Rassias (1978) [31] and for $K = \{-id, id\}$ by Skof (1983) [35] in Banach spaces. Cholewa (1984) [12] extended Skof's result to an abelian group. Czerwik (1992) [13], in the spirit of Hyers–Ulam–Rassias generalized Skof's theorem.

Recently, the stability problem of the K-quadratic functional equation has been investigated by a number of mathematicians, the interested reader should refer to Ait Sibaha et al. [3], Bouikhalene et al. [8], Charifi et al. [9,10] and Lukasik [26], see also [6,20,22–24,31].

In 1897, Hensel [16] discovered the p-adic numbers. Let p be a fixed prime number and x a nonzero rational number, there exists a unique integer $v_p(x) \in \mathbb{Z}$ such that $x = p^{v_p(x)} \frac{a}{b}$ where a and b are integers co-prime to p. The function defined in $\mathbb{Q}$ by $|x|_p = p^{-v_p(x)}$ is called a p-adic, a ultrametric or simply a non-Archimedean absolute value on $\mathbb{Q}$. So, with the p-adic absolute value $\mathbb{Q}$ is called a p-adic or a non-Archimedean field. The completion, denoted by $\mathbb{Q}_p$ of $\mathbb{Q}$ with respect to the metric defined by the p-adic absolute value is called the p-adic numbers. Their elements are the formal series $p^{v_p(x)}\sum_{i\geqslant 0} a_i p^i$, with $a_0 \neq 0$ and $|a_i| \leqslant p-1$ are integers.

In general, by a non-Archimedean field, we mean a field k equipped with a function $|\ | : k \to [0, +\infty)$, called a non-Archimedean absolute value on k and satisfying the following conditions

i. $|x| = 0 \iff x = 0$
ii. $|xy| = |x||y|, \ x, y \in k$
iii. $|x + y| \leqslant max(|x|, |y|), \ x, y \in k.$

We assume, throughout this paper that this value absolute is non-trivial i.e., there exists an element λ of k such that, $|\lambda| \neq 0, 1$.

By a non-Archimedean vector space, we mean a vector space E over a non-Archimedean field k equipped with a function $\|\ \| : E \to [0, +\infty)$ called a non-Archimedean norm on E and satisfying the following properties

i. $\|x\| = 0 \iff x = 0,$
ii. $\|\lambda x\| = |\lambda|\|x\|, \ (\lambda, x) \in k \times E,$
iii. $\|x + y\| \leqslant max(\|x\|, \|y\|), \ x, y \in E.$

The particularity of a non-Archimedean norm is the fact that they do not satisfy the Archimedean axiom and a sequence $\{x_n\}$ is Cauchy if and only if $\{x_{n+1} - x_n\}$ converges to zero.

In 2005 Arriola and Beyer in [2], initiated the stability of Cauchy functional equation over p-adic fields. In 2007, Sal Moslehian and Rassias [29] studied the stability of Cauchy and quadratic functional equations in non-Archimedean spaces. In [30], the author investigated, by using the fixed point method the non-Archimedean stability of a quadratic functional equation.

Following this investigation, we deal with the operatorial approach, in a non-Archimedean space, the Hyers–Ulam stability of a Pexiderized version of the K-quadratic functional equation,

$$\sum_{k\in K} f(x+k\cdot y) = \kappa g(x) + \kappa h(y), \;\; x,y\in E, \tag{1.1}$$

where $f,g,h : E \to F$ are applications from a normed space E into a non-Archimedean space F, K is a finite abelian subgroup of the group of automorphisms of E and κ denotes the order of K.

The present paper is a continuation, in a non-Archimedean space of the previous work by Charifi et al. [9,10].

The paper is organized as follows: in the second section we give some notions, notations and preliminary results. In the third section, we derive the non-Archimedean stability of Eq. (1.1).

2. Notations and preliminary results

In this section, we introduce some notions and notations. We give necessary results for the proof of Theorem 2.6. They are a faithful translation, in terms of a non-Archimedean norm of results which were given in the case of a usual norm by Hyers in [21].

A function $A : E \to F$ between vector spaces E and F is said to be additive provided if $A(x+y) = A(x) + A(y)$ for all $x,y\in E$; in this case it is easily seen that $A(rx) = rA(x)$ for all $x\in E$ and all $r\in\mathbb{Q}$.

Let $k\in\mathbb{N}$ and $A : E^k \to F$ be a function, then we say that A is k-additive provided if it is additive in each variable; in addition we say that A is symmetric provided if

$$A(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)}) = A(x_1, x_2, \ldots, x_k)$$

whenever $x_1, x_2, \ldots, x_k \in E$ and σ is a permutation of $(1,2,\ldots,k)$.

Let $k\in\mathbb{N}$ and $A : E^k \to F$ be symmetric and k-additive and let $A_k(x) = A(x,x,\ldots,x)$ for $x\in E$ and note that $A_k(rx) = r^k A_k(x)$ whenever $x\in E$ and $r\in\mathbb{Q}$.

In this way a function $A_k : E\to F$ which satisfies for all $\lambda\in\mathbb{Q}$ and $x\in E$, $A_k(\lambda x) = \lambda^k A_k$ will be called a rational-homogeneous form of degree k (assuming $A_k \neq 0$).

A function $p : E\to F$ is called a generalized polynomial (GP) function of degree $m\in\mathbb{N}$ if there exist $a_0\in E$ and a rational-homogeneous form $A_k : E\to F$ (for $1\leqslant k\leqslant m$) of degree k, such that

$$p(x) = a_0 + \sum_{k=1}^{m} A_k(x)$$

for $x\in E$.

Let F^E denote the vector space (over a field K) consisting of all maps from E into F. For $h\in E$ define the linear difference operator Δ_h on F^E by

$$\Delta_h f(x) = f(x+h) - f(x) \tag{2.1}$$

for $f \in F^E$ and $x \in E$. Notice that these difference operators commute ($\Delta_{h_1}\Delta_{h_2} = \Delta_{h_2}\Delta_{h_1}$ for all $h_1, h_2 \in E$) and if $h \in E$ and $n \in \mathbb{N}$, then Δ_h^n the nth iterate of Δ_h satisfies

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(x+kh)$$

for $f \in F^E$ and $x, h \in E$.

The following theorems were proved by Mazur and Orlicz [27,28], and in greater generality by Djokovic [14].

Theorem 2.1. *Let $n \in \mathbb{N}$ and $f : E \to F$ be a function between a vector space E and F, then the following assertions are equivalent,*

(1) $\Delta_h^n f(x) = 0$ for all $x, h \in E$.
(2) $\Delta_{h_n} \dots \Delta_{h_1} f(x) = 0$ for all $x, h_1, \dots, h_n \in E$.
(3) *f is a GP function of degree at most $n-1$.*

Theorem 2.2. *Let $A_k : E \to F$ be a rational-homogeneous form of degree k, then there exists a unique symmetric k-additive transformation $A : E^k \to F$ such that*

$$A_k(x) = A(x, \dots, x).$$

The k-additive transformation is often called the polar of transformation A_k and it is given by the formula

$$A(x_1, \dots, x_k) = \frac{1}{k!} \Delta^k_{x_1 \dots x_k} A_k(x).$$

Lemma 2.3. *Let E be a vector space, F a non-Archimedean Banach space and $p \in \mathbb{N}^*$, $|p| \neq 1$. Let δ be a fixed positive number and $f : E \to F$ be a function satisfying one of two conditions*

$$\begin{aligned} &1)\ \left\|\Delta_h^2 f(x)\right\| \leqslant \delta,\ \ x, h \in E, \\ &2)\ \|\Delta_h f(x) - \Delta_h f(0)\| \leqslant \delta,\ \ x, h \in E, \end{aligned} \tag{2.2}$$

then there exists an additive mapping $A : E \to F$ given by

$$A(x) = \lim_{n \to +\infty} p^n f(p^{-n} x)$$

and such that

$$\|A(x) - f(x) + f(0)\| \leqslant \delta$$

Proof. The proof is the same on the Assumption 1) or 2). Assume that 1) is true and put $g = f - f(0)$, so by (2.2) we have $\left\|\Delta_h^2 g(x)\right\| \leqslant \delta$ for all x and h in E. Replacing x by 0 and h by x, we get

$$\|g(2x) - 2g(x)\| \leqslant \delta \tag{2.3}$$

for all x in E. Replacing h by x we obtain

$$\|g(3x) - 2g(2x) + g(x)\| \leqslant \delta. \tag{2.4}$$

Therefore, taking into account (2.3) and (2.4), we obtain

$$\|g(3x) - 3g(x)\| \leqslant \delta. \tag{2.5}$$

We will prove by mathematical induction that

$$\|g(px) - pg(x)\| \leqslant \delta. \tag{2.6}$$

We suppose that (2.6) true for all $k \leqslant p$. Replacing x by $(p-1)x$ and h by x in (2.2) we get

$$||g((p+1)x) - 2g((p)x) + g((p-1)x)|| \leqslant \delta \tag{2.7}$$

and by hypothesis of induction we have

$$||g((p-1)x) - (p-1)^2 g(x)|| \leqslant \delta. \tag{2.8}$$

By using the inequalities 2.6, 2.7 and 2.8, we get the result

$$\|g(px) - pg(x)\| \leqslant \delta, \ p \in \mathbb{N}^*, \ x \in E.$$

We put $q_n(x) = p^n g(p^{-n}x)$, we have when replaced x by $p^{-n-1}x$ in (2.6)

$$\|g(p^{-n}x) - pg(p^{-(n+1)}x)\| \leqslant \delta. \tag{2.9}$$

By multiplying this inequality by p^n we get

$$\|q_{n+1}(x) - q_n(x)\| \leqslant |p^n|\delta. \tag{2.10}$$

Thus, since $|p| \neq 0, 1$, $q_n(x)$ is a Cauchy sequence, as F is complete hence $q_n(x)$ converge to $A(x)$. Now we have

$$\left\|\Delta_h^2 A(x)\right\| = \lim_{n\to+\infty} \left\|p^n \Delta^2_{p^{-n}h} g(p^{-n}x)\right\| \leqslant \lim_{n\to+\infty} |p^n|\delta = 0.$$

We see that $\Delta_h^2 A(x) = 0$ for all x and h in E. Thus, from Theorem 1.1 A is additive on E. By using (2.6) we have $\|q_n(x) - g(x)\| \leqslant \delta$, and taking limits as $n \to \infty$, we obtain

$$\|A(x) - f(x) + f(0)\| \leqslant \delta$$
$$A(x) = \lim_{n\to+\infty} p^n \Delta_{p^{-n}x} f(0).$$

This ends the proof of the lemma. □

Lemma 2.4. *Let E be a vector space, F a non-Archimedean space and $p \in \mathbb{N}^*$, $|p| \neq 1$. Let $h : E^2 \to F$ be either identically zero or else a rational-homogeneous form of degree $k-1$ $(k > 1)$ in x for each y, $q : E^2 \to F$ a transformation of degree at most $k-2$ in x which vanishes for $x = 0$ and $f : E \to F$ be a function satisfying the inequality*

$$\| f(x+y) - f(x) - f(y) + f(0) - q(x,y) - h(x,y)\| \leqslant \delta, \ x, y \in E, \tag{2.11}$$

Then $h(x,x) = kA_k(x)$, where $A_k : E \to F$ is either identically zero or else a homogeneous form of degree k,

$$A_k(x) = \frac{1}{k!} \lim_{n\to+\infty} |p^{kn}| \Delta^k_{p^{-n}x} f(0).$$

Moreover $h(x, y)$ *is given by the formula*

$$h(x, y) = \frac{1}{(k-1)!} \lim_{n\to+\infty} |p^{(k-1)n}| \Delta^{k-1}_{p^{-n}x} \Delta_y(f)(0).$$

Proof. By the hypothesis made on h, there exists a map $A : E^k \to F$ which is additive and symmetric in its first $k-1$ arguments, such that

$$h(x, y) = \frac{1}{(k-1)!} A(x, \ldots, x, y). \tag{2.12}$$

In view of (2.1) and (2.11), treating y as a constant and using the increments $x_1, \ldots, x_{k-1}$, we have

$$\left\| \Delta_{x_1^k \ldots x_{k-1} y} f(x) - \Delta_{x_1^{k-1} \ldots x_{k-1}} q(x, y) - \Delta^{k-1}_{x_1 \ldots x_{k-1}} h(x, y) \right\| \leqslant \delta.$$

Since $q(x, y)$ is of degree at most $k-2$ in x, by Theorem (2.1)

$$\Delta^{k-1}_{x_1 \ldots x_{k-1}} q(x, y) = 0.$$

Also from (2.12) and Theorem (2.2) it follows that

$$\Delta^{k-1}_{x_1 \ldots x_{k-1}} h(x, y) = A(x_1, \ldots, x_{k-1}, y).$$

Thus we have

$$\left\| \Delta^k_{x_1 \ldots x_{k-1} y} f(x) - A(x_1, \ldots, x_{k-1}, y) \right\| \leqslant \delta. \tag{2.13}$$

Using the fact that, for each j, $1 \leqslant j \leqslant k-1$ the kth difference in (2.13) is symmetric in all of its increments, then we obtain that

$$\left\| \Delta^k_{x_1 \ldots x_{k-1} y} f(x) - A(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_k, x_j) \right\| \leqslant \delta. \tag{2.14}$$

Now, from (2.13) and (2.14) we get

$$\| A(x_1, \ldots, x_{k-1}, y) - A(x_1, \ldots, x_{j-1}, y, x_{j+1}, \ldots, x_k, x_j) \| \leqslant \delta \tag{2.15}$$

In order, to prove that A is additive in its last argument and symmetric in all of its arguments we distinguish two cases.

(1) Case $k > 2$. Since $k > 2$, there exists an index i, $1 \leqslant i \leqslant k-1$ such that $i \neq j$. In (2.13), replacing x_i by $p^{-n}x_i$, multiplying this inequality by p^n and taking the limit as $n \to \infty$, we obtain that A is symmetric in all of its arguments. Obviously A must be necessarily additive in its last argument. From inequality (2.13) we have

$$\left\| \Delta^{k-1}_{x_1 \ldots x_{k-1}} \Delta_y f(0) - A(x_1, \ldots, x_{k-1}, y) \right\| \leqslant \delta.$$

Now, take each $x_j = p^{-n}x$, multiply the last inequality by $\frac{p^{n(k-1)}}{(k-1)!}$, and then let n tend to infinity. By (2.12) we get

$$h(x,y) = \frac{1}{(k-1)!} \lim_{n\to+\infty} p^{(k-1)n} \Delta^{k-1}_{p^{-n}x} \Delta_y f(0).$$

In a similar way, if we define $A_k(x) = k^{-1}h(x,x)$ and use the fact that A is additive in each of its arguments, from (2.13) we obtain that

$$A_k(x) = \frac{1}{k!} \lim_{n\to+\infty} p^{kn} \Delta^{k}_{p^{-n}x} f(0), \tag{2.16}$$

which gives the sought result.

(2) Case $k = 2$. Then (2.15) becomes

$$\|A(x_1,y) - A(y,x_1)\| \leqslant \delta$$

for all x_1 and y in E , where A is additive in the first argument. Replacing x_1 by $p^{-n}x_1$, and multiplying by p^n, where n and p are any positive integer, we obtain

$$\|A(x_1,y) - p^n A(y,p^{-n}x)\| \leqslant |p|^n \delta$$

and so by letting n tend to infinity

$$A(x_1,y) = \lim_{n\to+\infty} p^n A(y,p^{-n}x_1). \tag{2.17}$$

Thus

$$\begin{aligned} A(x_1,y+z) &= \lim_{n\to+\infty} p^n A(y+z,p^{-n}x_1) \\ &= \lim_{n\to+\infty} p^n A(y,p^{-n}x_1) + \lim_{n\to+\infty} p^n A(z,p^{-n}x_1) \\ &= A(x_1,y) + A(x_1,z) \end{aligned}$$

so that A is additive in its second argument. Now, the symmetry is given by (2.17) and additivity of A, which completes the proof of Lemma 2.2. □

Proposition 2.5. *Let E be a vector space, F be a non-Archimedean Banach space and $p \in \mathbb{N}^*$, $|p| \neq 1$. Let δ be a fixed positive number and $f : E \to F$ be a function satisfying the inequality*

$$\left\| \Delta^m_{h_1 \dots h_m} f(x) \right\| \leqslant \delta, \; x, h_1, \dots, h_m \in E. \tag{2.18}$$

Then there exists a GP function $p_{m-1} : E \to F$ which is of degree at most $m-1$, such that,

$$\|f(x) - p_{m-1}(x)\| \leqslant \delta \text{ for all } x \text{ in } E. \tag{2.19}$$

Moreover p_{m-1} is given by the formula

$$p_{m-1}(x) = f(0) + A_1(x) + \ldots + A_{m-1}(x) \tag{2.20}$$

where each A_k is either a homogeneous form of degree k or else identically zero. In addition, the A_k are given by the formulas

$$A_{m-1}(x) = \frac{1}{(m-1)!} \lim_{n\to+\infty} p^{(m-1)n} \Delta^k_{p^{-n}x} f(0), \tag{2.21}$$

$$A_k(x) = \frac{1}{k!} \lim_{n\to+\infty} p^{kn} \left\{ \Delta^k_{p^{-n}x} f(0) - \sum_{j=k+1}^{m-1} \Delta^k_{p^{-n}x} A_j(0) \right\}, \tag{2.22}$$

for $1 \leqslant k \leqslant m-2$.

Proof. We shall proceed by induction on m. From Lemma (2.1), the proposition holds for $m = 2$, with $A_1(x) = A(x)$. Assuming that the theorem holds for a given positive integer m, we shall prove it for $m+1$. By the hypothesis, we have

$$\left\| \Delta^{m+1}_{h_1 \ldots h_{m+1}} f(x) \right\| \leqslant \delta$$

for all x and h_j in E, $(j = 1 \ldots m+1)$.

Put

$$g(x, y) = \Delta_y f(x) = f(x+y) - f(x). \tag{2.23}$$

Then, treating y as a fixed parameter we have

$$\left\| \Delta^m_{h_1 \ldots h_m} g(x, y) \right\| = \left\| \Delta^m_{h_1 \ldots h_m} \Delta_y f(x) \right\| \leqslant \delta \tag{2.24}$$

for each fixed y and all x and h_j in E, $(j = 1 \ldots m+1)$.

By (2.24) and the induction hypothesis, there exists, for each fixed $y \in E$, a map $p : E \to F$ defined by $p(x, y)$ for all x in E which is of degree at most $m-1$ in x such that

$$\|g(x, y) - p(x, y)\| \leqslant \delta \tag{2.25}$$

for all x and y in E. More precisely $p(x, y)$ has the form

$$p(x, y) = g(0, y) + q(x, y) + h(x, y) \tag{2.26}$$

where $h(x, y)$ is a homogeneous form of degree $m-1$ or else is identically zero, while $q(x, y)$ is a map of degree at most $m-2$ in x, and $q(0, y) = 0$. From, (2.23) and (2.26) and (2.25) we obtain

$$\|f(x+y) - f(x) - f(y) + f(0) - q(x, y) - h(x, y)\| \leqslant \delta \tag{2.27}$$

for all x and y in E.

Now, in view of (2.27) and Lemma (2.24), the map $A_m : E \to F$ defined by $A_m(x) = m^{-1} H(x, x)$, $x \in E$, is either zero or else a homogeneous form of degree m. In addition, we have

$$A_m(x) = \frac{1}{m!} \lim_{n\to+\infty} p^{mn} \Delta^m_{p^{-n}x} f(0), \tag{2.28}$$

According to Lemma 2 [21] if we put

$$f_1(x) = f(x) - A_m(x) \tag{2.29}$$

then the map f_1 satisfies the conditions of Lemma (2.4) for $k = m - 1$; consequently, there exists the map $A_{m-1} : E \to F$ given by

$$A_{m-1}(x) = \frac{1}{(m-1)!} \lim_{n\to+\infty} p^{(m-1)n}\left\{\Delta^{m-1}_{p^{-n}x} f(0) - \Delta^{m-1}_{p^{-n}x} A_m(0)\right\} \tag{2.30}$$

which is either identically zero or else a homogeneous form of degree $m - 1$. Again by Lemma 2 of [21], if we put

$$f_2(x) = f_1(x) - A_{m-1}(x) \tag{2.31}$$

then f_2 satisfies the conditions of Lemma (2.4) for $k = m - 2$ which leads to the existence of the limit

$$A_{m-1}(x) = \frac{1}{m!} \lim_{n\to+\infty} p^{(m-1)n}\Delta^{m-1}_{p^{-n}x} f_2(0), \tag{2.32}$$

and

$$A_{m-2}(x) = \frac{1}{(m-2)!} \lim_{n\to+\infty} p^{(m-2)n}\left\{\Delta^{m-2}_{p^{-n}x} f(0) - \Delta^{m-2}_{p^{-n}x} A_{m-2}(0)\right\} \tag{2.33}$$

continuing in this way, we arrive at the map

$$f_{m-2}(x) = f(x) - A_3(x) - \ldots - A_m(x) \tag{2.34}$$

where the $A_k(x)$ are given by formula (2.22) in the statement of our theorem and where f_{m-2} satisfies the inequality

$$\| f_{m-2}(x+y) - f_{m-2}(x) - f_{m-2}(y) + f_{m-2}(0) - h(x,y)\| \leqslant \delta \tag{2.35}$$

in which $h(x, y)$ is either identically zero or a homogeneous form of degree one in x. Applying Lemma (2.4) once more and putting $A_2(x) = \frac{1}{2}h(x, x)$, we have

$$A_2(x) = \frac{1}{2} \lim_{n\to+\infty} p^{2n}\Delta^2_{p^{-n}x} f_{m-2}(0)$$

which, in view of (2.34), also agrees with formula (2.22) of the theorem. Finally on putting

$$f_{m-1}(x) = f_{m-2}(x) - A_2(x) = f(x) - A_2(x) - \ldots - A_m(x) \tag{2.36}$$

and in view of Lemma (2) [14] for the case $k = 2$, we get the inequality

$$\| f_{m-1}(x+y) - f_{m-1}(x) - f_{m-1}(y) + f_{m-1}(0)\| \leqslant \delta \tag{2.37}$$

for all x and y in E.

Since f_{m-1} satisfies (2.37), it follows from the Lemma (2.3) that there exists an additive map

$$A_1(x) = \lim_{n\to+\infty} p^n \Delta_{p^{-n}x} f_{m-1}(0) \tag{2.38}$$

satisfying the inequality

$$\| f_{m-1}(x) - f_{m-1}(0) - A_1(x)\| \leqslant \delta \tag{2.39}$$

for all x in E. Obviously $A_1(x)$ agrees with formula (2.22) by (2.36) and (2.34). By substituting (2.36) into (2.39) and observing that $f_{m-1}(0) = f(0)$, we obtain

$$\| f(x) - f(0) - A_1(x) - \ldots - A_m(x)\| \leqslant \delta$$

which is equivalent to conditions (2.21) and (2.22) of our proposition with m replaced by $m+1$. Thus the induction proof has been completed and Proposition (2.5) established. □

Theorem 2.6. *Let E be a vector space, F be a non-Archimedean Banach space and $p \in \mathbb{N}^*$, $|p| \neq 1$. Let δ be a fixed positive number and $f : E \to F$ be a function satisfying the inequality*

$$\left\|\Delta_h^m f(x)\right\| \leqslant \delta, \ x, h \in E. \tag{2.40}$$

Then there exists a GP function $p_{m-1} : E \to F$ which is of degree at most $m-1$, such that,

$$\| f(x) - p_{m-1}(x)\| \leqslant \delta \text{ for all } x \text{ in } E. \tag{2.41}$$

Proof. We have f satisfy

$$\left\|\Delta_h^m f(x)\right\| = \left\|\sum_{k=0}^{m}(-1)^{m-k}\binom{m}{k}f(x+kh)\right\| \leqslant \delta, \ x, h \in E.$$

By putting $g_k = (-1)^{m-k}\binom{m}{k}f$, we have

$$\left\|\sum_{k=0}^{m} g_k(x+kh)\right\| \leqslant \delta, \ x, h \in E. \tag{2.42}$$

For $0 \leqslant j \leqslant k \leqslant m$ let $\alpha_{jk} = k - j$ so that $\alpha_{jk} \neq 0$ if $j < k$ and $\alpha_{kk} = 0$.
For $0 \leqslant k \leqslant m$ and $x, y, h_1 \in E$,

$$(x + mh_1) + k(y - h_1) = x + ky + \alpha_{km}h_1$$

From (2.40) and (2.42) we get that

$$\begin{aligned}
&\left\|\sum_{k=0}^{m}(g_k(x+ky+\alpha_{km}h_1) - g_k(x+ky))\right\| \\
&\leqslant max\left\{\left\|\sum_{k=0}^{m} g_k(x+ky+d_{km}h_1)\right\|, \left\|\sum_{k=0}^{m} g_k(x+ky)\right\|\right\} \\
&\leqslant \delta
\end{aligned}$$

and since $\alpha_{mm} = 0$, we obtain that

$$\left\|\sum_{k=0}^{m-1}\Delta_{\alpha_{km}h_1} g_k(x+ky)\right\| \leqslant \delta \tag{2.43}$$

Repeating the argument that led from (2.40) to (2.41) we find that

$$\left\|\sum_{k=0}^{m-2}\Delta_{\alpha_{k,m-1}h_2}\Delta_{\alpha_{km}h_1} g_k(x+ky)\right\| \leqslant \delta \tag{2.44}$$

for all $x, y, h_1, h_2 \in E$. Repeating this reasoning $m-2$ times, we obtain that

$$\|\Delta_{\alpha_{01}h_m} \ldots \Delta_{\alpha_{0m}h_1} g_0(x)\| \leqslant \delta \tag{2.45}$$

for all $x, y, h_1, \ldots h_m \in E$. Since $g_0 = (-1)^m f$ and $\alpha_{0k} \neq 0$ for $1 \leqslant k \leqslant m$, the inequality (2.45) simply asserts that

$$\|\Delta_{h_m} \ldots \Delta_{h_1} f(x)\| \leqslant \delta \tag{2.46}$$

for all $x, h_1, \ldots, h_m \in E$. Thus, by Proposition (2.5), there exists a GP function $p_{m-1} : E \to F$, of degree at most $m-1$ such that

$$\|f(x) - p_{m-1}(x)\| \leqslant \delta, \tag{2.47}$$

which completes the proof of Theorem 2.6. □

3. Main result

In this section we obtain the non-Archimedean Hyers–Ulam stability of the K-quadratic functional equation.

Lemma 3.1. *Let E be a vector space, F a non-Archimedean Banach space, K a finite subgroup of the group of automorphisms of E and $\kappa = cardK$. Let $f : E \to F$ satisfy*

$$\|\Sigma_{k\in K} f(x + k\cdot y) - \Sigma_{k\in K} f(k\cdot y) - \kappa f(x)\| \leqslant \delta, \; x, y \in E. \tag{3.1}$$

Then

$$\left\|\Delta_v^{\kappa} f(u) - g(v)\right\| \leqslant \frac{\delta}{|\kappa|}, \; u, v \in E, \tag{3.2}$$

with $g(x) = -\sum_{i=0}^{\kappa-1} (-1)^{\kappa-i} \sum_{j=1}^{\binom{\kappa}{i}} f(\Sigma_{k\in K_{ij}} k\cdot x)$ and $K_{ij} \subset K$ are pairwise different sets such that $cardK_{ij} = \kappa - i$ for $j \in \left\{1, \ldots, \binom{\kappa}{i}\right\}, \; i \in \{0, \ldots, \kappa\}$.

Proof. We have

$$\sum_{j=1}^{\binom{\kappa}{i}} \sum_{\lambda\in K} f\left(\sum_{k\in K_{ij}} \lambda k\cdot x\right) = \kappa \sum_{j=1}^{\binom{\kappa}{i}} f\left(\sum_{k\in K_{ij}} k\cdot x\right), \; x \in E, \tag{3.3}$$

since for all $\beta \in K$,

$$\beta K_{ij} = K_{ik}, \; i \in \{0, \ldots, \kappa\}, \; j, \; k \in \left\{1, \ldots, \binom{\kappa}{i}\right\}.$$

Now, fix $u, \; v \in E$. Let

$$x_i = u + iv,\ \ y_{ij} = \sum_{k \in K_{ij}} k \cdot v,\ \ i \in \{0, \ldots, \kappa\},\ j \in \left\{1, \ldots, \binom{\kappa}{i}\right\}.$$

For all $\beta \in K,\ i \in \{0, \ldots, \kappa\},\ j \in \left\{1, \ldots, \binom{\kappa}{i}\right\}$ we have the two following cases

Case 1, $\beta^{-1} \in K_{ij}$. Thus $i \neq \kappa$, let $k \in \left\{1, \ldots, \binom{\kappa}{i+1}\right\}$ be such that $K_{ij} = K_{(i+1)k} \bigcup \{\beta^{-1}\}$. So, we have

$$\begin{aligned} x_i + \beta y_{ij} &= u + iv + \sum_{l \in K_{ij}} \beta l \cdot v = u + (i+1)v + \sum_{l \in K_{(i+1)k}} \beta l \cdot v \\ &= x_{i+1} + \beta y_{(i+1)k} \end{aligned}$$

Case 2, $\beta^{-1} \notin K_{ij}$. Since $i \neq 0$, let $k \in \left\{1, \ldots, \binom{\kappa}{i-1}\right\}$ be such that$K_{(i-1)k} = K_{ij} \bigcup \{\beta^{-1}\}$. By a similar calculation to the previous we obtain

$$x_i + \beta y_{ij} = x_{i-1} + \beta y_{(i-1)k}$$

Consequently, from the above consideration we get

$$\sum_{i=0}^{\kappa-1} (-1)^{\kappa-i} \sum_{j=1}^{\binom{\kappa}{i}} \sum_{\lambda \in K} f(x_i + \lambda y_{ij}) = 0. \tag{3.4}$$

Now, in view of (3.1), (3.3) and (3.4) we have

$$\begin{aligned} \left\| \kappa \Delta_v^\kappa f(u) - \kappa g(v) \right\| &= \left\| \kappa \sum_{i=0}^{\kappa} (-1)^{\kappa-i} \binom{\kappa}{i} f(u + iv) + \kappa \sum_{i=0}^{\kappa-1} \sum_{j=1}^{\binom{\kappa}{i}} (-1)^{\kappa-i} f\left(\sum_{k \in K_{ij}} k \cdot v \right) \right\| \\ &= \left\| \kappa \sum_{i=0}^{\kappa} (-1)^{\kappa-i} \binom{\kappa}{i} f(u + iv) + \sum_{0}^{\kappa-1} \sum_{j=1}^{\binom{\kappa}{i}} \sum_{\lambda \in K} (-1)^{\kappa-i} f\left(\sum_{k \in K_{ij}} \lambda k \cdot v \right) \right\| \\ &= \left\| \sum_{i=0}^{\kappa-1} (-1)^{\kappa-i} \sum_{j=1}^{\binom{\kappa}{i}} \left[\sum_{\lambda \in K} f(x_i + \lambda y_{ij}) - \kappa f(x_i) - \sum_{\lambda \in K} f(\lambda y_{ij}) \right] \right\| \\ &\leqslant \delta. \end{aligned}$$

This ends the proof. □

Theorem 3.2. *Let E be a vector space, F a non-Archimedean Banach space, K a finite subgroup of the group of automorphisms of E and* $\kappa = cardK$. *Let* $f : E \to F$ *satisfy*

$$\|\Sigma_{k\in K} f(x+k\cdot y) - \Sigma_{k\in K} f(k\cdot y) - \kappa f(x)\| \leqslant \delta, \; x,y \in E. \tag{3.5}$$

Then there exists a unique GP function $p : E \to F$ *solution of (1,1), of degree at most* κ, *such that*

$$\|f(x) - f(0) - p(x)\| \leqslant \frac{\delta}{|\kappa|}. \tag{3.6}$$

Proof. According to (3.5), we have

$$\|\Delta_v^\kappa f(u) - g(v)\| \leqslant \frac{\delta}{|\kappa|}, \; u,v \in E. \tag{3.7}$$

Replacing u by $u+v$ we get

$$\|\Delta_v^\kappa f(u+v) - g(v)\| \leqslant \frac{\delta}{|\kappa|}. \tag{3.8}$$

By (3.7) and (3.8) we obtain

$$\|\Delta_v^{\kappa+1} f(u)\| \leqslant \frac{\delta}{|\kappa|}. \tag{3.9}$$

Then by Theorem (2.6) there exists a GP function $q : E \to F$, of degree at most κ, such that

$$\|f(x) - q(x)\| \leqslant \frac{\delta}{|\kappa|}. \tag{3.10}$$

For $0 \leqslant k \leqslant \kappa$, there is a rational-homogeneous form of degree k $A_k : E \to F$ such that

$$q(x) = f(0) + \sum_{k=1}^{m=\kappa} A_k(x). \tag{3.11}$$

By (3.5) and (3.10), for all $x, y \in E$,

$$\left\|\sum_{k\in K} q(x+k.y) - \kappa q(x) - \kappa q(y)\right\| \tag{3.12}$$

$$\leqslant max\left\{\left\|\sum_{k\in K}(q(x+k.y) - f(x+k.y))\right\|, \left\|\sum_{k\in K}(q(k.y) - f(k.y))\right\|,\right.$$

$$\left.\|\kappa(q(x) - f(x))\|, \|\Sigma_{k\in K} f(x+k\cdot y) - \Sigma_{k\in K} f(k\cdot y) - \kappa f(x)\|\right\}$$

$$\leqslant \frac{\delta}{|\kappa|}. \tag{3.13}$$

Now (3.11) says, in light of (3.12) that, for all $x, y \in E$,

$$\left\| -\kappa f(0) + \sum_{j=1}^{\kappa} \sum_{k\in K} A_j(x+k.y) - Aj(k.y) - \sum_{j=1}^{m=\kappa} A_j(x)) \right\| \leqslant \frac{\delta}{|\kappa|}. \tag{3.14}$$

In (3.13) replace x by rx and y by ry ($r \in \mathbb{Q}$) to conclude that, for all $x, y \in E$ and all $r \in \mathbb{Q}$,

$$\left\| -\kappa f(0) + \sum_{j=1}^{\kappa} r^j \sum_{k\in K} (A_j(x+k.y) - \sum_{j=1}^{\kappa} r^j \kappa A_j(x) - \sum_{k\in K} \sum_{j=1}^{\kappa} r^j A_j(k.y)) \right\| \leqslant \frac{\delta}{|\kappa|} \tag{3.15}$$

By continuity (3.14) holds for all real r and all $x, y \in E$. Now suppose that $\phi : F \to \mathbb{R}$ is a continuous linear functional. Then by (3.14),

$$\left\| -\phi(\kappa f(0)) + \sum_{j=1}^{m=\kappa} r^j \phi \left\{ \sum_{k\in K} (A_j(x+k.y) - \sum_{j=1}^{\kappa} \kappa A_j(x) - \sum_{k\in K} \sum_{j=1}^{\kappa} A_j(y)) \right\} \right\| \leqslant \frac{\delta}{|\kappa|} ||\phi|| \tag{3.16}$$

for all $x, y \in E$ and all $r \in \mathbb{R}$.

Since a real polynomial function is bounded if and only if it is constant, from the last inequality we surmise that, for $1 \leqslant j \leqslant \kappa$,

$$\phi \left\{ \sum_{k\in K} (A_j(x+k.y) - \kappa A_j(x) - \sum_{k\in K} A_j(k.y)) \right\} = 0 \tag{3.17}$$

for all $x, y \in E$. Since this is so for every continuous linear functional $\phi : F \to \mathbb{R}$, by the Hahn-Banach theorem,

$$\sum_{k\in K} (A_j(x+k.y) - \kappa A_j(x) - \sum_{k\in K} A_j(y)) = 0 \ \textit{for} \ x, y \in E \ \textit{and} \ 1 \leqslant j \leqslant \kappa. \tag{3.18}$$

Letting $p(x) = q(x) - q(0)$ then p is a GP function of degree at most κ and by (3.17) it is a solution of Eq. (3.5),

$$\sum_{k\in K} (p(x+k.y) - \kappa p(x) - \sum_{k\in K} p(k.y)) = 0 \ \textit{for} \ x, y \in E. \tag{3.19}$$

Finally, by (3.10) and (3.18) we get the result, $\|f(x) - f(0) - p(x)\| < \frac{\delta}{|\kappa|}, \ x \in E$.

Let p' be another GP function solution of (1.1) of degree at most κ such that

$$\|f(x) - f(0) - p'(x)\| < \frac{\delta}{|\kappa|}, \ x \in E.$$

Then we get $\|p(x) - p'(x)\| < \frac{\delta}{|\kappa|}, \ x \in E$. Thus, necessarily $p = p'$. This ends the proof. □

Theorem 3.3. *Let E be a vector space, F a non-Archimedean Banach space, K a finite subgroup of the group of automorphisms of E and $\kappa = cardK$. Let $f, g, h : E \to F$ be functions satisfying*

$$\|\Sigma_{k\in K} f(x + k \cdot y) - \kappa g(x) - \kappa h(y)\| \leqslant \delta, \ x, y \in E. \tag{3.20}$$

Then there exists a unique GP function $p : E \to F$ solution of (1,1), of degree at most κ, such that

$$\|f(x) - f(0) - p(x)\| \leqslant \frac{\delta}{|\kappa|}, \; x \in E, \tag{3.21}$$

$$\left\| \kappa h(x) - \kappa h(0) - \sum_{k \in K} p(k.x) \right\| \leqslant \frac{\delta}{|\kappa|}, \; x \in E \tag{3.22}$$

and

$$\|g(x) - g(0) - p(x)\| \leqslant \frac{\delta}{|\kappa|}, \; x \in E. \tag{3.23}$$

Proof. By posing that $f' = f - f(0)$, $g' = g - g(0)$, and $h' = h - h(0)$, it is clear that f', g', h' satisfy (3.20). First we observe that:

$$\left\| \kappa h'(y) - \sum_{k \in K} f'(k.y) \right\| \leqslant \delta \tag{3.24}$$

and

$$\|\kappa g'(x) - \kappa f'(x)\| \leqslant \delta. \tag{3.25}$$

From the above inequality (3.20), (3.24) and (3.25) we have

$$\|\Sigma_{k \in K} f'(x + k \cdot y) - \Sigma_{k \in K} f'(k \cdot y) - \kappa f'(x)\| \leqslant \delta. \tag{3.26}$$

By Theorem (2.5) and inequality (3.24) and (3.25) the result follows. □

Corollary 3.4. *Let E be a vector space, F a non-Archimedean Banach space, K a finite subgroup of the group of automorphisms of E and $\kappa = cardK$. Let $f, h : E \to F$ be functions satisfying*

$$\|\Sigma_{k \in K} f(x + k \cdot y) - \kappa g(x)\| \leqslant \delta, \; x, y \in E. \tag{3.27}$$

Then there exists a unique GP function $p : E \to F$, solution of K-Jensen equation,

$$\Sigma_{k \in K} p(x + k \cdot y) = \kappa p(x), \; x, y \in E,$$

of degree at most κ, such that

$$\|f(x) - f(0) - p(x)\| \leqslant \frac{\delta}{|\kappa|}, \; x \in E, \tag{3.28}$$

and

$$\|g(x) - g(0) - p(x)\| \leqslant \frac{\delta}{|\kappa|}, \; x \in E. \tag{3.29}$$

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